Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 3, 421--442

Persistent URL: `http://dml.cz/dmlcz/106384`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,3 (1885)

A FUZZY MODIFICATION OF THE CATEGORY OF LINEARLY ORDERED SPACES

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Abstract: Generalizing the well-known Hutton's construction of the fuzzy unit interval we define a functor F from the category Ord of linearly ordered spaces into the category Fuz of fuzzy topological spaces. Some properties of this functor are established. Specifically, the connections between the properties of the linear order on X and the fuzzy topological properties of F(X) are studied. In case when X is connected, the space F(X) is fuzzy homeomorphic to the space K(X) constructed by A. Klein.

Key words: Fuzzy topological space, fuzzy unit interval, linearly ordered space.

Classification: 54A40, 54F05

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§ 0. Introduction. In [16] we offered a construction which for a given linearly ordered topological space associates in a definite way a fuzzy topological space F(X) - the so called fuzzy modification of a linearly ordered topological space X. In the case when X = I (= [0,1]), the space F(I) is fuzzy homeomorphic with the fuzzy closed unit interval [7 which is one of the most important and interesting examples of fuzzy topological spaces (see e.g. [7],[8],[5],[13],[14],[15] e.a.). The fuzzy spaces F(ℝ) and F(]0,1[) are fuzzy homeomorphic with the fuzzy real line [5] and the fuzzy open unit interval [5] respectively. In [16] we began to study the properties of F(X). In particular, there were established some connections between the topological properties of X and the fuzzy topological properties of $F(X)$.

The principal aim of the present paper is to impart the categorical character to this construction. Namely, the main object of the paper is a functor F from the category Ord of linearly ordered topological spaces and increasing continuous mappings into the category Fuz of fuzzy topological spaces and fuzzy continuous mappings (Section 4).

The paper begins with Section 1 containing the preliminary information employed in the text. In Section 2 the definition of the fuzzy modification $F(X)$ of a linearly ordered topological space X from [16] is reproduced. Here we state also the main results from [16] concerning the fuzzy topological properties of $F(X)$. The third section is devoted to a construction which allows to associate with an increasing continuous mapping $f: X \longrightarrow Y$ a fuzzy continuous mapping $F(f) = \hat{f}: F(X) \longrightarrow$ $\longrightarrow F(Y)$. The relation F appears to be functorial (Section 4). Section 5 contains the construction of a fuzzy modification for the case of a decreasing mapping.

Our definition of the fuzzy modification of a linearly ordered space is essentially based on the generalization of the fundamental idea of B. Hutton [7] which he has used for the construction of the fuzzy closed unit interval. An interesting and quite different extension of B. Hutton's construction was carried out by A. Klein [10]. He associates with a connected topological space X a fuzzy topological space $K^o(X)$ (our denotation) in such a way that $K^o(I)$ and $K^o(\mathbb{R})$ are equivalent with the fuzzy closed unit interval and the fuzzy real line. Moreover, the space X is contained in $K^o(X)$ as a fuzzy subspace. The aim of the last, sixth section is to show that if a topological space is both linearly ordered and connected, then the both constructions $F(X)$ and $K^o(X)$ may be considered as equivalent.

§ 1. Preliminaries. A. Linearly ordered spaces. Let X be a set and $<$ a linear order on it (see e.g. [4], p. 17). As usual, we write $x \leq a$ if $x<a$ or $x = a$. For $a,b \in X$ let $]a, \rightarrow[= \{x: x \in X,\ a<x\}$, $]\leftarrow, b[= \{x: x \in X,\ x<b\}$, $]a,b[= \{x: x \in X,\ a<x<b\}$, $[a,b] = \{x: x \in X,\ a \leq x \leq b\}$, $[a,b[= \{x: x \in X,\ a \leq x<b\}$, etc.

A subset X_o of X is called bounded in X if there exist $a,b \in X$ such that $X_o \subset [a,b]$. Specifically, X is bounded if it has a maximal and a minimal elements. By a cofinal character of X we understand the least cardinal number k for which there exists a subset $X_o \subset X$ of cardinality k such that for every $x \in X$ there are $y \in X_o$, $y \leq x$, and $z \in X_o$, $x \leq z$ (cf. [4], p. 22).

One can easily check that $\mathcal{B} = \{]a,b[: a,b \in X\}$ is a base for some topology $\mathcal{T}$ on the set X; it will be called "the topology generated by the linear order $<$ ". Throughout the paper, by a linearly ordered (topological) space, we understand a triple $(X,<,\mathcal{T})$. It will be usually abbreviated as $(X,<)$ or just as X if there can be no confusion. Thus in our context the linear order in a linearly ordered (topological) space is assumed to be fixed (in contrast with the usual terminology according to which a linearly ordered topological space is defined as a pair $(X,\mathcal{T})$ where the topology $\mathcal{T}$ can be generated by some linear order $<$ on X (see e.g. [4], p. 82)).

Let $(X,<)$ and $(Y,<)$ be two linearly ordered spaces. A mapping $f: X \rightarrow Y$ will be called increasing if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$. Decreasing mappings are defined analogously.

It is obvious that linearly ordered spaces and increasing continuous mappings between them form a category; this category will be denoted Ord.

