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Extension of differentiable functions

EXTENSION OF DIFFERENTIABLE FUNCTIONS
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Abstract. Let $H \subset \mathbb{R}^n$ be closed and let $F: H \to \mathbb{R}^n$ be differentiable with respect to H. It is shown that

(i) F' is Baire 2 on H;

(ii) F' is not necessarily Baire 1;

(iii) F can be extended to $\mathbb{R}^n$ as an everywhere differentiable function if and only if F' is Baire 1 on H.

Key words: Differentiable functions of several variables, extensions.

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1. Introduction. Let H be a perfect subset of $\mathbb{R}$ and let $f: H \to \mathbb{R}$ be differentiable with respect to H. It is easy to see that f' is Baire 1 on H and it is also well-known that f can be extended to $\mathbb{R}$ as an everywhere differentiable function (see e.g. [3],[4]). In this paper we are going to investigate the analogous problems in the n dimensional Euclidean space $\mathbb{R}^n$.

Let $\mathscr{L}(\mathbb{R}^n)$ denote the linear space of all linear forms on $\mathbb{R}^n$ endowed with the usual norm. Let H be a subset of $\mathbb{R}^n$ and $F: H \to \mathbb{R}$. F is said to be differentiable at $a \in H$ if there is $L(a) \in \mathscr{L}(\mathbb{R}^n)$ such that

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$$\lim_{\substack{x \to a \\ x \in H}} \frac{F(x) - F(a) - (L(s), x - a)}{|x - a|} = 0.$$

The linear form $L(a)$ is called the derivative of F at a; if it is determined uniquely, it is also denoted by $F'(a)$. The function F is said to be differentiable on H, if it is differentiable at every point of H.

Let $H \subset \mathbb{R}^n$ be closed and let $F: H \to \mathbb{R}$ be differentiable on H. Obviously, if its derivative is not determined uniquely, it need not be in the first class. For example, it suffices to consider the segment $H = [0,1]$ as a subspace of $\mathbb{R}^2$, the function $F = 0$ and its derivative $(L(x),(u_1,u_2)) = 0$ if $x \in A$ and $(L(x),(u_1,u_2)) = u_2$ if $x \notin A$, where A is, say, a nonmeasurable subset of H.

A natural conjecture seems to be that the derivative of F is in the first class provided that it is determined uniquely. We prove that this is not the case (Theorem 5). However, if F' is determined uniquely, it is of Baire class 2 (Theorem 4(i)). Also, if the tangent space of H is sufficiently rich, then F' is in the first class (Theorem 4(ii)).

Since the derivative of an everywhere differentiable function is Baire 1, a function $F: H \to \mathbb{R}$, differentiable on the closed set H can be extended to an everywhere differentiable function only if its derivative is Baire 1 on H. We show that this condition is sufficient as well (Theorem 7). For the proof we will need a generalization of the following theorem of L.E. Snyder [5]. If f is Baire 1 on the compact metric space X then there is a function $g: (X \times \mathbb{R}^+) \to \mathbb{R}$ such that f is the limit of g along the Stolz cones $\{(x,y); y > \text{dist}(x,a)\}$ ($a \in X$). We prove that the assertion remains valid if we replace $X \times \mathbb{R}^+$ and

$X \times \{0\}$ by an arbitrary metric space and a nowhere dense closed
subset, respectively (Theorem 6).

2. **Baire class of derivatives.** Let H be a subset of $\mathbb{R}^n$
and let $x \in \mathbb{R}^n$. A vector $u \in \mathbb{R}^n$ is called a tangent vector to H
at x if

$$\lim_{r \to 0+} \inf \text{ dist}(x + ru, H)/r = 0.$$

The set of all tangent vectors to H at x is denoted by $Tan(H,x)$.

