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EQUIVALENCE OF SOME GEOMETRIC AND RELATED RESULTS OF NONLINEAR FUNCTIONAL ANALYSIS

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Abstract: The author's Drop theorem, Generalized drop theorem, Krasnoselskii-Zabreiko renorming theorem, Browder's generalization of the Bishop-Phelps theorem, Ekeland's variational principle and Caristi fixed point theorem are mutually equivalent.

Key words: drop, fixed point.

Classification: 47H10, 47H15, 46B99

Introduction and notation. The goal of this paper is to show that the theorems listed in the abstract (see theorems D, GD, KZD, B and B', E and C below) are mutually equivalent.

If D is a convex set, M a set and s a point in a linear space, then co(M) denotes the convex hull of M, K(D) the convex cone generated by D (i.e. the set $\{tx : x \text{ in } D, t \text{ in } R^+\}$), $K(D,s) = co(D \cup \{s\})$ (the generalized drop with vertex s and basis D). If (X,d) is a metric space, D and S sets in X, s a point in X and $r > 0$, then B(s,r) denotes the closed r-ball centered at s, $d(s,D) = \inf\{d(x,s) : x \text{ in } D\}$ (the distance of s from D), $d(S,D) = \inf\{d(x,y) : x \text{ in } S, y \text{ in } D\}$ (the distance of the sets S and D) and $diam(D) = \sup\{d(x,y) : x,y \text{ in } D\}$ (the diameter of D).

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1. Preliminary (geometric) results.

Lemma 1. Let D be a convex set in a linear space X, s_0 in X and s in $K(D,s_0)$ (so that $s = as_0 + (1-a)u$ for some a in

$[0,1]$ and u in D). Then:

(1) $s + t(K(D,s_0)-s_0) \subset K(D,s) \subset K(D,s_0)$ for all t in $[0,a]$;

(2) if X is a normed linear space, $d(s_0,D) > 0$ and $t_0 = d(s,D)/d(s_0,D)$, then $t_0 \leq a$ and

$$(s + K(D-s_0)) \cap B(s,t_1 d(s,D)) \subset s + t_2 t_0(K(D,s_0)-s_0) \subset K(D,s) \subset K(D,s_0)$$

for all $0 \leq t_1 \leq t_2 \leq 1$.

Proof. (1) Let x in $s + t(K(D,s_0)-s_0)$ be given ($t \in [0,a]$). Then $x = s + t(bs_0 + (1-b)z - s_0) = s + t(1-b)(z-s_0)$ for some b in $[0,1]$ and z in D. Assume $a > 0$. Then $s_0 = a^{-1}s - a^{-1}(1-a)u$, so that $x = (1 - a^{-1}t(1-b))s + a^{-1}t(1-b)(az + (1-a)u) = (1-c)s + cw$, where $c = a^{-1}t(1-b)$ and $w = az + (1-a)u \in D$. But $0 \leq c \leq 1-b \leq 1$ and hence x is in K(D,s). Then the inclusion $K(D,s) \subset K(D,s_0)$ is trivial (both D and s are contained in $K(D,s_0)$). Assume $a = 0$. Then $t = 0$, $s = u \in D$ and (1) is trivial.

(2) As $(s + K(D-s_0)) \cap B(s,t_1 d(s,D)) = s + t_1(K(D-s_0) \cap B(0,d(s,D))) \subset s + t_2(K(D-s_0) \cap B(0,d(s,D))) = (s + K(D-s_0)) \cap B(s,t_2 d(s,D))$ and similarly $s + t_2 t_0(K(D,s_0)-s_0) \subset s + t_0(K(D,s_0)-s_0)$ for $0 \leq t_1 \leq t_2 \leq 1$, it is sufficient to show that

$$(s + K(D-s_0)) \cap B(s,d(s,D)) \subset s + t_0(K(D,s_0)-s_0) \subset K(D,s).$$