B. Fuzzy topological spaces. The terminology used in fuzzy topology is rather unsteady yet and various authors proceed sometimes from different basic definitions. Therefore everyone working in this field has to specify first the frames in which he carries his studies out. As in our previous papers [16],[17], [18], we work chiefly in the R. Lowen's category Fuz of fuzzy topological spaces (see the definition (1.3) below).

(1.1) Remark. Our preference of R. Lowen's definition on the whole was explained in [16] and [17]. However, all the results of this paper have obvious equivalents in the more general category Fuz^{o} of fuzzy topological spaces in the sense of C. Chang (definition $(1.3)^{o}$ below). The most important of these equivalents are formulated explicitly and numerated with the same number but with an additional superscript "o". The proofs of theorems in the case of Fuz^{o} are ommited since they can be obtained just by obvious and insignificant changes in the proofs of the corresponding theorems for Fuz. Notice, however, that both the versions are logically independent.

(1.2) Remark. The question whether the main results of this paper can be transferred to the category of L-fuzzy topological spaces ([6], see also [7],[5] e.a.) is more problematic. The author has only partial results in this direction and they are not reflected in this paper.

(1.3) Definition [11],[12]. A fuzzy topology on a set X is a family τ of its fuzzy subsets (i.e. $\tau \subset I^X$), satisfying the following three axioms:

(1) if $\mu, \nu \in \tau$ then $\mu \wedge \nu \in \tau$;

(2) if $\mu_a \in \tau$ for all $a \in A$, then $\vee\{\mu_a : a \in A\} = \mu \in \tau$

(3) τ contains all constants $c: X \longrightarrow I$.

A fuzzy topological space is a pair (X, τ) where X is a set and τ is a fuzzy topology on it.

$(1.3)^{\circ}$ Definition [3]. A fuzzy topology on a set X is a family τ of its fuzzy subsets, satisfying the axioms (1) and (2) of Definition (1.3) and the following axiom

$(3)^{\circ}$ τ contains the constants $0: X \longrightarrow I$ and $1: X \longrightarrow I$.

A fuzzy topological space is a pair (X, τ) where X is a set and τ is a fuzzy topology on it.

(1.4) Definition [3],[11]. Let (X, τ) and (Y, σ) be fuzzy topological spaces (either in the sense of R. Lowen or in the sense of C.L. Chang). A mapping $f: X \longrightarrow Y$ is called fuzzy continuous if $f^{-1}(\nu) \in \tau$ for all $\nu \in \sigma$.

(1.5) Denotation. The category of fuzzy topological spaces in the sense of R. Lowen and fuzzy continuous mappings between them will be denoted Fuz.

$(1.5)^{\circ}$ Denotation. The category of fuzzy topological spaces in the sense of C. Chang and fuzzy continuous mappings between them will be denoted Fuz°.

§ 2. Fuzzy modification of a linearly ordered space. Basing on the fundamental idea of B. Hutton [7] we have defined in [16] a construction which in a definite way associates with every linearly ordered space X a fuzzy topological space F(X). In this section we first reproduce the construction and then following [16] state the theorems which establish some connections between the properties of the space X and the fuzzy topological properties of its fuzzy modification F(X). All the proofs are omitted since they can be found in [16].

(2.1) <u>The construction of</u> F(X) [16]. Let Z(X) denote the set of all decreasing functions $z: X \longrightarrow I$ such that $\sup_X z(x) = 1$ and $\inf_X z(X) = 0$. For every $x \in X$ let

$$z(x^-) = \begin{cases} \inf_{t<x} z(t), & \text{if } x \neq \min X \\ z(x) = 1, & \text{if } x = \min X \end{cases} \quad \text{and}$$

$$z(x^+) = \begin{cases} \sup_{t>x} z(t), & \text{if } x \neq \max X \\ z(x) = 0, & \text{if } x = \max X. \end{cases}$$

For $z, z' \in Z(X)$ we write $z \sim z'$ iff $z(x^-) = z'(x^-)$ and $z(x^+) = z'(x^+)$ for every $x \in X$. Obviously, $\sim$ is an equivalence relation on Z(X). Let $[z] = \{z' \in Z(X) : z \sim z'\}$ and let F(X) denote the set of all equivalence classes $[z]$, i.e. $F(X) = Z(X)/_\sim$.

For all $a, b \in X$ let fuzzy sets λ_b and ρ_a of F(X) be defined by the equalities $\lambda_b[z] = 1 - z(b^-)$, and $\rho_a[z] = z(a^+)$. If $c \in I$, then we use the same symbol for the constant function $c: F(X) \longrightarrow I$. Let τ be the fuzzy topology on F(X) having $\sigma = \{\lambda_b, \rho_a : a, b \in X\} \cup \{c : c \in I\}$ as a subbase. The fuzzy topological space $(F(X), \tau)$ will be usually written just as F(X) and called the fuzzy modification of the linearly ordered space X.

$(2.1)^\circ$ <u>The construction of</u> $F^\circ(X)$. In the category Fuz°, the fuzzy modification $F^\circ(X)$ of a linearly ordered space X is defined just as in (2.1) with the only difference that the fuzzy topology τ° on F(X) is defined by the subbase $\sigma^\circ = \{\lambda_b, \rho_a : a, b \in X\}$ (instead of σ).