Lemma 1. Let $L(a)$ be a derivative of the function $f: H \to \mathbb{R}$
at $a \in H$. Then $A \in \mathcal{L}(\mathbb{R}^n)$ is a derivative of F at a if and only
if

$$Tan(H,a) \subset Ker(A - L(a)).$$

Proof. Let $B = A - L(a)$ and suppose first that A is a de-
rivative of F at a. Then B is a derivative of 0 at a and hence

(1)
$$\lim_{\substack{x \to a \\ x \in H}} \frac{(B,x - a)}{|x - a|} = 0.$$

Whenever $u \in Tan(H,a)$, we can find a sequence r_n of positive num-
bers converging to zero and a sequence $x_n \in H$ such that

$$\lim_{n \to \infty} |x_n - (a + r_n u)|/r_n = 0.$$

Then

$$|(B,u)| = \lim_{n \to \infty} \frac{|(B,x_n - a)|}{r_n} = \lim_{n \to \infty} \frac{|(B,x_n - a)|}{|x_n - a|} \cdot \frac{|x_n - a|}{r_n} = 0$$

since $|x_n - a| \leq |x_n - (a + r_n u)| + |r_n u| \leq r_n(1 + |u|)$ for n
large enough. Therefore $(B,u) = 0$ for every $u \in Tan(H,x)$ and hen-
ce $Tan(H,x) \subset Ker B$.

Now let $Tan(H,x) \subset Ker B$; first we prove (1). Suppose indi-
rectly that there are $\varepsilon > 0$ and a sequence $x_n \in H \setminus \{a\}$, $x_n \to a$

such that

$$\frac{|(B, x_n - a)|}{|x_n - a|} \geq \varepsilon .$$

There is a subsequence $\{x_{n_k}\}$ such that $\dfrac{x_{n_k} - a}{|x_{n_k} - a|}$ converges to

a unit vector u. It is easy to check that $u \in \mathrm{Tan}(H,x)$ and
$\langle B,u \rangle \neq 0$, a contradiction. Therefore (1) holds true and hence
B is a derivative of 0 and A = L(a) + B is a derivative of F at
a.

Corollary 2. Whenever $a \in H \subset \mathbb{R}^n$, the following statements
are equivalent.

(i) For every function $F : H \longrightarrow \mathbb{R}$ differentiable at a, the
derivative is determined uniquely.

(ii) $\mathrm{Tan}(H,a)$ spans $\mathbb{R}^n$.

Proposition 3. Let H be a subset of $\mathbb{R}^n$ and let, for each
$x \in H$, $a_H(x) = \sup \{ \det(u^1, \ldots, u^n); u^1, \ldots, u^n$ are unit vectors
from $\mathrm{Tan}(H,x) \}$. Then

(i) for every $a > 0$ the set $E_a = \{ x \in H; a_H(s) \geq a \}$ is a G_σ
subset of H, and

(ii) whenever $F : H \longrightarrow \mathbb{R}$ is differentiable on H and $a > 0$,
then F' as a map from E_a to $\mathscr{L}(\mathbb{R}^n)$ is of Baire class 1 relati-
ve to E_a.

Proof. Let F be a function differentiable on H and let M
be a closed subset of $\mathscr{L}(\mathbb{R}^n)$. We intend to prove that the set
$B = \{ x \in E_a; F'(x) \in M \}$ is a G_σ subset of H. This clearly implies
the second statement of the proposition and, since one may choo-
se F = 0 and M = {0}, also its first statement.

For each $x \in B$ and each k = 1,2,... we find numbers
$c_k(x) \in (0, 2^{-k})$, $t_k^1(x), \ldots, t_k^n(x) \in (0, c_k(x))$ and unit vectors

$u_k^1(x),\ldots,u_k^n(x) \in \mathbb{R}^n$ such that $|F(y)-F(x)-\langle F'(x),y-x\rangle| <$

$< 2^{-k}\|y-x\|$ whenever $y \in H$ and $0 < \|y-x\| < c_k(x)$,

$x+t_k^i(x)u_k^i(x) \in H$ for each $i = 1,\ldots,n$, and

$\det(u_k^1(x),\ldots,u_k^n(x)) > a - 2^{-k}$.

Next we use the continuity of F on H to find

$d_k(x) \in (0,2^{-k} \min(t_k^1(x),\ldots,t_k^n(x)))$ such that

$|F(x + t_k^i(x)u_k^i(x)) - F(y) - \langle F'(x),t_k^i(x)u_k^i(x)\rangle| \leq 2^{-k}t_k^i(x)$

for each $i = 1,\ldots,n$ and each $y \in H$ with $\|y-x\| < d_k(x)$.