Let x in $(s + K(D-s_0)) \cap B(s,d(s,D))$ be given. Then $x = s + r(z-s_0)$ for some $r \geq 0$ and z in D. As x is in B(s,d(s,D)), we have $d(s,D) \geq \|x-s\| = r\|z-s_0\|$, so that $r \leq d(s,D)/\|z-s_0\| \leq d(s,D)/d(s_0,D) = t_0$. Assume $t_0 > 0$ and set $b = 1 - rt_0^{-1}$. Then b is in $[0,1]$ and $x = s + t_0(bs_0 + (1-b)z - s_0)$ is in $s + t_0(K(D,s_0) - s_0)$. If $t_0 = 0$, then $r = 0$ and $x = s$ is in $s + t_0(K(D,s_0)-s_0)$. We have proved that $(s + K(D-s_0)) \cap B(s,d(s,D))$ is contained in $s + t_0(K(D,s_0)-s_0)$.

By the convexity of D we have $d(s,D) = d(as_0 + (1-a)u,D) \leq ad(s_0,D) + (1-a)d(u,D) = ad(s_0,D)$, so that $t_0 \leq a$. This inequality and (1) imply that $s + t_0(K(D,s_0)-s_0) \subset K(D,s)$.

Remark 1. $(s + K(D-s_0)) \cap B(s,d(s,D))$ is a "natural" ball section of the cone $s + K(D-s_0)$ parallel to the generalized drop $K(D,s_0)$. Similarly, $s + t_0(K(D,s_0)-s_0) = K(D_{t_0},s)$ is a generalized drop parallel and similar to the generalized drop $K(D,s_0)$, where $D_{t_0} = s + t_0(D-s_0)$.

Lemma 2. Let D be a convex set and s_o a point in a linear space. If s_o is not in D, then

$$(s + K(D-s_o)) \cap K(D,s_o) = K(D \cap (s + K(D-s_o)),s)$$

for each s in $K(D,s_o)$.

Proof. Let $s = as_o + (1-a)u$ with u in D and a in $(0,1]$. If $0 \neq h \in K(D-s_o)$, then the ray $s_o + R^+h$ intersects D and hence $s_o + bh$ is in D for some $b > 0$. Set $w = a(s_o + bh) + (1-a)u$. Then w is in D and $w = s + abh$ is in $s + R^+h$. We have shown that each ray $s + R^+h$, $0 \neq h \in K(D-s_o)$, intersects D.

Let x in $(s + K(D-s_o)) \cap K(D,s_o)$ be given. We show that x is in $K(D \cap (s + K(D-s_o)),s)$. We may assume that $x \neq s$. Then $0 \neq x-s$ is in $K(D-s_o)$ and hence there exist $b > 0$ and $c \geqslant 0$ such that $v = s_o + b(x-s)$ is in D and $w = s + c(x-s)$ is in D. If the set $(s + R^+(x-s)) \cap D$ is unbounded, then one easily sees that $x \in s + R^+(x-s) \subset K(D \cap (s + K(D-s_o)),s)$. Now assume that the set $(s + R^+(x-s)) \cap D$ is bounded; this set is nonempty because it contains the point w. The set $J = \{t \geqslant 0 : s + t(x-s) \in \in D\}$ contains c and is bounded. We show that $s + t(x-s) \notin \notin K(D,s_o)$ for each $t > J$ (i.e. for each t satisfying $t > j$ for each j in J). Assume the contrary. Then $y = s + t(x-s)$ is in $K(D,s_o)$ for some $t > J$ (and hence $t > c$), so that $y = gs_o + (1-g)z$ for some z in D and g in $(0,1]$. It is easy to see that

$$y = (bg + t - c)^{-1}(bgw + g(t-c)v + (1-g)(t-c)z),$$

which shows that y is in D, i.e. t is in J, a contradiction. Hence y is not in $K(D,s_o)$. This implies that

$$x \in co(\{s\} \cup (D \cap (s + R^+(x-s)))) \subset K(D \cap (s + K(D-s_o)),s).$$

The above considerations give the inclusion

$$(s + K(D-s_o)) \cap K(D,s_o) \subset K(D \cap (s + K(D-s_o)),s).$$

As the opposite inclusion is trivial, the lemma is proved.