(2.2) <u>Examples</u> [16]. The fuzzy spaces $F(\mathbb{R})$, F(I) and $F(]0,1[)$ are fuzzy homeomorphic with the stratified fuzzy real line [14], the stratified fuzzy closed unit interval [14] and the stratified fuzzy open unit interval [14] respectively.

(2.2)^{o} Examples. The fuzzy spaces $F^{o}(\mathbb{R})$, $F^{o}(I)$ and $F^{o}(]0,1[)$ are fuzzy homeomorphic with the fuzzy real line [5], the fuzzy closed unit interval [7] and the fuzzy open unit interval [7] respectively.

R. Lowen has defined and widely used the embedding functor ω: Top $\rightarrow$ Fuz (see e.g. [12]). For a topological space X the fuzzy topological space $\omega(X)$ can be in a natural way considered as a fuzzy copy of X.

(2.3) Theorem [16]. If X is a linearly ordered space, then $\omega(X)$ is fuzzy homeomorphic to a (proper) fuzzy subspace of $F(X)$.

Since the category Top of topological spaces and continuous mappings may be in an obvious way considered as a subcategory of Fuz^{o}, the corresponding equivalent of the previous theorem is even more lucid:

(2.3)^{o} Theorem. If X is a linearly ordered space, then X is fuzzy homeomorphic to a (proper) fuzzy subspace of $F^{o}(X)$.

(2.4) Theorem [16]. If X is an infinite linearly ordered space, then its weight is equal to the fuzzy weight of $F(X)$.

(The fuzzy weight of a fuzzy topological space is naturally defined as the minimal cardinality of the bases of its fuzzy topology [16].)

We shall not state explicitly the equivalent of (2.4) as well as the equivalents of (2.5) - (2.9) below for the category Fuz^{o} because one can obtain them just by replacing $F(X)$ with $F^{o}(X)$.

(2.5) Corollary (cf. [13]). The fuzzy spaces F(R), F(I) and F(]0,1[) have countable fuzzy weights.

(2.6) Theorem [16]. If a linearly ordered space X is bounded, then F(X) is fuzzy α-compact for all $\alpha \in [0,1[$. Conversely, if F(X) is fuzzy α-compact for some $\alpha \in [0,1[$, then X is bounded.

(For the definition of fuzzy α-compactness see [5] or [12].)

(2.7) Corollary (cf. [5],[13]). F(I) is fuzzy α-compact for all $\alpha \in [0,1[$; F(R) and F(]0,1[) are not fuzzy α-compact for any $\alpha \in [0,1[$.

(2.8) Theorem [16]. If X is an unbounded linearly ordered space, then the fuzzy Lindelöf number of F(X) is equal to the cofinal character of X (see § 1.A).

(The fuzzy Lindelöf number of a fuzzy space Y is defined as the minimal cardinal k such that for every $\alpha \in [0,1[$ every α-shading [5] has an α-subshading of cardinality less or equal to k.)

(2.9) Theorem [16]. The following conditions are equivalent for a linearly ordered space X:

(a) X has a G_δ-diagonal;

(b) X is stratifiable;

(c) X is metrizable;

(d) F(X) is fuzzy stratifiable.

(The equivalence of the first three conditions is a well-known fact of general topology (see e.g. [4] and [1].) For the definition of a topological stratifiable space see [2] and [1]; fuzzy stratifiable spaces were introduced and studied in [17],

[18]. The author is sorry about the confusing consonance of the two completely different notions of a fuzzy stratifiable space and a stratified fuzzy topological space ([19],[13] e.a.).)

§ 3. Fuzzy modification of an increasing continuous mapping

(3.1) Let $(X,<)$ and $(Y,<)$ be linearly ordered spaces and $f:X \longrightarrow Y$ an increasing continuous mapping. For every $z \in Z(X)$ let $f^*(z) = u: Y \longrightarrow I$ be defined as follows:

$$u(y) = \begin{cases} \inf\limits_{f(x) \leq y} z(x), & \text{if }]\leftarrow, y] \cap f(X) \neq \emptyset \\ 1, & \text{if }]\leftarrow, y] \cap f(X) = \emptyset. \end{cases}$$

(3.2) Proposition. Let $z_1, z_2 \in Z(X)$ and $u_1 = f^*(z_1)$, $u_2 = f^*(z_2)$. If $z_1 \sim z_2$ in $Z(X)$, then $u_1 \sim u_2$ in $Z(Y)$.

