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CANONICAL LIPSCHITZ STRUCTURES ON COMPACT HILBERT CUBE MANIFOLDS

Jouni LUUKKAINEN

Abstract: Let d be one of the Lipschitz homogeneous metrics on the Hilbert cube Q discovered by Väisälä and Hohti. We show that every compact Q-manifold is homeomorphic to a Lipschitz (Q,d)-manifold of the form M × (Q,d) where M is a compact PL n-submanifold of $\mathbb{R}^n$ for some n and that such an M × (Q,d) is unique up to a Lipschitz homeomorphism.

Key words: Hilbert cube, Lipschitz homeomorphism, Lipschitz manifold, Q-manifold.

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1. Existence. A homeomorphism $f: (X,d) \to (Y,d')$ of metric spaces is a *Lipschitz homeomorphism* if there is $L \geq 1$ such that

$$d(x,y)/L \leq d'(f(x),f(y)) \leq Ld(x,y) \qquad (x,y \in X).$$

Let s be a sequence $s_1 \geq s_2 \geq \dots$ of positive real numbers converging to zero such that ($\mathbb{N} = \{1,2,\dots\}$)

$$R(s) = \sup \{s_k/s_{k+1}: k \in \mathbb{N}\} < \infty.$$

Let Q_s denote the Hilbert cube $Q = [-1,1]^{\mathbb{N}}$ equipped with the compatible metric d,

$$d(x,y) = \sup \{s_k|x_k - y_k|: k \in \mathbb{N}\}.$$

Definition 1. A *Lipschitz Q_s-manifold* is a separable metric space whose every point has a neighborhood Lipschitz homeomorphic to Q_s.

There is an essentially equivalent alternative definition based on atlases; cf. [7, 3.3-3.7]. Definition 1 is natural because A. Hohti [4, 5.3] has proved that every connected Lipschitz Q_s-manifold (and thus, in particular, Q_s itself) is homogeneous with respect to Lipschitz homeomorphisms and because, on the other hand, J. Väisälä [12, 3.5] has proved that this is never true of Q_s if the condition $R(s) < \infty$ is not satisfied. The model cube Q_s is natural also because it is an absolute extensor for Lipschitz maps [8, Theorem 1], which implies, as in [7, 5.12], that every Lipschitz Q_s-manifold is an absolute neighborhood extensor for locally Lipschitz maps.

Example 2. For the cartesian product of finitely many metric spaces use any of the standard Lipschitz equivalent metrics. Define a *Lipschitz* n-*manifold* ($n \in \mathbb{N} \cup \{0\}$) by means of the model cube $I^n = [-1,1]^n$ (see [7]). Since $R(s) < \infty$, the natural homeomorphism

$$I^n \times Q_s \to Q_s, \quad (x,y) \rightsquigarrow (x_1,\ldots,x_n,y_1,y_2,\ldots),$$

is a Lipschitz homeomorphism. Hence, if M is a Lipschitz n-manifold, $M \times Q_s$ is a Lipschitz Q_s-manifold.

Note that every PL homeomorphism of compact polyhedra in $\mathbb{R}^n$ is a Lipschitz homeomorphism [7, 2.18] and that, thus, every PL manifold in $\mathbb{R}^n$ is a Lipschitz manifold. (For PL topology we refer to [10].) It now immediately follows from well-known results on (topological) Q-manifolds [2] that there exists a Lipschitz Q_s-manifold structure on every compact Q-manifold:

Proposition 3. *If* X *is a compact* Q-*manifold, there is a compact* PL n-*manifold* M *in* $\mathbb{R}^n$ *for some* n *such that* X *is homeomorphic to the Lipschitz* Q_s-*manifold* $M \times Q_s$.

Proof. By [2, 36.2] there is a compact polyhedron P in some $\mathbb{R}^n$ such that X is homeomorphic to $P \times Q$. Choose a regular neighborhood M of P in $\mathbb{R}^n$. Then M is a compact PL n-manifold. Since P and M are simple homotopy equivalent, $P \times Q$ and $M \times Q$ are homeomorphic by [2, 29.4]. Hence, X is homeomorphic to $M \times Q_s$, which is a Lipschitz Q_s-manifold by the above. □

Proposition 3 is an observation of Hohti and it is published with his permission. Hohti has since constructed a Lipschitz Q_s-manifold structure on every Q-manifold [5].