Lemma 3. Let D be a convex set and s_o a point in a linear space. If s_o is not in D, then

$$s + K(D-s_o) \subset s + K(D-s)$$

for all s in $K(D,s_o)$.

Proof. Let $s = as_o + (1-a)u$ for some u in D and a in $(0,1]$ and let x in $s + K(D-s_o)$ be given. Then $x = s + c(v-s_o)$ for some v in D and $c \geq 0$. We have $w = av + (1-a)u \in D$, $s_0 = a^{-1}s - (a^{-1}-1)u$ and hence $x = s + c(v-s_0) = s + a^{-1}c((av + (1-a)u) - s) = s + a^{-1}c(w-s)$ is in $s + K(D-s)$.

<u>Lemma 4</u>. Let D be a convex set, S a set and s_0 a point in a linear space. If $S \quad K(D,s_0) = S \cap (s_0 + K(D-s_0))$ and s is a point in $K(D,s_0)$ such that $S \cap K(D,s) = \{s\}$, then

$$S \cap (s + K(D-s_0)) = \{s\}.$$

<u>Proof</u>. By lemma 2 we have

$$s \in (s + K(D-s_0)) \cap S = (s + K(D-s_0)) \cap (s_0 + K(D-s_0)) \cap S =$$
$$= (s + K(D-s_0)) \cap K(D,s_0) \cap S = K(D \cap (s + K(D-s_0)),s) \cap S \subset$$
$$\subset K(D,s) \cap S = \{s\}.$$

<u>Lemma 5</u>. Let D be a closed bounded convex body in a normed linear space X such that $D-z_0$ is symmetric for some z_0 in D, $s_0 \in X$, $S \subset X$, $s \in K(D,s_0)$, $K(D,s) \cap S = \{s\}$ and $S \cap K(D,s_0) = S \cap (s_0 + K(D-s_0))$. Then

$$|u_t - x| > |u_t - s|$$

for all x in S with $x \neq s$ and all $t \geq 0$, where $|\cdot|$ is the (equivalent) norm in X defined as the Minkowski functional of the set $K(D-z_0,s_0-z_0) \cup K(D-z_0,z_0-s_0)$ and $u_t = s + t(z_0-s_0)$ for $t \geq 0$.

<u>Proof</u>. Let $t \geq 0$ and let B_t be the t-ball in the norm $|\cdot|$ centered at u_t, i.e. $B_t = K(D_t,s) \cup K(D_t,2u_t-s)$, where $D_t = s + t(D-s_0)$. As $D_t \subset s + K(D-s_0)$ and $2u_t-s \in s + K(D-s_0)$ (because $(2u_t-s) - s = 2t(z_0-s_0) \in K(D-s_0)$), we have $B_t \subset s + K(D-s_0)$. By lemma 4, $(s + K(D-s_0)) \cap S = \{s\}$, so that $B_t \cap S = \{s\}$. Hence for any x in S with $x \neq s$ we have $x \notin B_t$ and this implies $|u_t - x| > |u_t - s| = t$.