Whenever $y \in H \cap \bigcap_{k=1} \bigcup_{x \in B}\{y \in \mathbb{R}^n;\ \|y-x\| < d_k(x)\}$, we find

a sequence $x_k \in B$ such that $\|x_k-y\| < d_k(x_k)$. There is a subse-

quence $k_1 < k_2 < \ldots$ such that

$u^i = \lim_{j \to \infty} u_{k_j}^i(x_{k_j})$ exists for each $i = 1,\ldots,n$. To simplify the

notation, we put $z_j = x_{k_j}$, $t_j^i = t_{k_j}^i(z_j)$, $u_j^i = u_{k_j}^i(z_j)$, $v_j^i =$

$= t_j^i u_j^i$, and $d_j = d_{k_j}(z_j)$.

From $z_j + v_j^i \in H$ and from

$\lim_{j \to \infty} (t_j^i)^{-1} \|(z_j + v_j^i - y) - t_j^i u^i\| \leq \lim_{j \to \infty} [(t_j^i)^{-1} \|z_j-y\| +$

$+ \|u_j^i-u^i\|] \leq \lim 2^{-j} = 0$

we infer that $u^i \in \operatorname{Tan}(H,y)$ for each $i = 1,\ldots,n$. Since clearly

$\det(u^1,\ldots,u^m) \geq a$, $y \in E_a$.

Whenever $\varepsilon > 0$ and j is sufficiently large, we have

$|F(z_j + v_j^i) - F(y) - \langle F'(y),(z_j + v_j^i - y)\rangle| \leq \varepsilon \|z_j + v_j^i - y\| \leq$

$\leq \varepsilon(t_j^i + d_j) \leq 2 \varepsilon t_j^i$.

Hence

$|\langle F'(z_j),v_j^i\rangle - \langle F'(y),v_j^i\rangle| \leq |F(z_j + v_j^i) - F(y) - \langle F'(z_j),v_j^i\rangle|$

$+ 2 \varepsilon t_j^i + |\langle F'(y),z_j-y\rangle| \leq (2^{-j} + 2\varepsilon)t_j^i + \|F'(y)\|d_j$

$\leq (2^{-j} + 2\varepsilon + 2^{-j}\|F'(y)\|)t_j^i$ for each $i = 1,\ldots,n$.

Consequently, $\lim_{j \to \infty} |\langle F'(z_j),u_j^i\rangle - \langle F'(y),u_j^i\rangle| = 0$ for

each $i = 1,\ldots,n$, which, together with $\lim\limits_{j\to\infty} \det(u_j^1,\ldots,u_j^n) > 0$

imply that $F'(y) = \lim F'(z_j)$. Since $F'(z_j) \in M$ for each $j =$

$= 1,2,\ldots,$ $F'(y) \in M$ and therefore $y \in B$. Thus

$$B \supset H \cap \bigcap_{k=1} \bigcup_{x \in B} \{y \in \mathbb{R}^n; \ \|y-x\| < d_k(x)\},$$

and, since the converse inclusion is obvious, B is a G_σ subset
of H.

Theorem 4. Let H be a subset of $\mathbb{R}^n$ such that $\mathrm{Tan}(H,x)$
spans $\mathbb{R}^n$ for every $x \in H$.

(i) If $F:H \to \mathbb{R}$ is differentiable on H, then F' as a map
from H to $\mathcal{L}(\mathbb{R}^n)$ is of Baire class 2 relative to H.

(ii) If H can be covered by countably many relatively
closed subsets H_k such that $\inf\{a_H(x); \ x \in H_k\} > 0$ for each k,
then the derivative of every function differentiable on H is
of the first class on H.

Proof. Both statements follow immediately from Proposition 3.

Theorem 5. There exist a compact set $H \subset \mathbb{R}^2$ and a function $F:H \to \mathbb{R}$ such that F is differentiable on H, $F'(x)$ is uniquely determined at every point of H and F' is not Baire 1.

Proof. Let C denote the Cantor ternary set in $[0,1]$ and
let $]a_n,b_n[$ be the components of $]0,1[\setminus C$. We denote $T_n =$
$= \{a_1,\ldots,a_n, b_1,\ldots,b_n\}$, $T = \bigcup_{n=1}^{\infty} T_n$ and $C' = C \setminus T$.