Proof. Let $y_0 \in Y$ and let $u \in Z(Y)$. To show that $u_1(y_0^-) = u_2(y_0^-)$ consider the following possible cases:

a) $f^{-1}]\leftarrow, y_0[\, \neq \emptyset$ and there is no maximal element in $f^{-1}]\leftarrow, y_0[$. Then $u(y_0^-) = \inf\limits_{y<y_0} u(y) = \inf\limits_{y<y_0} \inf\limits_{f(x) \leq y} z(x) = \inf\limits_{f(x)<y_0} z(x) = \inf\limits_{f(x)<y_0} z(x^-)$.

b) There exists $x_1 = \max f^{-1}]\leftarrow, y_0[$. Let $y_1 = f(x_1) < y_0$. Then either x_1 is the maximal element in X and hence $u(y_0^-) = u(y_1) = z(x_1) = 0$, or there exists $x_2 \in X$, $x_1 < x_2$, such that there is a jump [4] between x_1 and x_2 (otherwise f cannot be continuous) and therefore $u(y_0^-) = u(y_1) = z(x_1) = z(x_2^-)$.

c) $f^{-1}]\leftarrow, y_0[\, = \emptyset$. Then $u(y) = 1$ for every $y < y_0$ and hence $u(y_0^-) = 1$.

Thus in every case we conclude that $u_1(y_0^-) = u_2(y_0^-)$.

It remains to show that $u_1(y_0^+) = u_2(y_0^+)$. Consider the next possible cases:

a) $f^{-1}]y_0, \rightarrow[\, \neq \emptyset$ and there is no minimal element in this set. Then $u(y_0^+) = \sup\limits_{y>y_0} u(y) = \sup\limits_{y>y_0} \inf\limits_{f(x) \leq y} z(x) = \sup\limits_{f(x)>y_0} z(x) =$

$= \sup_{f(x) > y_0} z(x^+)$.

b) There is $x_1 = \min f^{-1}]y_0, \rightarrow[$. Let $y_1 = f(x_1)$. Then either x_1 is the minimal element of X and hence $u(y_0^+) = u(y_1) = z(x_1) = 1$, or there exists $x_2 \in X$, $x_2 < x_1$ such that there is a jump between x_2 and x_1 (otherwise f cannot be continuous) and therefore $u(y_0^+) = \sup_{y > y_0} u(y) = \sup_{y > y_0} \inf_{f(x) \leq y} z(x) = \inf_{f(x) \leq y_1} z(x) = z(x_1) = z(x_2^+)$.

c) $f^{-1}]y_0, \rightarrow[= \emptyset$. Then obviously $u(y) = 1$ for every $y > y_0$ and hence $u(y_0^+) = 1$.

Thus again in every case $u_1(y_0^+) = u_2(y_0^+)$.

This proposition ensures the correctness of the following definition:

(3.3) <u>Definition</u>. Let $f: X \longrightarrow Y$ be an increasing continuous mapping. The equality $\hat{f}[z] = [u]$ where $[z] \in F(X)$, $u = f^*(z)$ and $[u] \in F(Y)$ defines a mapping $\hat{f}: F(X) \longrightarrow F(Y)$.

The mapping $\hat{f}$ will be called a fuzzy modification of the mapping f.

(3.4) <u>Theorem</u>. The mapping $\hat{f}: F(X) \longrightarrow F(Y)$ is fuzzy continuous.

<u>Proof</u>. Let $\mathcal{T} = \{\lambda_b, \rho_a : a, b \in X\} \cup \{c : c \in I\}$ be the standard subbase of the topology on F(X) (see (2.1)), and let analogously $\Pi = \{L_e, R_d : e, d \in Y\} \cup \{c : c \in I\}$ be the standard subbase of the topology on F(Y). (Here the fuzzy sets $L_e, R_d : F(Y) \longrightarrow I$ are defined by the equalities $L_e[u] = 1 - u(e^-)$, $R_d[u] = u(d^+)$ for all $[u] \in F(Y)$.) Since the preimage of every constant $c: F(Y) \longrightarrow I$ under $\hat{f}$ is obviously the same constant $c = \hat{f}^{-1}(c): F(X) \longrightarrow I$, to show the continuity of $\hat{f}$ it suffices to check that the preimage of all L_e and R_d are open in F(X).

Take some L_e and let $[z] \in F(X)$. Then $\hat{f}^{-1}(L_e)[z] =$

$= L_e\hat{f}[z] = L_e[u] = 1 - u(e^-)$ where $u = f^*(z)$. Consider the following possible cases:

a) $f^{-1}]\leftarrow, e[\neq \emptyset$ and there is no maximal element in $f^{-1}]\leftarrow, e[$. Then $u(e^-) = \inf_{y<e} \inf_{f(x)<y} z(x) = \inf_{f(x)<e} z(x) =$ $= \inf_{f(x)<e} z(x^-)$. Hence $\hat{f}^{-1}(L_e)[z] = 1 - \bigwedge_{f(x)<e} z(x^-) =$ $= \bigvee_{f(x)<e} (1 - z(x^-)) = \bigvee_{f(x)<e} \lambda_x[z]$,

b) $x_1 = \max f^{-1}]\leftarrow, e[$. Then the continuity of f implies that either x_1 is the maximal element of X and hence $u(e^-) =$ $= z(x_1) = 0$, i.e. $\hat{f}^{-1}(L_e)[z] = 1$, or there exists $x_2 \in X$, $x_1 < x_2$ such that there is a jump between x_1 and x_2. In this case $u(e^-) =$ $= z(x_1) = z(x_2^-)$ and therefore $f^{-1}(L_e)[z] = 1 - z(x_2^-) = \lambda_{x_2}[z]$.

c) $f^{-1}]\leftarrow, e[= \emptyset$. Then $u(y) = 1$ for every $y < e$, hence $u(e^-) = 1$ and $\hat{f}^{-1}(L_e) = 0$.