<u>Lemma 6</u>. Let D be a convex set and s_0 a point in a linear space satisfying $s_0 \notin D$, $s \in K(D,s_0)$ (so that $s = as_0 + (1-a)u$ for some u in D and a in $(0,1]$), $z_0 \in D$, $u_t = s + t(z_0-s_0)$ and $D_t = s + t(D-s_0)$ for $t \geq 0$. Then:

(1) $2u_t - s \in K(D,s) \cup K(D,2z_0-s_0)$ and

$$K(D_t,s) \cup K(D_t,2u_t-s) \subset K(D,s) \cup K(D,2z_0-s_0)$$

for all $t \in [0,a]$;

(2) $2u_t - s \in K(D,s) \cup K(D,2z_o-s)$ and
$K(D_t,s) \cup K(D_t,2u_t-s) \subset K(D,s) \cup K(D,2z_o-s) \subset$
$\subset K(D,s) \cup K(D,2z_o-s_o)$
for all $t \in [0,a(2-a)^{-1}]$.

Proof follows from the identities

$$2u_t - s = (1-a^{-1}t)s + t(2z_o-s_o) + t(a^{-1}-1)u$$

and

$$2u_t - s = (1+t-2a^{-1}t)s + 2t(a^{-1}-1)u + t(2z_o-s).$$

Under the corresponding restrictions on t, the right hand sides of these identities are convex combinations of points s, $2z_o-s_o$, u and s, $2z_o-s$, u, respectively. This proves the first part of (1) and (2). The second part of both (1) and (2) follows from the first part and the inclusions $D_t \subset K(D_t,s) \subset K(D,s)$ (see lemma 1).

Lemma 7. Let D be a closed bounded convex body in a normed linear space X such that $D-z_o$ is symmetric for some z_o in D, S a set in X, $s_o \in X \setminus D$, $s \in K(D,s_o)$ (so that $s = as_o + (1-a)u$ for some u in D and a in (0,1]), $u_t = s + t(z_o-s_o)$ for $t \geqslant 0$. Assume that $K(D,s) \cap S = \{s\}$. Then:
(1) If $|\cdot|$ is the (equivalent) norm on X defined as the Minkowski functional of the set $K(D-z_o,s_o-z_o) \cup K(D-z_o,z_o-s_o)$, then

$$|u_t - x| > |u_t - s|$$

for all x in $S \setminus K(D,2z_o-s_o)$ with $x \neq s$ and all t in $[0,a]$.
(2) If $|||\cdot|||$ is the (equivalent) norm on X defined as the Minkowski functional of the set $K(D-z_o,s-z_o) \cup K(D-z_o,z_o-s)$, then

$$|||u_t - x||| > |||u_t - s|||$$

for all x in $S \setminus K(D,2z_o-s)$ with $x \neq s$ and all t in $[0,a(2-a)^{-1}]$.
(Note that $a(2-a)^{-1} < a$ for all a in (0,1).)

Proof follows from lemma 6 (compare with the proof of lemma 5).

2. Main results. Several years ago we have proved the following

Theorem D ([DANEŠ], Drop theorem). Let X be a Banach space, S a nonempty closed subset of X, z_0 a point in $X \setminus S$, $\varepsilon > 0$ and $0 < r < R = d(z_0,S)$. Then there exists a point s in S such that

$$\|s - z_0\| < R + \varepsilon \quad \text{and} \quad K(B(z_0,r),s) \cap S = \{s\}.$$

The proof of theorem D has been given by means of the

Lemma KZ ([KRASNOSELSKIĬ, ZABREĬKO]). Let X be a Banach space, x,y in X, $0 < r < p < \|x-y\|$. Then
$$\operatorname{diam}(K(B(x,r),y) \setminus B(x,p)) \leq 2(\|x-y\|-r)^{-1}(\|x-y\|+r)(\|x-y\|-p).$$

A natural generalization of lemma KZ is the following

Lemma GKZ. Let D be a bounded convex set in a normed linear space X, s_0 in X with $d(s_0,D) > 0$ and s in $K(D,s_0)$. Then
$$\|s-s_0\| \leq (1 - d(s,D)d(s_0,D)^{-1}).(d(s_0,D) + \operatorname{diam}(D)).$$
Hence, if $d(s,D) \geq q$, then
$$\|s-s_0\| \leq (1 - qd(s_0,D)^{-1}).(d(s_0,D) + \operatorname{diam}(D)).$$