For every fixed n we construct a set $S_n \subset C'$ with $\bar{S}_n \setminus S_n \subset$
$\subset T_n \cup \{0,1\}$ and such that for every $t \in C \setminus T_n$ there is an $s \in S_n$
with $|t - s| < \mathrm{dist}^4(t,T_n)$.

Let $]\alpha_j, \beta_j[$ $(j = 1,\ldots,n+1)$ denote the components of
$]0,1[\setminus \bigcup_{i=1}^{n} [a_i,b_i]$. For every $j = 1,\ldots,n+1$ we choose an incre-

asing sequence $\{x_k^{(j)}\}_{k=-\infty}^{\infty}$ such that $\lim_{k\to-\infty} x_k^{(j)} = \alpha_j$, $\lim_{k\to\infty} x_k^{(j)} = \beta_j$ and $0 < x_{k+1}^{(j)} - x_k^{(j)} < \min((x_k^{(j)} - \alpha_j)^4, (\beta_j - x_{k+1}^{(j)})^4)$ for every k. We select a point $s \in C' \cap [x_k^{(j)}, x_{k+1}^{(j)}]$ whenever this intersection is nonempty and we denote by S_n the set of these points. It is easy to check that S_n satisfies our requirements.

We denote

$$S_n^* = \bigcup_{s \in S_n} \{(x,y) \in \mathbb{R}^2; \ |x - s| \le y \le \tfrac{1}{2} \operatorname{dist}^2(s, T_n)\}$$

and

$$H_n = \{(x,y); \ a_n \le x \le b_n, \ 0 \le y \le (b_n - a_n)\min(x - a_n, b_n - x)\}$$

$$(n = 1,2,\dots).$$

Finally, we define

$$H = \bigcup_{m=1}^{\infty} (H_n \cup S_n^*) \cup (C \times \{0\})$$

and

$$F(x,y) = \begin{cases} 0 \text{ if } (x,y) \in H \smallsetminus \bigcup_{m=1}^{\infty} H_n \\[2mm] y \text{ if } (x,y) \in \bigcup_{m=1}^{\infty} H_n. \end{cases}$$

It is easy to see that H is a compact subset of $\mathbb{R}^2$. We show that $\operatorname{Tan}(H,x)$ spans $\mathbb{R}^2$ for every $x \in H$. This is obvious for $x \in H \smallsetminus (C \times \{0\})$. If $x = (t,0) \in (C \times \{0\})$ and if $t \in T_n$ then x is a vertex of H_n and the assertion is also clear. If $t \in C'$ then, obviously, $(1,0) \in \operatorname{Tan}(H,x)$. We prove that $(0,1) \in \operatorname{Tan}(H,x)$. Let $r_n = \operatorname{dist}^2(t, T_n)$ and choose an $s_n \in S_n$ with $|s_n - t| < r_n^2$. Since $(s_n, \tfrac{1}{8}r_n) \in H$ if n is sufficiently large,

$$\lim_{n\to\infty} \operatorname{dist}(x + \tfrac{1}{8}r_n(0,1), H)/r_n \le \lim_{n\to\infty} |(t,\tfrac{1}{8}r_n) - (s_n, \tfrac{1}{8}r_n)| / r_n$$

$$= \lim_{n\to\infty} |t - s_n| / r_n \le \lim_{n\to\infty} r_n = 0.$$

Hence $(0,1)$ belongs to $\operatorname{Tan}(H,x)$.

We claim that $F'(x,y) = 0$ for $(x,y) \in H \setminus \bigcup_{m=1}^{\infty} H_n$ and

$(F'(x,y), (u,v)) = v$ for $(x,y) \in \bigcup_{m=1}^{\infty} H_n$. This is obvious for all

points $(x,y) \in H \setminus (C \times \{0\})$. To prove the remaining case, we first

note that, whenever $t \in C$ and $(u,v) \in H_n$, then $0 \leqslant v \leqslant (b_n - a_n) \cdot$

$\cdot \operatorname{dist}(u, \{a_n, b_n\}) \leqslant (b_n - a_n)|u - t|$. Since $b_n - a_n \to 0$, this shows

$F'(t,0) = 0$ for every $t \in C'$. Now let $t \in \{a_m, b_m\}$, we have to show

$$\lim_{\substack{(u,v) \to (t,0) \\ (u,v) \in H}} \frac{F(u,v) - v}{|(u,v) - (t,0)|} = 0.$$