Thus in every case the preimage $f^{-1}(L_e)$ is an open fuzzy subset of $F(X)$.

Now take some R_d. Then $f^{-1}(R_d)[z] = R_d\hat{f}[z] = R_d[u] =$ $= u(d^+)$, where $u = f^*(z)$. Consider the following possible cases:

1) $f^{-1}]d, \rightarrow[\neq \emptyset$ and there is no minimal element in it. Then $u(d^+) = \sup_{y>d} u(y) = \sup_{y>d} \inf_{f(x)\leq y} z(x) = \sup_{f(x)>d} z(x) =$ $= \sup_{f(x)>d} z(x^+)$, and hence $\hat{f}^{-1}(R_d) = \bigvee_{f(x)>d} \rho_x$.

2) $x_1 = \min f^{-1}]d, \rightarrow[$. Then the continuity of f implies that either $x_1 = \min X$ and hence $u(d^+) = z(x_1) = 1$, i.e. $\hat{f}^{-1}(R_d) = 1$, or there is $x_2 < x_1$ such that there is a jump between x_2 and x_1. In this case $u(d^+) = z(x_1) = z(x_2^+)$, i.e. $\hat{f}^{-1}(R_d) = \rho_{x_2}$.

3) $f^{-1}]d,\rightarrow[= \emptyset$. Then $u(y) = 0$ for every $y > d$ and hence $u(d^+) = 0$. Therefore $\hat{f}^{-1}(R_d) = 0$.

Thus in every case the preimage $\hat{f}^{-1}(R_d)$ is fuzzy open in X. This completes the proof of the theorem.

(3.3)° Since the fuzzy modifications F(X) and $F^o(X)$ (see (2.1)°) coincide as sets, the mapping $\hat{f}$ defined in (3.3) may be considered also as a mapping $\hat{f}:F^o(X) \rightarrow F^o(Y)$.

(3.4)° Theorem. The mapping $\hat{f}:F^o(X) \rightarrow F^o(Y)$ is fuzzy continuous.

(3.5) Proposition. If $f:X \rightarrow Y$ is an increasing homeomorphism, then $\hat{f}:F(X) \rightarrow F(Y)$ is a fuzzy homeomorphism.

Proof. Let $f^{-1}:Y \rightarrow X$ be the inverse of f. We shall first show that $(f^{-1})^* \circ f^*(z) = z$ for every $z \in Z(X)$ and $f^*(f^{-1})^*(u) = u$ for every $u \in Z(Y)$. This will precisely mean that $(f^{-1})^*:Z(Y) \rightarrow Z(X)$ is the inverse of $f^*:Z(X) \rightarrow Z(Y)$.

Since f is a bijection, the equality $f^*(z) = u$ means in this case that $z(x) = u(y)$ for $x = f(y)$. Hence $(f^{-1})^* f^*(z)(x) = z(x)$ for every $z \in Z(X)$ and all $x \in X$; thus, $(f^{-1})^* f^*(z) = z$. The equality $f^*(f^{-1})^*(u) = u$ for every $u \in Z(Y)$ can be proved similarly.

Since $(f^{-1})^*$ is the inverse of f^*, it is easy to conclude now that $(\widehat{f^{-1}}):F(Y) \rightarrow F(X)$ is the inverse of $\hat{f}:F(X) \rightarrow F(Y)$. Moreover, since the mappings f and f^{-1} are continuous from Theorem (3.4) it follows that $\hat{f}$ and $\hat{f}^{-1}$ are fuzzy continuous. Hence $\hat{f}$ is a fuzzy homeomorphism.

(3.5)° Proposition. If $f:X \rightarrow Y$ is an increasing homeomorphism, then $\hat{f}:F^o(X) \rightarrow F^o(Y)$ is a fuzzy homeomorphism.

(3.6) Remark. One may consider that it is more natural to define $f^*(z) = u: Y \longrightarrow I$ in the following way (which is obviously not equivalent with (3.1)):

$$u(y) = \begin{cases} \inf\limits_{f(x) < y} z(x), & \text{if }]\leftarrow, y[\cap f(X) \neq \emptyset \\ 1 & , \text{if }]\leftarrow, y[\cap f(X) = \emptyset. \end{cases}$$

The analog of Proposition (3.2) holds for f^* defined in such a way, too, and, moreover, the assumption of continuity of f is superfluous in this case. However, on the other hand, the analog of Theorem (3.4) does not hold for the corresponding $\hat{f}$ even if f is continuous. This is one of the reasons for our choice of Definition (3.1) as the basic one.

§ 4. Functor F. Let Ord be the category of linearly ordered topological spaces and increasing continuous mappings. In this section we define basing on the results of the two previous sections an embedding functor $F: \mathrm{Ord} \longrightarrow \mathrm{Fuz}$ (see 1.5)). Incidentally we consider also an embedding functor $F^o: \mathrm{Ord} \longrightarrow \mathrm{Fuz}^o$ (see (1.5^o) which is the natural analog of F for the case of Chang's definition of fuzzy topological spaces.