Proof. As s is in $K(D,s_0)$, we have $s = as_0 + (1-a)u$ for some u in D and a in $(0,1]$. Then
$$\|s-s_0\| = (1-a)\|u-s_0\| \leq (1-a)(\|v-s_0\| + \|u-v\|) \leq$$
$$\leq (1-a)(\|v-s_0\| + \operatorname{diam}(D))$$
for each v in D, so that
$$\|s-s_0\| \leq (1-a)(d(s_0,D) + \operatorname{diam}(D)).$$
By lemma 1, $a \geq d(s,D)/d(s_0,D)$ and the result follows.

Remark 2. Setting $D = B(x,r)$, $s_0 = y$ and $s = x$ in lemma GKZ, one obtains a slight refinement of lemma KZ.

Theorem GD (Generalized drop theorem). Let X be a Banach space, S a closed subset of X, D a closed bounded convex subset of X with $d(S,D) > 0$ and s_0 a point in S. Then there exists a point s in $S \cap K(D,s_0)$ such that
$$K(D,s) \cap S = \{s\}.$$

Proof. Theorem GD follows from lemma GKZ in the same way as theorem D does from lemma KZ (see [DANEŠ]).

Remark 3. It is clear that theorem GD implies theorem D.

Remark 4. Putting theorem GD and lemma 7 (or lemma 5) together one may obtain futher results.

Remark 5. The assumption on the boundedness of the set D is essential as the following example shows. Let X be a non-reflexive Banach space. By the James' theorem there exists a linear continuous functional f on X such that $\|f\| = 1$ and f does not attain its supremum on the closed unit ball S of X. Set $D = \{x \in X : f(x) = 2\}$. For each s in S we have $f(s) < 1$, $K(D,s) = \{x \in X : f(s) < f(x) \leq 2\} \cup \{s\}$ and $K(D,s) \cap S = \{x \in S : f(s) < f(x) < 1\} \cup \{s\} \neq \{s\}$.

Theorem GD may be also derived from

Theorem E ([EKELAND]). Let (X,d) be a complete metric space, $f: X \to (-\infty, +\infty]$ a l.s.c. function with finite infimum $i = \inf f(X)$, u in X, $\varepsilon > 0$ with $\varepsilon \geq f(u) - i$ and $\lambda > 0$. Then there exists a point v in X such that

$d(u,v) \leq 1/\lambda$, $f(v) \leq f(u)$,

$f(w) - f(v) > -\lambda\varepsilon d(w,v)$ for all $w \neq v$.

Theorem E implies theorem GD. Set $Y = K(D,s_0) \cap S$ and define $f: Y \to R$ by $f(x) = d(x,D)$. Then Y is a complete metric space and f is a continuous function on Y with $\inf f(Y) = d(Y,D)$. Take $\varepsilon, \lambda > 0$ such that

(§) $(1 - \lambda\varepsilon)d(Y,D) > \lambda\varepsilon\, \mathrm{diam}(D)$

(for example, one may take $\varepsilon = \lambda = \delta^{1/2}$, where δ is any number satisfying $0 < \delta < d(Y,D).(d(Y,D) + \mathrm{diam}(D))^{-1}$). By theorem E there exists s in Y such that $f(s) \leq f(s_0)$, $\|s-s_0\| \leq 1/\lambda$ and $f(x) - f(s) > -\lambda\varepsilon\|x-s\|$ for all x in Y with $x \neq s$, i.e.

(§§) $d(x,D) + \lambda\varepsilon\|x-s\| > d(s,D)$ for all x in Y with $x \neq s$.