Since $F(u,v) = v$ if $(u,v) \in \bigcup_{m=1}^{\infty} H_n$ and $F(u,v) = 0$ otherwise, it

is enough to prove that

$$\lim_{\substack{(u,v) \to (t,0) \\ (u,v) \in \cup S_n^*}} \frac{v}{u - t} = 0.$$

Since a_m and b_m do not belong to the closure of the set $\bigcup_{m=1}^{m-1} S_n^*$,

thus $\delta = \operatorname{dist}((t,0), \bigcup_{m=1}^{m-1} S_n^*) > 0$. If $(u,v) \in \bigcup_{m=1}^{\infty} S_n^*, \operatorname{dist}((u,v),$

$(t,0)) < \delta$, then $(u,v) \in \bigcup_{n=m}^{\infty} S_n^*$ and hence there is $n \geqslant m$ and

$s \in S_n$ such that $|u - s| \leqslant v \leqslant \frac{1}{2} \operatorname{dist}^2(s, T_n) \leqslant \frac{1}{2}(s - t)^2$.

Therefore $|u - t| \geqslant |s - t| - |u - s| \geqslant |s - t| - \frac{1}{2}(s - t)^2 \geqslant$

$\geqslant \frac{1}{2}|s - t|$ and $v \leqslant \frac{1}{2}(s - t)^2$, which proves our assertion.

Finally, we note that F' is not Baire 1 on H, since $C \times \{0\}$
contains no point of continuity of the restriction of F' to
$C \times \{0\}$.

3. **Extension of differentiable functions.** Our next result
will be used in the proof of the extension theorem, but may have
some interest in itself. Let H be a nowhere dense, closed subset
of the metric space (X,d). By a Stolz cone with vertex $a \in H$ we

mean a set

$$\{x \in X; \ \mathrm{dist}(x,H) \gtrless c \cdot d(x,a)\},$$

where c is a positive constant. Our theorem implies that if
$f:H \to \mathbb{R}$ is of Baire class 1 then there is a function $g:X \to$
$\to \mathbb{R}$ such that for every $a \in H$, $\lim\limits_{x \to a} g(x) = f(a)$ relative to
any Stolz cone with vertex a. Using a locally finite, continu-
ous partition of unity of $X \setminus H$ subordinated to the system of
balls with center $x \in X \setminus H$ and radius $\frac{1}{2}$ dist (x,H), one can ea-
sily show that g can be chosen to be continuous. This result
is a generalization of a theorem by L.E. Snyder [5].

Theorem 6. Let (X,d) be a metric space, let H be a closed
subset of X and let $f:H \to \mathbb{R}$ be of Baire class 1 on H. Then the-
re exists a function $g(X \setminus H) \to \mathbb{R}$ such that

(3) $\lim\limits_{\substack{x \to a \\ x \in X \setminus H}} |g(x) - f(a)| \ \frac{\mathrm{dist}(x,H)}{d(x,a)} = 0$

for every $a \in \partial H$.

Proof. First we remark that if $f:Y \to \mathbb{R}$ is a Baire 1 func-
tion defined on the metric space Y then f is the pointwise li-
mit of a sequence of bounded Lipschitz functions. This has been
proved by Hausdorff in a more general setting (see [2], § 41,
pp. 264-276); or it follows more directly from [1], Propositi-
on 3.9.

Applying this result to $Y = H$ as a subspace of X and to
$f:H \to \mathbb{R}$, we get a sequence $f_n:H \to \mathbb{R}$ of bounded Lipschitz
functions converging to f on H. Let $1 \leqq M_1 \leqq M_2 \leqq \ldots$ and $0 < K_1 \leqq$
$\leqq K_2 \leqq \ldots$ be such that $|f_n| \leqq M_n$ and $|f_n(x) - f_n(y)| \leqq K_n \ d(x,y)$
$(n = 1,2,\ldots; \ x,y \in H)$.