(4.1) Definition of $F: \mathrm{Ord} \longrightarrow \mathrm{Fuz}$. For every object X of Ord let F(X) be the fuzzy modification of X (see (2.1)) and for every morphism $f: X \longrightarrow Y$ in Ord let $F(f) = \hat{f}: F(X) \longrightarrow F(Y)$ (see (3.3)).

(4.2) Theorem. F is a functor from the category Ord into the category Fuz.

Proof: follows immediately from the next two lemmas.

(4.3) Lemma. Let $(X,<)$, $(Y,<)$ and $(T,<)$ be linearly ordered spaces and $f: X \longrightarrow Y$, $g: Y \longrightarrow T$ increasing continuous mappings. Then $F(g \circ f) = F(g) \circ F(f)$.

Proof. Denote $h = g \circ f$ and let $z \in F(X)$, $u = f^*(z) \in Z(Y)$, $v = g^*(u) \in Z(T)$, $w = h^*(z) \in Z(T)$. To prove the lemma it suffices to show that $v = w$. From the definition (3.1) it follows that for every $t \in T$

$$w(t) = \begin{cases} \inf\limits_{h(x) \le t} z(x), & \text{if }]\leftarrow, t] \cap h(X) \neq \emptyset \\ 1 & , \text{if }]\leftarrow, t] \cap h(X) = \emptyset , \end{cases}$$

$$v(t) = \begin{cases} \inf\limits_{g(y) \le t} u(y), & \text{if }]\leftarrow, t] \cap g(Y) \neq \emptyset \\ 1 & , \text{if }]\leftarrow, t] \cap g(y) = \emptyset . \end{cases}$$

Moreover, for every $y \in Y$

$$u(y) = \begin{cases} \inf\limits_{f(x) \le y} z(x), & \text{if }]\leftarrow, y] \cap f(X) \neq \emptyset \\ 1 & , \text{if }]\leftarrow, y] \cap f(X) = \emptyset . \end{cases}$$

Fix $t \in T$. The two possible cases are:

a) $]\leftarrow, t] \cap h(X) = \emptyset$. Then $w(t) = 1$. If $]\leftarrow, t] \cap g(Y) = \emptyset$, then $v(t) = 1$, too. Otherwise $]\leftarrow, t] \cap g(Y) \neq \emptyset$ and hence $]\leftarrow, y] \cap f(X) = \emptyset$ for every $y \in g^{-1}]\leftarrow, t]$, and therefore also $v(t) = \inf\limits_{g(y) \le t} u(y) = 1$.

b) $]\leftarrow, t] \cap h(X) \neq \emptyset$. In this case $w(t) = \inf\limits_{h(x) \le t} z(x)$. On the other hand, in this case $]\leftarrow, y] \cap f(X) \neq \emptyset$ for some $y \in g^{-1}]\leftarrow, t]$. Therefore $v(t) = \inf\limits_{g(y) \le t} u(y) = \inf\limits_{g(y) \le t} \inf\limits_{f(x) \le y} z(x) = \inf\limits_{h(x) \le t} z(x)$. Hence in every case $v(t) = w(t)$.

(4.4) *Lemma.* If $i: X \longrightarrow X$ is the identical mapping, then $F(i): F(X) \longrightarrow F(Y)$ is also the identity.

Proof: is obvious.

(4.5) *Proposition.* If $f_1, f_2: X \longrightarrow Y$ are two increasing continuous mappings and $f_1 \neq f_2$, then $\hat{f}_1 \neq \hat{f}_2$.

Proof. Take $x_o \in X$ such that $f_1(x_o) = y_1$, $f_2(x_o) = y_2$ and assume for definiteness that $y_1 < y_2$. Let $z \in Z(X)$ be defined by the equality

$$z(x) = \begin{cases} 1, & \text{if } x < x_0 \\ 0, & \text{if } x \geq x_0 \end{cases},$$

and let $u_1 = f_1^*(z)$, $u_2 = f_2^*(z)$. It is obvious that $u_1(y_2^-) \leq \leq u_1(y_1) \leq z(x_0) = 0$ while $u_2(y_2^-) = \inf_{y < y_2} u_2(y) = = \inf_{y < y_2} \inf_{f_2(x) \leq y} z(x) = 1$. Thus $u_1(y_2^-) \neq u_2(y_2^-)$ and hence $\hat{f}_1 \neq \hat{f}_2$.

Theorem (4.2) and Proposition (4.5) immediately imply the main result of this section:

(4.6) Corollary. $F: \mathrm{Ord} \to \mathrm{Fuz}$ is an embedding functor.

(4.1)° Definition of $F^o: \mathrm{Ord} \to \mathrm{Fuz}^o$. For every object X of Ord let $F^o(X)$ be defined as in (2.1)° and for every morphism $f: X \to Y$ in Ord let $F^o(f) = \hat{f}: F^o(X) \to F^o(Y)$ (see (3.3)°).

(4.6)° Theorem. F^o is an embedding functor from the category Ord into the category Fuz^o.

