W. Poreda; Elzbieta Wagner-Bojakowska; Władysław Wilczyński
Remarks on I-density and I-approximately continuous functions

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 3, 553--563

Persistent URL: http://dml.cz/dmlcz/106392

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
26,3 (1885)

REMARKS ON I-DENSITY AND I-APPROXIMATELY CONTINUOUS FUNCTIONS

W. POREDA, E. WAGNER-BOJAKOWSKA and W. WILCZYŃSKI

Abstract: Special properties of I-density points, topologies $\mathcal{T}_I$ and I-approximately continuous functions are investigated on reals, where I is a proper σ-ideal of sets.

Key words: Density point, approximately continuous function.

Classification: 54H05, 26A99

This paper is a refinement of a paper [2] in which we have introduced the notion of I-density point and proved that it leads in a natural way to an interesting topology on the real line.

We have found also basic properties of real functions which are continuous with respect to this topology. Here we present some special properties of I-density point, topology $\mathcal{T}_I$ and I-approximately continuous functions.

Let (X,S) be a measurable space and let $I \subset S$ be a proper σ-ideal of sets. We shall say that some property holds I-almost everywhere (in abbr. I-a.e.) if and only if the set of points which do not have this property belongs to I. We shall say that the sequence $\{f_n\}_{n\in N}$ of S-measurable real functions defined on X converges with respect to I to some S-measurable real function f defined on X if and only if every subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges to f I-a.e. We shall use the denotation $f_n \xrightarrow[n\to\infty]{I} f$.

Now let $X = R$ (the real line), let S be a σ-algebra of Lebesgue measurable sets and m-a Lebesgue linear measure. A point 0 is a density point of a set $A \in S$ if and only if $\lim_{h \to 0^+} [(2h)^{-1} \cdot m(A \cap [-h,h])] = 1$. Observe that this condition is fulfilled if and only if $\lim_{n \to \infty} [(2^{-1}n) \cdot m(A \cap [-\frac{1}{n}, \frac{1}{n}]) = 1$. The last limit can be described in terms of convergence in measure in the following way: 0 is a density point of A if and only if the sequence $\{\chi_{(n \cdot A) \cap [-1,1]}\}_{n \in N}$ of characteristic functions (where $n \cdot A = \{nx : x \in A\}$) converges in measure to 1 on the interval $[-1,1]$. This fact is the basis for the following definition, where $X = R$, S is a σ-algebra of subsets of R invariant with respect to linear transformations and $I \subset S$ is a σ-ideal, which is also invariant with respect to linear transformations.

Definition 1. We shall say that 0 is an I-density point of a set $A \in S$ if and only if $\chi_{(n\,A) \cap [-1,1]} \xrightarrow[n \to \infty]{I} 1$.

We shall say that x_0 is an I-density point of $A \in S$ if and only if 0 is an I-density point of $A - x_0 = \{x - x_0 : x \in A\}$. We shall say that x_0 is an I-dispersion point of $A \in S$ if and only if x_0 is an I-density point of $R - A$. Observe that 0 is an I-dispersion point of A if and only if $\chi_{(n \cdot A) \cap [-1,1]} \xrightarrow[n \to \infty]{I} 0$.

Similarly one can define right- and left-hand I-density points. We can take some interval $[-a,a]$, $a > 0$, instead of $[-1,1]$.

In the sequel we shall consider only sets having the Baire property as the σ-algebra S and for I we shall always take the family of meager sets. Under these assumptions we have:

Lemma 1. If A is an open set and the sequences $\{i_n\}_{n \in N}$ and $\{j_n\}_{n \in N}$ have the following properties: $i_n > 0$, $j_n > 0$ for each $n \in N$, $\lim_{n \to \infty} i_n = \infty$, $\lim_{n \to \infty} j_n = \infty$, $\lim_{n \to \infty} \frac{j_n}{i_n} = 1$ and if

$$\chi_{(i_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0 \quad I\text{ - a.e., then also}$$
$$\chi_{(j_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0 \quad I\text{ - a.e.}$$