We show that $K(D,s) \cap S = \{s\}$. As $K(D,s) \subset K(D,s_0)$, this is equivalent to showing that $K(D,s) \cap Y = \{s\}$. Assume $K(D,s) \cap Y \neq \{s\}$. Then there exist u in D and a in (0,1) such that $x = as + (1-a)u$ is in Y. By the convexity of D we have

$$d(x,D) + \lambda\varepsilon\|x-s\| = d(as + (1-a)u,D) + \lambda\varepsilon(1-a)\|s-u\| \leq$$
$$\leq ad(s,D) + (1-a)d(u,D) + \lambda\varepsilon(1-a)\|s-u\| \leq$$
$$\leq ad(s,D) + \lambda\varepsilon(1-a)(d(s,D) + \mathrm{diam}(D)),$$

because $\|s-u\| \leq d(s,D) + \mathrm{diam}(D)$.

The above inequality and (§§) give

$$d(s,D) < ad(s,D) + \lambda\varepsilon(1-a)(d(s,D) + diam(D)),$$

i.e. (as $1-a > 0$)

$$(1 - \lambda\varepsilon)d(s,D) < \lambda\varepsilon\, diam(D)$$

and hence $(1 - \lambda\varepsilon)d(Y,D) < \lambda\varepsilon\, diam(D)$, which is a contradiction to (§).

As theorem E works on metric spaces and theorem GD only on normed spaces, it may be somewhat surprising that theorem GD implies E. We show that even theorem D (a weaker form of theorem GD) implies theorem E. The trick is to imbed a metric space isometrically in a normed linear space. We will use the well-known

<u>Lemma 8</u>. Let (X,d) be a metric space, x_0 in X and $C_b(X)$ the Banach space of all continuous bounded (real-valued) functions on X (with the supremum norm). For x in X, let $f_x(y) = d(x,y) - d(x_0,y)$, $y \in X$. Then f_x is in $C_b(X)$ and the mapping $T:X \to C_b(X)$ defined by $Tx = f_x$ is an isometry.

<u>Corollary 1</u>. Let (X,d) be a complete metric space and $f:X \to (-\infty,+\infty]$ a l.s.c. function. Define a function $F:C_b(X) \to (-\infty,+\infty]$ as follows

$$F(u) = \begin{cases} f(T^{-1}u) & \text{if } u \text{ is in } T(X) \\ +\infty & \text{otherwise,} \end{cases}$$

where T is as in lemma 8. Then the set $T(X)$ is closed in $C_b(X)$ and F is an l.s.c. function on $C_b(X)$.

<u>Proof</u>. If $t = +\infty$, then $\{u \in C_b(X) : F(u) \leq t\} = C_b(X)$. If t is finite, then $\{u \in C_b(X) : F(u) \leq\}t = T(\{x \in X : f(x) \leq t\})$, so that the set $\{u \in C_b(X) : F(u) \leq t\}$ is complete as the image of the complete set $\{x \in X : f(x) \leq t\}$ under the isometry T. The same argument shows that $T(X)$ is complete.

<u>Theorem D implies theorem E</u>. Let (X,d) be a complete metric space, $f:X \to (-\infty,+\infty]$ a l.s.c. function with finite $i = \inf f(X)$, u in X with $f(u) \leq i + \varepsilon$ and $\lambda > 0$. By corollary 1 we may assume that X is a Banach space and d is the distance given by the norm of X. Let $Y = X \times R$, $S = epi(f) =$

$= \{(x,t) : x \text{ in } X,\ +\infty > t \geq f(x)\}$, $s_0 = (u,f(u))$, $a > 0$, $0 < r < 1$, $z_0 = (u,i-1)$ and let the norm on Y be defined by

$$\|(x,t)\| = a\|x\| + |t|.$$

Then

$$0 < r < 1 \leq \check{d}(z_0,S) \leq \|z_0-s_0\| = f(u) - i + 1,$$

where $\check{d}(z_0,S)$ is the distance from z_0 to S in the norm of Y. One easily sees that