2. Uniqueness. We next show that a Lipschitz Q_s-manifold structure on a compact Q-manifold X induced by a homeomorphism $X \approx M \times Q_s$ as in Proposition 3 is unique up to a Lipschitz homeomorphism and may thus be called *canonical*.

Theorem 4. *Let* $M_i \subset \mathbb{R}^{n_i}$ *be a compact* PL n_i*-manifold,* $i=1,2$*, such that* $M_1 \times Q_s$ *and* $M_2 \times Q_s$ *are homeomorphic. Then* $M_1 \times Q_s$ *and* $M_2 \times Q_s$ *are Lipschitz homeomorphic.*

Proof. By [2, 38.1] (and the proof of [2, 29.5]), M_1 and M_2 are simple homotopy equivalent. Hence, by [13, Theorem 25], if we choose a sufficiently large $n \geq \max(n_1,n_2)$ and identify $\mathbb{R}^{n_i}$ with $\mathbb{R}^{n_i} \times 0 \subset \mathbb{R}^n$, every regular neighborhood of M_1 in $\mathbb{R}^n$ is PL homeomorphic to every regular neighborhood of M_2 in $\mathbb{R}^n$. Choose a regular neighborhood N_i of M_i in $\mathbb{R}^{n_i}$. Then $N_i' = N_i \times I^{n-n_i}$ is a regular neighborhood of M_i in $\mathbb{R}^n$, because it collapses onto N_i and, thus, onto M_i. Hence, N_1' and N_2' are PL homeomorphic. Since M_i is PL homeomorphic to N_i, it follows that $M_1 \times I^{n-n_1}$ and $M_2 \times I^{n-n_2}$ are PL homeomorphic. Since Q_s is Lipschitz homeomorphic to $I^{n-n_i} \times Q_s$, this implies that $M_1 \times Q_s$ and $M_2 \times Q_s$ are Lipschitz homeomorphic. □

It is not known whether every two compact Lipschitz Q_s-manifolds are Lipschitz homeomorphic if they are homeomorphic. This problem is equivalent to the problem whether every compact Lipschitz Q_s-manifold is Lipschitz homeomorphic to a Lipschitz Q_s-manifold with a canonical structure. Our final result shows that this is the case for some manifolds of Example 2.

Theorem 5. *Suppose that* M *is a compact Lipschitz manifold and that either* $\dim M = n$ *or* $\dim M = n-1$, $n \neq 4$*, and* $\partial M = \emptyset$*. Suppose also that* M *can be topologically embedded into* $\mathbb{R}^n$*. Let* $p = 6$ *if* $n = 4$ *or* 5 *and let* $p = n$ *otherwise. Then there is a compact* PL p*-manifold* N *in* $\mathbb{R}^p$ *such that* $M \times Q_s$ *is Lipschitz homeomorphic to* $N \times Q_s$.

Proof. Suppose first that M is an n-manifold. We reduce the case $n = 4$ or 5 to the case $n = 6$ replacing M by $M \times I^{6-n}$. It suffices to find a PL n-manifold N in $\mathbb{R}^n$ homeomorphic to M, because then M and N are Lipschitz homeomorphic by the generalization [11, 4.8] of a theorem of D. Sullivan. Choose a manifold $S \subset \mathbb{R}^n$ homeomorphic to M and an open collar $c\colon \partial S \times [0,1) \to S$ of ∂S in S. Then $T = S \setminus c[\partial S \times [0,1/2)]$ is homeomorphic to S and $c[\partial S \times (0,1)]$ is an open bicollar neighborhood of $\partial T = c[\partial S \times \{1/2\}]$ in $\mathbb{R}^n$. Hence, by [6, I, 5.1 and 4.1] if $n \geq 6$ or by classical results [9] if $n \leq 3$, there is a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $N = fT$ is a PL submanifold of $\mathbb{R}^n$.

Suppose now that M is an (n-1)-manifold. By [1, p. 61] if $n \geq 5$ or by classical results if $n \leq 3$, there is a locally flat embedding $f\colon M \to \mathbb{R}^n$. If S is a component of fM, by [3, 27.10] $\mathbb{R}^n \setminus S$ consists of two components, whose closures are n-manifolds with boundary S. Hence, there is an embedding

$g: M \times I^1 \to \mathbb{R}^n$. Thus, replacing M by $M \times I^1$ reduces the situation to the first case of the theorem. □

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