Proof. Suppose that $\chi_{(i_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0$ I - a.e. Then the set of all $x\in[-1,1]$, for which the sequence $\{\chi_{(i_n A)\cap[-1,1]}(x)\}_{n\in N}$ does not converge to zero, belongs to I. The last set is equal to $[-1,1]\cap \limsup_{n} (i_n\cdot A)$, so we have $[-1,1]\cap\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}(i_k\cdot A)\in I$. Since the last set is of type G_δ and meager, it must be nowhere dense. We shall prove that $[-1,1]\cap\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}(j_k\cdot A)$ is also nowhere dense. Let $[a,b]\subset[-1,1]-\{0\}$ be an arbitrary non-degenerate interval. For convenience suppose that $a>0$ (in the case $b<0$ the proof is analogous). It follows that there exists a non-degenerate interval $[c,d]\subset[a,b]$ such that $[c,d]\cap\bigcap_{m=1}^{\infty}\bigcup_{k=m}^{\infty}(i_k\cdot A)=\emptyset$. So for every $x\in[c,d]$ there exists a natural number $n(x)$ such that for every natural number $k\geq n(x)$ we have $x\notin i_k\cdot A$. Let $E_n=\{x\in[c,d];n(x)\leq n\}$. The sequence $\{E_n\}_{n\in N}$ of sets is increasing and $\bigcup_{n=1}^{\infty}E_n=[c,d]$. Hence there exists a number $n_0\in N$ and non-degenerate interval $[e,f]\subset[c,d]$ such that E_{n_0} is dense in $[e,f]$. Then for every $k\geq n_0$ we have $(E_{n_0}\cap[e,f])\cap(i_k\cdot A)=\emptyset$, so $(\overline{E_{n_0}\cap[e,f]})\cap(i_k\cdot A)=\emptyset$ (it follows immediately from the fact that $i_k\cdot A$ is open). But $\overline{E_{n_0}\cap[e,f]}=[e,f]$ so at last we have $[e,f]\cap\bigcup_{k=n_0}^{\infty}(i_k\cdot A)=\emptyset$. Take $\varepsilon>0$ such that $g=(1+\varepsilon)\cdot e<(1-\varepsilon)\cdot f=h$. Let $N_0\geq n_0$ be such a number that for $n>N_0$ we have $(1-\varepsilon)i_n<j_n<(1+\varepsilon)i_n$ (such N_0 does exist since $\frac{j_n}{i_n}\xrightarrow[n\to\infty]{}1$). Observe that for each $m\geq N_0$ and for arbitrary $y\in A$ we have $i_m\cdot y<e$ or $i_m\cdot y>f$. Hence for $m\geq N_0$ and for $y\in A$ we have $j_m\cdot y<(1+\varepsilon)\cdot i_m y<(1+\varepsilon)\cdot e=g$, when $i_m\cdot y<e$ or $j_m\cdot y>(1-\varepsilon)\cdot i_m\cdot y>(1-\varepsilon)\cdot f=h$, when $i_m\cdot y>f$. So $[g,h]\cap j_m\cdot A=\emptyset$ for $m\geq N_0$ and $[g,h]\cap\bigcap_{m=1}^{\infty}\bigcup_{m=m}^{\infty}(j_m\cdot A)=\emptyset$.

From the above reasoning it follows that for every interval $[a,b]\subset[-1,1]-\{0\}$ there exists a non-degenerate interval $[g,h]\subset[a,b]$ such that $[g,h]\cap\limsup_m(j_n\cdot A)=\emptyset$ so $\limsup_m(j_n\cdot A)$ is nowhere dense and obviously belongs to I. So we obtained that $\chi_{(j_n\cdot A)\cap[-1,1]}\xrightarrow[m\to\infty]{}0$ I-a.e. which ends the proof of the lemma.

Corollary 1. A point x_0 is an I-density point of the set $B\in S$ if and only if for every increasing sequence $\{t_n\}_{n\in N}$ of positive real numbers tending to infinity there exists a subsequence $\{t_{n_m}\}_{m\in N}$ such that $\chi_{(t_{n_m}\cdot(B-x_0))\cap[-1,1]}\xrightarrow[m\to\infty]{}1$ I-a.e.