Let $x \in X \setminus H$ be fixed. If dist $(x,H) \gtrless (K_1(M_1 + 2))^{-1}$, we

define $g(x) = 0$. If, for a natural number n,

(4) $[(n + 1)K_{n+1}((n+1)M_{n+1} + 2)]^{-1} \leqslant \text{dist}(x,H) < [nK_n(nM_n + 2)]^{-1}$

then we select a point $u(x) \in H$ with $d(x,u(x)) < 2\,\text{dist}(x,H)$ and define $g(x) = f_n(u(x))$. We prove that if $a \in H$, $x \in X \setminus H$ and (4) holds then

(5) $|g(x) - f(a)|\,\dfrac{\text{dist}(x,H)}{d(x,a)} \leqslant \dfrac{1}{n} + \dfrac{|f(a)|}{n} + |f_n(a) - f(a)|$.

Since $f_n(a) \to f(a)$, this will prove (3). We distinguish between two cases.

If $\dfrac{\text{dist}(x,H)}{d(x,a)} \leqslant \dfrac{1}{nM_n}$ then we have $|g(x) - f(a)|\,\dfrac{\text{dist}(x,H)}{d(x,a)} \leqslant$

$$\leqslant |f_n(u(x)) - f(a)|$$

$\cdot \dfrac{1}{nM_n} \leqslant \dfrac{M_n}{nM_n} + \dfrac{|f(a)|}{nM_n}$ and thus (5) holds true.

If $\dfrac{\text{dist}(x,H)}{d(x,a)} > \dfrac{1}{nM_n}$ then $d(u(x),a) \leqslant d(u(x),x) + d(x,a) <$

$< 2\,\text{dist}(x,H) + nM_n\,\text{dist}(x,H) = (nM_n + 2)\,\text{dist}(x,H) < 1/nK_n$,

and hence

$$|g(x) - f(a)| = |f_n(u(x)) - f(a)| \leqslant |f_n(u(x)) - f_n(a)| +$$

$$+ |f_n(a) - f(a)| \leqslant K_n d(u(x),a) + |f_n(a) - f(a)| < \frac{1}{n} +$$

$$+ |f_n(a) - f(a)|.$$

Since $\text{dist}(x,H) \leqslant d(x,a)$, this implies (5), which completes the proof.

Theorem 7. Suppose that H is a closed subset of $\mathbb{R}^n$, F is a real valued function defined on H and $L \colon H \to \mathscr{L}(\mathbb{R}^n)$ is a derivative of F on H. Then F can be extended to a function $\hat{F}$ everywhere differentiable on $\mathbb{R}^n$ with

$$\hat{F}'(x) = L(x) \qquad (x \in H)$$

if and only if L is a map of the first class.

Proof. The necessity of the condition is obvious, since,
by Theorem 4(ii), the derivative of every function differentiab-
le on $\mathbb{R}^n$ is Baire 1. In order to prove the sufficiency, suppose
that $f:H \longrightarrow \mathbb{R}$ and L satisfy the conditions of the theorem.

Then there is a map $A:(\mathbb{R}^n \setminus H) \longrightarrow \mathcal{L}(\mathbb{R}^n)$ such that

(6)
$$\lim_{\substack{u \to a \\ u \notin H}} \|A(u) - L(a)\| \frac{\text{dist}(u,H)}{|u - a|} = 0$$

for every $a \in \partial H$. Indeed, let

$$(L(x),u) = \sum_{i=1}^{m} L_i(x)u_i \qquad (x \in H, \ u = (u_1,\ldots,u_n) \in \mathbb{R}^n),$$

then the functions $L_i:H \longrightarrow \mathbb{R}$ are of Baire class 1. By Theorem.
6, there are functions $g_i:(\mathbb{R}^n \setminus H) \longrightarrow \mathbb{R}$ such that (3) holds with
$X = \mathbb{R}^n$ and $f = L_i$. We define $(A(u),v) = \sum_{i=1}^{m} g_i(u)v_i$
$(u \in \mathbb{R}^n \setminus H, \ v = (v_1,\ldots,v_n) \in \mathbb{R}^n)$, then the map $A:(\mathbb{R}^n \setminus H) \longrightarrow$
$\longrightarrow \mathcal{L}(\mathbb{R}^n)$ satisfies (6).