§ 5. Fuzzy modification of a decreasing continuous mapping. Since the composition of two decreasing mappings is not decreasing except for some special cases, it is impossible to develop the previous theory to the full extent for the case of decreasing mappings. However, there are some ways in which one can partially extend the study of the fuzzy modification of a monotone continuous mapping to the case of a decreasing mapping. One of these ways is sketched below.

Let $(X, <)$ and $(Y, <)$ be two linearly ordered spaces and let $<'$ be the inverse order on Y, i.e. $y_1 <' y_2$ iff $y_2 < y_1$. The pair $(Y, <')$ will be usually abbreviated to Y'.
It is obvious that $u \in Z(Y)$ iff $1 - u \in Z(Y')$.

(5.1) Lemma. Let $u_1, u_2 \in Z(Y)$. Then $u_1 \sim u_2$ in $Z(Y)$ iff

$1 - u_1 \sim 1 - u_2$ in $Z(Y')$.

Proof is obvious.

This lemma ensures the correctness of the following definition:

(5.2) Definition. The equality $\varphi[u] = [1 - u]$ determines a mapping $\varphi : F(Y') \longrightarrow F(Y)$.

(5.3) Lemma. The mapping $\varphi^{-1} : F(Y) \longrightarrow F(Y')$ defined by the equality $\varphi^{-1}[u] = [1 - u]$ is the inverse of φ. Specifically, $\varphi : F(Y') \longrightarrow F(Y)$ is a bijection.

Proof is obvious.

(5.4) Proposition. $\varphi : F(Y') \longrightarrow F(Y)$ is a fuzzy homeomorphism.

Proof. Show directly that φ and φ^{-1} are fuzzy continuous and apply the previous lemma.

A decreasing continuous mapping $f : X \longrightarrow Y$ can be obviously considered as an increasing continuous mapping $f' : X \longrightarrow Y'$ with the same values. Applying (3.3) we obtain a fuzzy continuous mapping $\hat{f}' : F(X) \longrightarrow F(Y')$.

(5.5) Definition. Let $f : X \longrightarrow Y$ be a decreasing continuous mapping. Then its fuzzy modification $\hat{f} : F(X) \longrightarrow F(Y)$ is defined by the equality $\hat{f} = \varphi \circ \hat{f}'$.

(5.6) Theorem. If $f : X \longrightarrow Y$ is a decreasing continuous mapping, then its fuzzy modification $\hat{f} : F(X) \longrightarrow F(Y)$ is fuzzy continuous.

Proof follows immediately from (3.4) and (5.4).

(5.7) Proposition. If $f : X \longrightarrow Y$ is a decreasing homeomorphism, then $\hat{f} : F(X) \longrightarrow F(Y)$ is a fuzzy homeomorphism.

Proof. Use (3.5) and (5.4).

It is quite obvious how to reformulate the results of this section for the case of the category Fuz^o.

§ 6. <u>Fuzzy modification of a linearly ordered connected space</u>. Taking as a basis the fundamental ideas of B. Hutton [7], A. Klein [10] has generalized the construction of fuzzy unit interval in a completely different way than ours. For every connected topological space X he has defined a fuzzy topological space which we shall denote $K^o(X)$ and which has some important properties (see Definition $(6.9)^o$ and Remark (6.15) below). Specifically, the spaces $K^o(I)$, $K^o(]0,1[)$ and $K^o(\mathbb{R})$ are fuzzy homeomorphic with the fuzzy closed unit interval, the fuzzy open unit interval and the fuzzy real line respectively (cf. $(2.2)^o$). The space X is contained as a fuzzy subspace in $K^o(X)$ (cf. $(2.3)^o$).

If both constructions $K^o(X)$ and $F^o(X)$ are suitable and natural generalizations of Hutton's fuzzy unit interval, one could hope that for a linearly ordered connected space X the fuzzy spaces $K^o(X)$ and $F^o(X)$ are to be isomorphic. The aim of this section is to show that this is really the case. We begin with a brief outline of the construction from [10] but in a form appropriately modified for the case when the space is both linearly ordered and connected.

Thus, let X be a linearly ordered connected space and let M(X) denote the set of all monotone mappings (both increasing and decreasing) $z:X \longrightarrow I$ such that $\sup_{x\in X} z(x) = 1$ and $\inf_{x\in X} z(x) = 0$. (Specifically, $Z(X) \subset M(X)$.)

The following two lemmas can be easily proved.

(6.1) <u>Lemma</u> (cf. [10], Proposition 1.1). $z:X \longrightarrow I$ is monotone iff for all $\alpha, \beta \in I$ the set $z^{-1}[\alpha, \beta]$ is connected.

(6.2) Lemma. If $\alpha \in]0,1[$ and $z^{-1}(\alpha) \neq \emptyset$, then $X \setminus z^{-1}(\alpha)$ is disconnected.

(6.3) Definition (cf. [10], def. 1.3). For $z \in M(X)$ and $\alpha \in [0,1[$ let

$$H_\alpha(z) = \begin{cases} \overline{z^{-1}]\alpha,1]} \cap \overline{z^{-1}[0,1-\alpha[}, & \text{if } \alpha < \frac{1}{2} \\ \overline{z^{-1}[0,\alpha]} \cap \overline{z^{-1}[1-\alpha,1]}, & \text{if } \alpha \geq \frac{1}{2}. \end{cases}$$

(6.4) Definition. For $z_1, z_2 \in M(X)$ let $z_1 \approx z_2$ iff $H_\alpha(z_1) = H_\alpha(z_2)$ for every $\alpha \in [0,1[$.