Theorem 1. There exists an open set $E=\bigcup_{n=1}^{\infty}(a_n,b_n)$ where $\{b_n\}_{n\in N}$ tends decreasingly to zero, $a_{n+1}<b_n<a_n$ for each $n\in N$ such that 0 is an I-dispersion point of E.

Proof. Observe that from the definition it follows immediately that 0 is an I-dispersion point of some set $E\in S$ if and only if for every increasing sequence $\{n_m\}_{m\in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that $\chi_{(n_{m_p}E)\cap[-1,1]}\xrightarrow[p\to\infty]{}$ $\xrightarrow[p\to\infty]{}0$ I-a.e. So we ought to construct a set E fulfilling the above described condition. In virtue of the fact that E will consist only of positive numbers we shall consider the characteristic function in the interval $[0,1]$ instead of $[-1,1]$.

Let $(a_1,b_1)\subset(0,1)$, $a_1>0$ be an arbitrary interval. There exists exactly one natural number q_1 such that $(q_1a_1,q_1b_1)\cap$ $\cap[0,1]\neq\emptyset$ and $((q_1+1)\cdot a_1,(q_1+1)\cdot b_1)\cap[0,1]=\emptyset$. Choose $b_2\in(0,1)$ such that $(q_1+1)\cdot b_2<2^{-1}$. Let $a_2=\frac{2}{3}b_2$ and q_2 be a natural number such that $(q_2a_2,q_2b_2)\cap[0,1]\neq\emptyset$ and $((q_2+1)\cdot a_2,(q_2+1)\cdot b_2)\cap[0,1]=\emptyset$. There is exactly one such

q_2.

Suppose that we had already chosen a_i, b_i and q_i for $i = 1,2,\dots,k$. We choose a_{k+1}, b_{k+1} and q_{k+1}. Let $b_{k+1} \in (0,1)$ be such a number that $(q_k + 1) \cdot b_{k+1} < 2^{-k}$. Put $a_{k+1} = \frac{k+1}{k+2} \cdot b_{k+1}$ and let q_{k+1} be such that $(q_{k+1} \cdot a_{k+1}, q_{k+1} \cdot b_{k+1}) \cap [0,1] \neq \emptyset$, $((q_{k+1} + 1) \cdot a_{k+1}, (q_{k+1} + 1) \cdot b_{k+1}) \cap [0,1] = \emptyset$.

So by the induction we have defined a_n, b_n and q_n for each natural n. Observe that the sequence $\{q_n\}_{n\in N}$ is increasing. Put $E = \bigcup_{n=1}^{\infty} (a_n, b_n)$. We shall prove that E is the required set.

We shall start from proving that for every natural number i $\chi_{(n\cdot E)\cap[2^{-i},1]} \xrightarrow[n\to\infty]{I} 0$. Let i_o be a fixed natural number. From the construction it follows that for $n > q_{i_o}$ the set $(n \cdot E) \cap \cap [2^{-i_o},1]$ either is empty, or consists of one interval $(n \cdot a_{i(n)}, n \cdot b_{i(n)}) \cap [2^{-i_o},1]$ where $i(n) > i_o$ and $i(n) \xrightarrow[n\to\infty]{} \infty$. From the fact that $a_k = \frac{k}{k+1} b_k$ it follows that the length of the above mentioned interval (which may be open or half closed) tends to zero, when n tends to infinity. Let $\{n_m\}_{m\in N}$ be an increasing sequence of natural numbers. If for infinitely many natural numbers the set $(n_m \cdot E) \cap [2^{-i_o},1]$ is empty, then we can choose the subsequence $\{n_{m_p}\}_{p\in N}$ for which $\chi_{(n_{m_p}\cdot E)\cap[2^{-i_o},1]}$ tends to zero everywhere. In the opposite case we have a sequence $\{\chi_{(n_m\cdot E)\cap[2^{-i_o},1]}\}_{m>m_o}$ where m_o is a suitably chosen number of characteristic functions of intervals with lengths tending to zero. Choose a subsequence $\{n_{m_p}\}_{p\in N}$ to assure that the left-hand ends of those intervals are convergent to a number $x_o \in [2^{-i_o},1]$. Then we can see that $\chi_{(n_{m_p}\cdot E)\cap[2^{-i_o},1]}$ tends to zero everywhere except, perhaps, the point x_o. In both cases we have convergence I-a.e. on the interval $[2^{-i_o},1]$.