$$K(B(z_0,r),s_0) \cap H = (s_0 + K) \cap H,$$

and hence

$$K(B(z_0,r),s_0) \cap S = (s_0 + K) \cap S,$$

where $H = \{(x,t) : x \text{ in } X,\ t \geq i\}$ and $K = K(B(z_0-s_0,r))$ (because $S \subset H$, $K = K(B(0,r/a) \times (i-1-f(u))$ and $B(z_0,r) \cap H = \emptyset$). By theorem D applied to the set $S \cap K(B(z_0,r),s_0)$ rather than to S there exists a point s in $S \cap K(B(z_0,r),s_0)$ such that $S \cap K(B(z_0,r),s) = \{s\}$ and hence, by lemma 5 (set $D = B(z_0,r)$),

(§) $\qquad |u_t - \check{x}| > |u_t - s|$

for all $\check{x}$ in S with $\check{x} \neq s$ and all $t \geq 0$, where $|.|$ is the (equivalent) norm on Y defined as the Minkowski functional of the set $K(B(0,r),s_0-z_0) \cup K(B(0,r),z_0-s_0)$ and $u_t = s + t(z_0-s_0)$ for $t \geq 0$. It is clear that $s = (v,f(v))$ for some v in dom(f) $= \{x \in X : f(x) \text{ finite}\}$; note also that

$$|(x,t)| = ar^{-1}\|x\| + t\|s_0-s_0\|^{-1} = ar^{-1}\|x\| + t.(f(u)-i+1)^{-1}$$

for (x,t) in Y. Applying (§) to any $\check{x} = (w,f(w))$ with w in dom(f) and $w \neq v$, we have

$$|(v-w,f(v) - f(w) - t(f(u)-i+1)| > |(0,-t(f(u)-i+1))|,$$

i.e.

$$ar^{-1}\|w-v\| + |(f(u)-i+1)^{-1}(f(w)-f(v)) + t| > t \text{ for all } t \geq 0.$$

This gives

$$ar^{-1}\|w-v\| + (f(u)-i+1)^{-1}(f(w)-f(v)) > 0,$$

i.e.

$$f(w) - f(v) > -ar^{-1}(f(u)-i+1)\|w-v\| \geq -ar^{-1}(\varepsilon+1)\|w-v\|.$$

We have proved that

$$f(w) - f(v) > -a(\varepsilon+1)r^{-1}\|w-v\|$$

for all w in dom(f) with $w \neq v$ and hence for all w in X with $w \neq v$.

Now we show that $\|u-v\| \leq a^{-1}r\,\varepsilon(\varepsilon+1)^{-1}$. Let $y = (u,f(v))$ and let x be the intersection of the ray $s_0 + R^+(s-s_0)$ with the hyperplane $X \times (i-1)$ (take $x = s_0$ if $s = s_0$). Since

$|s-s_0| = |s_0-y|\,|s_0-x|\,|s_0-z_0|^{-1}$, $|s_0-z_0| = 1$, $|s_0-y| = (f(u)-f(v)).(f(u)-i+1)^{-1}$ and $|s_0-x| \leq 2$ (both s_0 and x lie in the unit ball in the norm $|.|$ centered at z_0), we have

$$ar^{-1}\|u-v\| + (f(u)-f(v)).(f(u)-i+1)^{-1} = |s-s_0| \leq$$
$$\leq 2(f(u)-f(v))-(f(u)-i+1)^{-1}$$

(here we have used the inequality $f(v) \leq f(u)$ which is an obvious consequence of the inclusion $(v,f(v)) \in K(B(z_0,r),s_0) \subset \subset s_0 + K)$ and thus

$$\|u-v\| \leq a^{-1}r(f(u)-f(v)).(f(u)-i+1)^{-1} \leq a^{-1}r(f(u)-i).(f(u)-i+1)^{-1} \leq$$
$$\leq a^{-1}r\,\varepsilon(\varepsilon+1)^{-1}.$$

The proof will be complete if, for given $\lambda > 0$, we are able to find positive a such that

$$1/\lambda = a^{-1}r\,\varepsilon(\varepsilon+1)^{-1} \quad \text{and} \quad \lambda\varepsilon = ar^{-1}(\varepsilon+1).$$

But this is easy: given $\lambda > 0$, set $a = \lambda\varepsilon\, r(\varepsilon+1)^{-1}$.

