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A NOTE TO E. MIERSEMANN'S PAPERS ON HIGHER EIGENVALUES OF VARIATIONAL INEQUALITIES

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Abstract: An improvement of E. Miersemann's result on higher eigenvalues of variational inequalities and some examples, for which the obtained criterion is sharp, are given.

Key words: variational inequality, eigenvalue problem

Classification: 49H05, 73H10

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1. INTRODUCTION

Let H be a real separable Hilbert space and $K \subset H$ a closed convex cone with its vertex at zero (see [3]). Let $A: H \to H$ be a linear, completely continuous, symmetric and positive operator. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots > 0$ be the eigenvalues of the operator A and let the corresponding eigenvectors $u_1, u_2, u_3, \ldots$ form an orthonormal basis of H.

We are interested in the eigenvalue problem for the variational inequality

(1) $\quad u \in K: \quad (\lambda u - Au, v-u) \geq 0 \quad$ for all $v \in K$,

where λ is a real eigenvalue parameter and we look for non-trivial solutions u of (1).

We shall denote $\sigma_K(A)$ the set of all eigenvalues to (1).

2. E. MIERSEMANN'S RESULT

Denote E_n the linear hull of $\{u_1,\dots,u_n\}$, L_n the eigenspace to λ_n, $B = \{u \in H;\ \|u\| \leq 1\}$, $S = \{u \in H;\ \|u\| = 1\}$, $S_n = E_n \cap S$. Further let P be an orthogonal projection of H onto E_n.

In [1,2,3] the following assertions are proved:

Theorem 1. Let $\tilde{H} \subset H$ be a closed subspace, $\tilde{H} \subset K$. Denote $\tilde{P}$ the orthogonal projection of H onto $\tilde{H}$. We consider the equation

$$u \in \tilde{H}: \quad \tilde{P}Au = \tilde{\lambda} u$$

and assume that there exist at least n positive eigenvalues $\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_n$. Let

$$(2) \qquad \tilde{\lambda}_n > \lambda_{n+1} .$$

Then there exists an eigenvalue $\lambda \in \sigma_K(A) \cap (\lambda_{n+1}, \lambda_n \rangle$.

Theorem 2. Let $V = \{v \in E_n^{\perp};\ u+v \in K \text{ for all } u \in S_n\}$ be nonempty and suppose

$$(3) \qquad \lambda_n > \lambda_{n+1} + \inf_{v \in V} \{\lambda_{n+1} \|v\|^2 - (Av,v)\} .$$

Then there exists $\lambda \in \sigma_K(A) \cap (\lambda_{n+1}, \lambda_n \rangle$.

Remark 1. The assumptions of Theorem 2 are fulfilled, if e.g. $u_{n+1} \in K^o$ (= interior of K).

Theorem 3. Let the assumptions of Theorem 1 or Theorem 2 be fulfilled and let, moreover, $L_n \not\subset K$.

Then there exists $\lambda \in \sigma_K(A) \cap (\lambda_{n+1}, \lambda_n)$.

The idea of the proof is following:

Define N_α the class of all compact sets $F \subset K \cap S$ such that

(a) $\min_{u \in F} (Au,u) \geq \lambda_{n+1} + \alpha$

(b) F is not contractible within the set $R = \{u \in H;\ Pu \neq 0\}$.

Using (2) or (3) it is proved that the class N_α is nonempty for a suitable $\alpha > 0$ and then using some topological technique (see [1]) it is proved that there exists $u \in K \cap S$ such that

$$(Au,u) = \sup_{F \in N_\alpha} \min_{v \in F} (Av,v)$$

which is also a solution of the variational inequality (1) with corresponding eigenvalue $\lambda \in \langle \lambda_{n+1} + \alpha, \lambda_n \rangle$ (resp. $\lambda \in \langle \lambda_{n+1} + \alpha, \lambda_n)$).

3. IMPROVEMENT OF E. MIERSEMANN'S RESULT

We shall weaken conditions (2),(3) in Theorems 1,2. A slight weaker version of Theorem 4 was obtained also by prof. Miersemann (personal communication).