Now we shall prove that 0 is I-dispersing point of E. Let $\{n_m\}_{m\in N}$ be an increasing sequence of natural numbers. Let $\{n_m^{(1)}\}_{m\in N}$ be a subsequence for which convergence holds I-a.e. on $[2^{-1},1]$, $\{n_m^{(2)}\}_{m\in N}$ - a subsequence of $\{n_m^{(1)}\}_{m\in N}$ which is good for $[2^{-2},1]$ and so on. For the diagonal subsequence $\{n_p^{(p)}\}_{p\in N}$ we obtain convergence I-a.e. on $[0,1]$. This ends the proof.

From the above theorem it follows that the notion of an I-density point is rather delicate and different from the notion of a residual point.

Let $\Phi(A) = \{x;\ x$ is an I-density point of $A\}$ for $A\in S$. It is obvious that $\Phi(A)\in S$ for $A\in S$ and it is known that the operation Φ is so called "lower density" and that $\mathcal{T}_I = \{\Phi(A) - N;\ A\in S,\ N\in I\}$ is a topology (see [2]).

Now we shall occupy ourselves with some properties of topology $\mathcal{T}_I$. In the sequel $\mathcal{T}_I$ - Int A and $\mathcal{T}_I$ - Cl A shall denote the interior and closure of the set A respectively in $\mathcal{T}_I$.

It is easy to prove that

Theorem 2. Every set of the first category is $\mathcal{T}_I$-isolated.

Theorem 3. The family of sets which are $\mathcal{T}_I$-Borel sets coincide with the family of sets having the Baire property.

Proof. If a set A is $\mathcal{T}_I$-open, then it has the Baire property. Hence every set which is a $\mathcal{T}_I$-Borel set has the Baire property.

Conversely, if a set A has the Baire property then $A = (G - P_1)\cup P_2$ where G is an open set in the natural topology and $P_1, P_2 \in I$. Hence $G - P_1 \in \mathcal{T}_I$ and P_2 is $\mathcal{T}_I$-closed so A is a $\mathcal{T}_I$-Borel set.

Corollary 2. Every set which is a $\mathcal{T}_I$-Borel set is the sum of $\mathcal{T}_I$-open set and $\mathcal{T}_I$-closed set.

Theorem 4. A set is $\mathcal{T}_I$-nowhere dense if and only if it is of the first category.

Proof. If A is a set of the first category then A is $\mathcal{T}_I$-closed and $\mathcal{T}_I$ - Int A = $\emptyset$ so A is $\mathcal{T}_I$-nowhere dense.

Conversely, if $A \notin I$ then $\mathcal{T}_I - \mathrm{Cl}\, A \in S - I$, so $\mathcal{T}_I$-Cl A $(G - P_1) \cup P_2$ where G is a non-empty open set in the natural topology, $P_1, P_2 \in I$. Obviously, $\emptyset \neq G - P_1 \subset \mathcal{T}_I\text{-Int}\,(\mathcal{T}_I - \mathrm{Cl}\, A)$. Hence A is not $\mathcal{T}_I$-nowhere dense.

Now we shall study some properties of continuous functions from $(R, \mathcal{T}_I)$ into R equipped with the natural topology.

Definition 3. We shall say that a function $f: R \longrightarrow R$ having the Baire property is I-approximately continuous at x_0 if and only if for every $\varepsilon > 0$ the set $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ has x_0 as an I-density point.

Definition 4. We shall say that a function $f: R \longrightarrow R$ is I-approximately continuous if and only if for every interval (y_1, y_2) the set $f^{-1}((y_1, y_2))$ belongs to $\mathcal{T}_I$.