In [BRÉZIS, BROWDER] there is shown that theorem B below implies theorem D.

<u>Theorem B</u> ([BRÉZIS, BROWDER], Theorem 4). Let S be a closed subset of a Banach space X and z_0 in $X \setminus S$. Let $0 < r < d(z_0,S)$ and s_0 in S. Then there exists a point s in $S \cap K(B(z_0,r),s_0)$ such that

$$S \cap (s + K) \cap B(s,\delta) = \{s\}$$

for all $\delta < d(z_0,S) - r$, where $K = K(B(z_0-s_0),r))$.

Now we show that

<u>Theorem D implies theorem B</u>. By theorem D applied to the set $S \cap K(D,s_0)$ rather than to S (where $D = B(z_0,r)$ and $\varepsilon > 0$ is arbitrary) there is s in $S \cap K(D,s_0)$ such that $S \cap K(D,s) = \{s\}$. By lemma 1 (with $t_1 = 1$) we have

$$(s + K) \cap B(s,d(s,B(z_0,r)) \subset K(D,s).$$

Since $d(s,B(z_0,r)) = \|s-z_0\| - r \geq d(z_0,S) - r$, we have also $(s + K) \cap B(s,d(z_0,S)-r) \subset K(D,s)$ and hence

$$S \cap (s + K) \cap B(s,d(z_0,S)-r) = \{s\}.$$

We have shown, in fact, that theorem D implies the following apter form of theorem B.

<u>Theorem B'</u>. Under the hypotheses of theorem B, there

exists a point s in $S \cap K(B(z_0,r),s_0)$ such that

$$S \cap (s + K) \cap B(s,d(z_0,S)-r) = \{s\}.$$

Let us remember the Caristi fixed point theorem:

Theorem C ([CARISTI], [CARISTI, KIRK]). Let (X,d) be a complete metric space, f a real l.s.c. function on X bounded below and $T:X \to X$ a mapping such that $d(x,Tx) \leq f(x) - f(Tx)$ for all x in X. Then T has a fixed point.

The equivalence of theorems C and E is evident (see also [BRÉZIS, BROWDER]).

Let us mention also a refinement of Krasnoselskiĭ-Zabreĭko renorming theorem [KRASNOSELSKIĬ, ZABREĬKO] as stated in [DANEŠ].

Theorem KZD. Let X be a Banach space, S a closed set in X and z_0 a point in $X \setminus S$. Let $0 < \delta < 1$. Then there exists an equivalent norm $|\cdot|$ on X such that the $|\cdot|$-distance of z_0 from S is attained at a point of S and at most at two points of S (in this case the sum of these two points equals $2z_0$), and such that

$$(1 - \delta)\|\cdot\| \leq |\cdot| \leq \|\cdot\|.$$

In [DANEŠ] we have shown that theorem D is equivalent to theorem KZD. Summarizing, we may state the following

CLAIM. Theorems D, GD, KZD, B, B', E and C are mutually equivalent.

Concluding remark. The manuscript of this paper has circulated since Dec. 1978. Originally, we have not intended to publish it. As the number of papers in which the same (or related) results are derived by one of the equivalent theorems considered in this paper is increasing, we have decided to publish the manuscript. In the meanwhile, J.-P. Penot sent me his preprint "The drop theorem, the petal theorem and Ekeland's variational principle" (Sept. 1984) where he also proves among others the basic implication theorem D $\Rightarrow$ theorem E. Unfortunately, a common publication is impossible because he has sent his preprint for publication.

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