<u>Lemma</u>. Let $A_k : H \to H$ be linear continuous operators, $A_k \to A$ in the operator norm (the operator A is supposed to satisfy the assumptions from Section 1). Let $u^k \in K \cap S$, $\lambda^k \in \langle c_1, c_2 \rangle$ (where c_1, c_2 are positive constants) and

$$(\lambda^k u^k - A_k u^k, v - u^k) \geq 0 \quad \text{for all } v \in K .$$

Then there exists a subsequence (we denote it as before) such that $\lambda^k \to \lambda$, $u^k \to u$ and

$$(\lambda u - Au, v-u) \geq 0 \quad \text{for all } v \in K .$$

Proof. We may suppose $\lambda^k \to \lambda$, $u^k \to u \in K \cap B$.
Then $\lambda^k = (A_k u^k, u^k) \to (Au,u)$, hence

(4) $\lambda = (Au,u)$, $u \neq 0$.

Further $0 \leqq (\lambda^k u^k - A_k u^k, v) \to (\lambda u - Au, v)$ for all $v \in K$, thus

(5) $(\lambda u - Au, v) \geqq 0$ for all $v \in K$.

Putting $v=u$ in (5) and using (4) we get

$$\lambda \|u\|^2 \geqq (Au,u) = \lambda ,$$

thus $u \in K \cap S$ and $u^k \to u$.

Theorem 4. Suppose that $E_n^{\perp} \cap K \cap S \neq \emptyset$ and put $c_n = \sup_{u \in E_n^{\perp} \cap K \cap S} (Au,u)$. Assume instead of the conditions (2), (3) in Theorems 1,2 the conditions

(2*) $\tilde{\lambda}_n > c_n$

(3*) $\lambda_n \geqq c_n + \inf_{v \in V^*} \{ c_n \|v\|^2 - (Av,v) \}$,

where $V^* = \{ v \in E_n^{\perp} ; u+v \in K \text{ for all } u \in S_n^* \}$,

$$S_n^* = \left\{ u \in E_n - \{0\} ; \|u\|^2 = \frac{\lambda_n - c_n}{\frac{(Au,u)}{\|u\|^2} - c_n} \right\}$$

and V^* is supposed to be nonempty.

Then there exists $\lambda \in \sigma_K(A) \cap \langle c_n, \lambda_n \rangle$ (and $\lambda < \lambda_n$ if $L_n \not\subset K$).

Remark 2. Obviously $c_n \leqq \lambda_{n+1}$ and it can be easily proved $V \subset V^*$, hence (2) $\Rightarrow$ (2*), (3) $\Rightarrow$ (3*).

If $V^* \neq \emptyset$ then $E_n^{\perp} \cap K \cap S \neq \emptyset$.

Proof of Theorem 4.

1. First suppose (2^*) or that in (3^*) strong inequality holds. Define N^*_α the class of all compact sets $F \subset K \cap S$ such that

(a^*) $\min_{u \in F} (Au,u) \geq c_n + \alpha$

(b) F is not contractible within the set $R = \{u \in H;\ Pu \neq 0\}$.

If (2^*) holds, then $S \cap \tilde{E}_n \in N^*_\alpha$ for some $\alpha > 0$ ($\tilde{E}_n$ denotes the linear hull of the first n eigenvectors of the equation $\tilde{P}Au = \tilde{\lambda} u$). If in (3^*) strong inequality holds, then the set $F = \left\{ \frac{u+v}{\|u+v\|} ;\ u \in S^*_n \right\}$ belongs to N^*_α for a suitable $v \in V^*$ and $\alpha > 0$ (cf. [2,3]). Hence in both cases $N^*_\alpha \neq \emptyset$ for some $\alpha > 0$ and the remaining part of the proof is nearly the same as in [1].