From the above definitions we obtain immediately the following theorem.

Theorem 5. A function $f: R \longrightarrow R$ is I-approximately continuous if and only if it is I-approximately continuous at every point.

Recall that in real analysis there are at least two frequently used definitions of (ordinary) approximate continuity at point x_0: first of them (similarly as def. 3 above) says that f is approximately continuous at x_0 if and only if for every $\varepsilon > 0$ the

set $f^{-1}((f(x_0) - \varepsilon, f(x_0) + \varepsilon))$ has x_0 as a density point (this set includes a neighbourhood of x_0 in the density topology); the second deals with some restriction of f, namely, f is approximately continuous at x_0 if and only if there exists in the density topology a neighbourhood E of x_0 such that $f \mid E$ is continuous at x_0 (in the natural topology relativised to E).
If we took any topology $\mathcal{T}$ instead of density topology, we should obtain the "topological" definition and "restrictional" definition of continuity at x_0. According to [1], th. 5 these conditions for topology $\mathcal{T}$ invariant with respect to translations are equivalent if and only if the following condition (W*) is fulfilled (we quote the condition in the formulation more convenient for our purposes): (W*) For every descending sequence $\{E_n\}_{n\in N}$ right-hand (left-hand) $\mathcal{T}$-neighbourhoods of 0 there exists a sequence $\{h_n\}_{n\in N}$ such that $h_n \searrow 0$ and the set $\{0\} \cup \bigcup_{n=1}^{\infty} (E_n \cap [h_{n+1}, h_n))$ $(\{0\} \cup \bigcup_{n=1}^{\infty}(E_n \cap (-h_n, -h_{n+1}]))$ includes a right-hand (left-hand) $\mathcal{T}$-neighbourhood of 0.

Since for any descending sequence $\{E_n\}_{n\in N}$ of sets having 0 as a point of right-hand density there exists a sequence $\{h_n\}_{n\in N}$, $h_n \searrow 0$ for which the set $\bigcup_{n=1}^{\infty} (E_n \cap [h_{n+1}, h_n))$ has also 0 as a point of right-hand density, the above quoted two definitions of approximate continuity are obviously equivalent. Observe also that the above condition can be also formulated in terms of points of dispersion.

From the following theorem we can conclude immediately that for $\mathcal{T}_I$ topology the "restrictional" and "topological" definitions are not equivalent. Obviously, "restrictional" continuity always implies "topological".

<u>Theorem 6</u>. There exists an increasing sequence $\{A_n\}_{n\in N}$

of sets having the Baire property such that for every natural n 0 is an I-dispersion point of A_n and for any sequence $\{h_n\}_{n\in N}$ of numbers tending decreasingly to zero a point 0 is not an I-dispersion point of the set $A = \bigcup_{n=1}^{\infty}(A_n \cap [h_{n+1}, h_n))$.

Proof. We shall use two obvious lemmas:

Lemma 2. If 0 is an I-dispersion point of B_1 and B_2, then 0 is an I-dispersion point of $B_1 \cup B_2$.

Lemma 3. If 0 is an I-dispersion point of B, then for every number $a \in R$ 0 is an I-dispersion point of $a \cdot B$.

Now let E be the open set having 0 as an I-dispersion point, $E = \bigcup_{i=1}^{\infty}(a_i, b_i)$, $b_i \searrow 0$ (see the proof of th. 1). We can (and shall) suppose that for every natural i we have $b_i = (l_i)^{-1}$, where l_i is a natural number. Put $A_1 = E$ and for $n > 1$ $A_n = A_{n-1} \cup \bigcup_{i=1}^{n-1}(\frac{1}{n} \cdot E)$. From lemmas 2 and 3 it follows immediately that for each n zero is an I-dispersion point of A_n.