Let $T: \mathbb{R}^n \longrightarrow H$ be a map with $|u - T(u)| = \text{dist}(u,H)(u \in \mathbb{R}^n)$.
Let Φ_j be a locally finite C^1 partition of unity on $\mathbb{R}^n \setminus H$ sub-
ordinated to the system of open balls with center $u \in \mathbb{R}^n \setminus H$ and
radius $\frac{1}{6}$ dist(u,H). For every j, let $u_j \in \mathbb{R}^n \setminus H$ be chosen such
that $\Phi_j(u_j) \neq 0$. We define $\hat{F}$ by

$$\hat{F}(u) = \begin{cases} F(u) \text{ if } u \in H, \\ \sum_j \Phi_j(u)[F(T(u_j)) + (A(u_j),u-T(u_j))] \text{ if } u \in H. \end{cases}$$

Let $a \in \partial H$. Then for every $u \notin H$

$|\hat{F}(u) - \hat{F}(a) - (L(a),u - a)|$

$= |\sum_j \Phi_j(u)[F(T(u_j))+(A(u_j),u-T(u_j))-\hat{F}(a)-(L(a),u-a)]|$

$= |\sum_j \Phi_j(u)[F(T(u_j))-F(a)-(L(a),T(u_j)-a)-(L(a)-A(u_j),$

$$u-T(u_j))]|$$

$\leq \sum_{\substack{j \\ \Phi_j(u) \neq 0}} \Phi_j(u)|F(T(u_j))-F(a)-(L(a),T(u_j)-a)| +$

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$$+ \sum_{\substack{j \\ \Phi_j(u) \neq 0}} \Phi_j(u) |(L(a)-A(u_j), u-T(u_j))| \overset{\text{def}}{=} \Sigma_1(u) + \Sigma_2(u).$$

Now $\Phi_j(u) \neq 0$ implies $|u - u_j| < \frac{1}{2} \text{dist}(u,H)$, from which

$$(7) \quad |u-T(u_j)| \leqslant |u-u_j| + |u_j-T(u_j)| \leqslant |u-u_j| + |u_j-T(u)|$$
$$\leqslant 2|u-u_j| + |u-T(u)| \leqslant 2 \text{ dist}(u,H).$$

Thus we have

$$|T(u_j)-a| \leqslant |u-T(u_j)| + |u-a| \leqslant 2 \text{ dist}(u,H) + |u-a| \leqslant 3|u-a|.$$

Since $F'(a) = L(a)$, there is a function $w: H \longrightarrow \mathbb{R}$ such that

$$\lim_{\substack{z \to a \\ z \in H}} w(z) = 0 \text{ and}$$

$$|F(z)-F(a)-(L(a),z-a)| = w(z)|z-a| \quad (z \in H).$$

Then

$$|\Sigma_1(u)| = \sum_{\substack{j \\ \Phi_j(u) \neq 0}} \Phi_j(u) w(T(u_j)) |T(u_j)-a| \leqslant 3|u-a|$$

$$\sum_{\substack{j \\ \Phi_j(u) \neq 0}} \Phi_j(u) \sup\{w(z); z \in H, |z-a| \leqslant 3|u-a|\} = 3|u-a| \sup\{w(z);$$

$$z \in H, |z-a| \leqslant 3|u-a|\}.$$

Hence

$$\lim_{\substack{u \to a \\ u \notin H}} \frac{|\Sigma_1(u)|}{u-a} = 0 .$$

On the other hand, by (7), we get

$$\frac{|\langle L(a)-A(u_j), u-T(u_j)\rangle|}{u-a} \leqslant \frac{\|L(a)-A(u_j)\| \cdot 2 \text{ dist}(u,H)}{|u-a|}$$

$$\leqslant \|L(a)-A(u_j)\| \frac{4 \cdot \text{dist}(u_j,H)}{\frac{1}{2}|u_j-a|}$$

which, taking (6) into consideration, implies

$$\lim_{\substack{u \to a \\ u \notin H}} \frac{\Sigma_2(u)}{|u-a|} = 0 .$$

Since, from the differentiability of F on H,

$$\lim_{\substack{u \to a \\ u \in H}} \frac{\hat{F}(u) - \hat{F}(a) - (L(a), u-a)}{u-a} = 0,$$

we obtain $\hat{F}'(a) = L(a)$. This finishes the proof of the theorem, since $\hat{F}$ is continuously differentiable on $\mathbb{R}^n \setminus H$.

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