2. If (3^*) holds and $\lambda_n = c_n + \inf_{v \in V^*} \{ c_n \|v\|^2 - (Av,v) \}$,

then put $A_k u = (1 + \frac{1}{k}) APu + A(I-P)u$, use the proved part of Theorem 4 for A_k and then use Lemma.

<u>Theorem 5</u>. Let $u_n \notin K$ and let the set V (see Theorem 2) be nonempty. Choose $v \in V$ and put

(6) $$d_n = \inf_{0 \leq s < \frac{1}{\sqrt{1+\|v\|^2}}} c_n(s), \quad \text{where} \quad c_n(s) = \sup_{\substack{u \in S \cap K \\ Pu = su_n}} (Au,u)$$

Suppose

(3^{**}) $\lambda_n > d_n + d_n \|v\|^2 - (Av,v)$.

Then there exists $\lambda \in \sigma_K(A) \cap (d_n, \lambda_n)$.

<u>Remark 3</u>. Obviously $d_n \leq c_n = c_n(0)$.

<u>Remark 4</u>. The assumption $v \in V$ guarantees that the set

$\{u \in S \cap K \; ; \; Pu = su_n\}$ is nonempty for all $|s| \leq \frac{1}{\sqrt{1+\|v\|^2}}$.

Remark 5. In (6) we could put $d_n = \inf_{\substack{z \in E_n \\ \|z\|^2 < \frac{1}{1+\|v\|^2}}} \sup_{\substack{u \in S \cap K \\ Pu = z}} (Au,u)$,

but then we would loose the estimate $\lambda \leq \lambda_n$.

Remark 6. There can be stated an analogous condition (2^{**}).

Idea of the proof of Theorem 5:

There exists $s \in \langle 0, \frac{1}{\sqrt{1+\|v\|^2}})$ such that

$$\lambda_n > c_n(s) + c_n(s)\|v\|^2 - (Av,v) .$$

We define N_α^{**} the class of all compact sets $F \subset K \cap S$ such that

(a^{**}) $\min_{u \in F} (Au,u) \geq c_n(s) + \alpha$

(b^{**}) F is not contractible within the set $R(s) = \{u \in H ; \; Pu \neq su_n\}$.

Then the set $F = \{ \frac{u+v}{\|u+v\|} \; ; \; u \in S_n \}$ belongs to N_α^{**} for a suitable $\alpha > 0$ and one can use the technique from [1] to obtain the desired result.

4. EXAMPLES AND REMARKS

Example 1. Let $H = R_3$, $A([x_1,x_2,x_3]) = [\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3]$, $\lambda_1 > \lambda_2 > \lambda_3 > 0$, $K = \{u \in H ; \; (u,w_1) \geq 0, \; (u,w_2) \geq 0\}$, where $w_1 = [M(a-1),-1,a]$, $w_2 = [M(a-1),a,-1]$ $(a > 1, \; M > 0)$.

Let us fix $a > 1$. Using elementary calculus we get that the problem (1) has an eigenvalue $\lambda \neq \lambda_1$ if and only if

$$M \leq M_1 = \sqrt{c(A)\,\frac{a+1}{a-1} - \frac{1}{2}}\ , \quad \text{where} \quad c(A) = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{2(\lambda_2 - \lambda_3)}$$

(and it has exactly two eigenvalues different from λ_1 iff $M < M_1$).

Theorem 5 is available (with n=1) if $M < M_1$, using Lemma we get the positive result also for $M = M_1$.

Theorem 4 is available if $M \leq M_2 = \sqrt{c(A)\,\frac{a^2+1}{a^2-1} - \frac{1}{2}}$ $(< M_1)$,

Theorem 3 is available only for $M < M_3 = \sqrt{c(A) - \frac{1}{2}}$ $(< M_2)$.

Unfortunately, using our variational approach we get (for $M < M_1$) only one of two existing eigenvalues different from λ_1. We do not get the eigenvector $u \in \partial K \cap S$, where the functional (Au,u) attains a local minimum on $\partial K \cap S$ (∂K denotes the boundary of K).