Let $\{h_n\}_{n\in N}$ be a sequence of numbers decreasing to zero. We shall show that 0 is not an I-dispersion point of $A = \bigcup_{n=1}^{\infty}(A_n \cap [h_{n+1}, h_n))$. Namely, we shall show that for such $\{h_n\}_{n\in N}$ there exists a sequence $\{m_k\}_{k\in N}$ such that for every subsequence $\{m_{k_p}\}_{p\in N}$ of $\{m_k\}_{k\in N}$ the sequence $\{\chi_{[0,1]\cap(m_{k_p} \cdot A)}\}_{p\in N}$ does not converge to zero I -a.e. (obviously it suffices to consider [0,1] instead of [-1,1], because A_n and A consist only of positive numbers). Now we construct a sequence $\{m_k\}_{k\in N}$. Let $m_1 = \frac{1}{b_i} = l_i$, where i is the smallest natural number such that $b_i < h_1$. Suppose that we have defined $m_1, \dots, m_k$. Let $m_{k+1} = \frac{1}{b_i} = l_i$, where i is the smallest natural number such that $b_i < h_{k+1}$ and $l_i > \max(m_1, \dots, m_k)$. So we have defined by induction an increasing sequence $\{m_k\}_{k\in N}$ of natural numbers.

Let $\{m_{k_p}\}_{p\in N}$ be a subsequence of $\{m_k\}_{k\in N}$. Denote $f_p = \chi_{[0,1]\cap(m_{k_p}\cdot A)}$. Observe that for each natural k we have $A\cap(0,h_k)\supset A_k\cap(0,h_k)$, because the sequence $\{A_n\}_{n\in N}$ was increasing. Hence, in virtue of the inequality $h_{k_p}\cdot m_{k_p}>1$ we have $[0,1]\cap(m_{k_p}\cdot A) = [0,1]\cap(m_{k_p}\cdot(A\cap(0,h_{k_p})))\supset[0,1]\cap(m_{k_p}\cdot(A_{k_p}\cap(0,h_{k_p})))$. From the definition of A_{k_p} we conclude that the set $[0,1]\cap(m_{k_p}\cdot(A_{k_p}\cap(0,h_{k_p})))$ includes the following intervals: $\frac{1}{k_p}\cdot(m_{k_p}\cdot a_i, m_{k_p}\cdot b_i)$, $\frac{2}{k_p}(m_{k_p}\, a_i,\, m_{k_p}\cdot b_i),\ldots$ $\ldots,\frac{k_p-1}{k_p}(m_{k_p}\cdot a_i,\, m_{k_p}\cdot b_i)$ where i is a number described during the construction of $\{m_k\}_{k\in N}$ (obviously $m_{k_p}\cdot b_i = 1$), hence the set $[0,1]\cap(m_{k_p}\cdot(A\cap(0,h_{k_p})))$ also includes the same intervals. Since $\lim_p \sup f_p(x) = 1$ if and only if

$$x\in\lim_p \sup\left([0,1]\cap(m_{k_p}\cdot(A\cap(0,h_{k_p})))\right) = \bigcap_{n=1}^{\infty}\bigcup_{p=n}^{\infty}\left([0,1]\cap(m_{k_p}\cdot(A\cap(0,h_{k_p})))\right),$$

and each union in the last expression is an open set which is dense in $[0,1]$ (this fact immediately follows from the above argument), we conclude that $\lim_p \sup f_p(x) = 1$ I-a.e. It means that the sequence $\{\chi_{[0,1]\cap(m_{k_p}\cdot A)}\}_{p\in N}$ does not converge I-a.e. to zero, which ends the proof.

<u>Theorem 7</u>. A function $f:R\to R$ has the Baire property if and only if it is "restrictionally" I-approximately continuous I-a.e.

For the proof compare the proof of the theorem 7 from [2].

R e f e r e n c e s

[1] J. JĘDRZEJEWSKI: On limit numbers of real functions, Fund. Math. LXXXIII(1974), 269-281.

[2] W. POREDA, E. WAGNER-BOJAKOWSKA, W. WILCZYŃSKI: A category analogue of the density topology, Fund. Math. (to appear).

Instytut Matematyki Univ. Lodž, S.Banacha 22, 90-228 Lodž, Poland

(Oblatum 6.8. 1984)