<u>Example 2</u>. Let H,K,A satisfy the general assumptions from Section 1. Let $u_1,\dots,u_n \in K^o$, $\lambda_n > \lambda_{n+1}$ and $\{u_1,\dots,u_n\}^{\perp} \cap K^o = \emptyset$ ($\Leftrightarrow V = \emptyset$). Suppose $u_k \notin K$ for $k > n$. Then the problem (1) has no eigenvalue λ with $\lambda < \lambda_n$.

Proof. Suppose $\lambda < \lambda_n$, $u \in K$, $(\lambda u - Au, v-u) \geq 0$ for all $v \in K$. Let us write $u = \sum_{i=1}^{n} \alpha_i u_i + w$, where $w \in E_n^{\perp}$.

Putting $v = u + u_i$ we get $(\lambda - \lambda_i)\alpha_i \geq 0$, thus $\alpha_i \leq 0$ $(i=1,\dots,n)$. Suppose $\alpha_i < 0$ for some i, then

$-\sum_{i=1}^{n} \alpha_i u_i \in K^o$, $\quad w = u - \sum_{i=1}^{n} \alpha_i u_i \in K^o$, which gives us a contradiction. Thus $\alpha_i = 0$ for all $i = 1,\dots,n$, $u = w$.

Putting $v=u+u_1+\tilde{w}$, $\tilde{w}\in E_n^{\perp}$ arbitrary (but small), we get $\lambda u=Au$. Since $u_k\notin K$ for $k>n$, we have $u=0$.

Remark 7. Let $f:H\to R$ be a weakly continuous functional of the class C^2, $f'(0)=0$ and let the second Fréchet derivative f'' be bounded (on bounded sets). Denote $A=f''(0)$ and suppose that A fulfils the assumptions of Section 1. Then the eigenvalue λ to (1), which we get in Theorems 1-5, is also a bifurcation point for the variational inequality

(7) $\quad u\in K: \quad (\lambda u - f'(u),\ v-u) \geq 0 \quad$ for all $v\in K$

(see [1]). The following example shows that a general eigenvalue λ to (1) (which is not an eigenvalue of the operator A) need not be a bifurcation point for (7).

Example 3. Let $H=R_3$, let $A:H\to H$ be a symmetric linear operator with eigenvalues $\lambda_1>\lambda_2>\lambda_3>0$ and corresponding eigenvectors u_1,u_2,u_3. Put

$K=\{u\in H;\ (u,u_1)\geq 0,\ (u,u_3-u_2)\geq 0\}$, $\quad f(u)=\frac{1}{2}(Au,u)+\|u\|^2(u,u_1)$.

Then $u=u_2+u_3$ is an eigenvector to (1) with $\lambda=\frac{\lambda_2+\lambda_3}{2}$, since $(\lambda u-Au,v-u)=\frac{1}{2}(\lambda_2-\lambda_3)(u_3-u_2,v)\geq 0$ for all $v\in K$.

Suppose $u\in K$, $\lambda\leq\lambda_1$ and $(\lambda u-f'(u),v-u)\geq 0$ for all $v\in K$. Putting $v=u+u_1$ we get

$$0\leq(\lambda u-f'(u),u_1)=(\lambda u-Au-\|u\|^2u_1-2u(u,u_1),u_1)=$$

$$=-\|u\|^2+(u,u_1)(\lambda-\lambda_1-2(u,u_1))\leq-\|u\|^2\ ,$$

thus $u=0$. Hence $\lambda=\frac{1}{2}(\lambda_2+\lambda_3)$ is not a bifurcation point for (7).

Remark 8. Suppose that the assumptions from Section 1 are fulfilled. Then the set $\sigma_K(A)$ is nonempty and closed in $R^+ = \{\lambda \in R; \lambda > 0\}$. It would be interesting to investigate the general structure of $\sigma_K(A)$. Example 2 shows that this set may consist only of one point (also for $\dim H = \infty$), Theorems 1-5 assure the existence of higher eigenvalues to (1). There can be constructed examples in R_3, for which the set $\sigma_K(A)$ has infinitely many accumulation points (see Example 5). Nevertheless, it can be proved that for $H=R_3$ the set $\sigma_K(A) \subset R$ has Lebesgue measure zero (this is not true for A nonsymmetric). It is also an open problem (to the author) to find reasonable assumptions on A and K (for $\dim H = \infty$) which would guarantee that the set $\sigma_K(A)$ consists of a sequence of eigenvalues which converge to zero (cf. the following example).

Example 4. Let H be the Hilbert space $W_0^{1,2}(0,\pi)$ with the inner product $(u,v) = \int_0^\pi u'(x)v'(x)\,dx$, let $A:H \to H$ be defined by $(Au,v) = \int_0^\pi u(x)v(x)\,dx$. Let $M \subsetneqq \langle 0,\pi \rangle$ be a closed set and put $K = \{u \in H;\ u \geq 0 \text{ on } M\}$. Then it can be shown that the eigenvalues to (1) form a sequence converging to zero.

Example 5. Let $H = R_3$, let $A:H \to H$ be a symmetric linear operator with eigenvalues $\lambda_1 = \lambda_2 > \lambda_3 > 0$ and corresponding eigenvectors u_1, u_2, u_3.

Put $w_n = \sqrt{1-\frac{1}{n}}\,u_1 + \sqrt{\frac{1}{n}}\,u_2$, $\quad v_n = u_3 + \frac{w_n + w_{n+1}}{1+(w_n, w_{n+1})}$,

$K = \{u \in H;\ (u, u_3 - w_n) \geq 0 \text{ for each } n = 1,2,3,\dots\}$.

Then $v_n \in K$, $\lambda^n = \frac{(Av_n, v_n)}{\|v_n\|^2} = \frac{2\lambda_1 + \lambda_3(1+(w_n, w_{n+1}))}{3 + (w_n, w_{n+1})} \searrow \frac{1}{2}(\lambda_1 + \lambda_3)$,

$$(\lambda^n v_n - Av_n, v) = \frac{\lambda_1 - \lambda_3}{3+(w_n, w_{n+1})}(2u_3 - w_n - w_{n+1}, v) \geq 0 \quad \text{for all} \quad v \in K,$$

hence $\lambda^n \in \sigma_K(A)$ and $\sigma_K(A)$ contains a non-zero accumulation point.

If we put $w_{n,k} = r_k\sqrt{1 - \frac{1}{k} - \frac{1}{n}}\, u_1 + \sqrt{\frac{1}{k} + \frac{1}{n}}\, u_2$,

where $n > k^2$ and $k > 1$ are natural numbers, $r_2 = 1$,

$r_{k+1}^2 = r_k^2 + \frac{1}{8(k+1)^5}$, and $v_{n,k} = u_3 + \frac{w_{n,k} + w_{n+1,k}}{1+(w_{n,k}, w_{n+1,k})}$,

$K = \{u \in H;\ (u, u_3 - w_{n,k}) \geq 0 \text{ for } n > k^2 > 1\}$, then again $v_{n,k}$ is an eigenvector to (1) and $\sigma_K(A)$ contains infinitely many accumulation points $\lambda(k)$, where

$$\lambda(k) = \lim_{n\to\infty} \lambda^{n,k} = \frac{\lambda_1(\frac{1}{k} + r_k^2(1 - \frac{1}{k})) + \quad _3}{\frac{1}{k} + r_k^2(1 - \frac{1}{k}) + 1} \nearrow \frac{\lambda_1 r^2 + \lambda_3}{r^2 + 1}$$

$(r = \lim_{n\to\infty} r_k)$.

Similar example can be constructed also for $\lambda_1 > \lambda_2 > \lambda_3$ (we start with $w_n = c\sqrt{1 - \frac{1}{n^2}}\, u_1 + \frac{1}{cn}\, u_2$, where $c^2(\lambda_2 - \lambda_3) = \lambda_1 - \lambda_3$).

R e f e r e n c e s

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