It is obvious that $\approx$ is an equivalence relation on $M(X)$. If $z \in M(X)$ ket $(z) = \{z' : z' \in M(X),\ z' \approx z\}$.

(6.5) Definition. Let $K(X)$ denote the set of all $\approx$-equivalence classes, i.e. $K(X) = M(X)/_\approx$.

Assume that $\alpha \geq \frac{1}{2}$. Since X is connected, there exist $a, b \in X$ such that $\overline{z^{-1}[0,\alpha]} = [a, \rightarrow[$ and $\overline{z^{-1}[1-\alpha,1]} =]\leftarrow, b]$ (see [4]). Moreover, $a \leq b$ in this case (otherwise for $y \in]b,a[$ the inequality $1 - \alpha < z(y) < \alpha$ would imply $\alpha > \frac{1}{2}$). Therefore $H_\alpha(z) = [a,b]$.

Applying similar reasonings, one can easily show that in case $\alpha < \frac{1}{2}$, $\alpha \neq 0$ there exist $a, b \in X$, $a < b$ such that $H_\alpha(z) = [a,b]$, but $H_0(z)$ has one of the following four forms: $[a,b]$, $[a, \rightarrow[$, $]\leftarrow, b]$ or X. Since X is connected, applying [4], pp. 281, 457, we get from the above the next

(6.6) Lemma (cf. [9], Lemma 3.5). If $\alpha \neq 0$, then $H_\alpha(z)$ is compact.

If $\alpha < \frac{1}{2}$ and $\alpha \neq 0$, then the monotoness of z allows to conclude that $]a,b[\subset z^{-1}]\alpha\ \ , 1-\alpha[\subset[a,b]$. Moreover, since X is connected, $]a,b[\ = \mathrm{Int}\ z^{-1}]\alpha\ \ , 1-\alpha[$. Similar inclusions may be written also for $H_0(z)$. For example, if $H_0(z) = [a,\rightarrow[$ then $]a,\rightarrow[\subset z^{-1}]\alpha\ \ , 1-\alpha[\subset[a,\rightarrow[$ and $\mathrm{Int}\ z^{-1}]\alpha\ \ , 1-\alpha[\ = \ ]\alpha,\rightarrow[$. Hence we derive the following two statements.

(6.7) Lemma. If $\alpha < \frac{1}{2}$, then either

$H_\alpha(z) = \overline{\mathrm{Int}\ z^{-1}]\alpha\ , 1-\alpha[}$ or $H_\alpha(z)$ is a singleton.

(6.8) Lemma. If $\alpha < \frac{1}{2}$ and $z_1, z_2 \in M(X)$ then $H_\alpha(z_1) = H_\alpha(z_2)$ implies that $\mathrm{Int}\ z_1^{-1}]\alpha\ , 1-\alpha[\ = \mathrm{Int}\ z_2^{-1}]\alpha, 1-\alpha[$.

From the previous three lemmas one notices that the set K(X) coincides with the set X(I) defined by A. Klein in [10]. Let τ^o be the fuzzy topology on K(X) = X(I) defined exactly as in [10].

$(6.9)^o$ Denotation. The fuzzy topological space $(K(X), \tau^o)$ will be denoted $K^o(X)$.

Our next goal is to establish a natural fuzzy homeomorphism between $F^o(X)$ and $K^o(X)$.

(6.10) Lemma. If $z \in M(X)$, then $z \approx 1 - z$.

Proof is easy (cf. also [10], Lemma 3.3).

(6.11) Corollary. Every class $(z) \in K(X)$ contains a decreasing member $z \in (z)$.

(6.12) Lemma. Let $z_1, z_2 \in Z(X)$. Then $z_1 \sim z_2$ iff $H_\alpha(z_1) = H_\alpha(z_2)$ for all $\alpha \in [0,1[$.

(6.13) Definition. Define the mapping $\varphi : F(X) \longrightarrow K(X)$ by the equality $\varphi[z] = (z)$.

Lemma (6.12), Definition (6.4) and Corollary (6.11) ensure that the mapping φ is one-to-one and onto.

Reasonings quite similar to the ones used in the proof of [10], Theorem 3.4, show that the mapping $\varphi : F^o(X) \to K^o(X)$ is fuzzy continuous and fuzzy open. Now we can sum up the obtained information in the following

(6.14)^o Theorem. Let X be a linearly ordered connected space. Then the mapping $\varphi : F^o(X) \to K^o(X)$ defined by the equality $\varphi[z] = (z)$ is a fuzzy homeomorphism.

(6.15) Remark. In this section as everywhere in the paper the superscript "o" is used to mark those statements and constructions which deal with the category Fuz^o (in contrast with the category Fuz (see (1.5)^o, (1.5))). Since the original construction of A. Klein was fulfilled just for Fuz^o, the exposition of this section is presented in the form of Fuz^o, too. However, quite obvious changes in the text allow to obtain the corresponding analogs for the category Fuz.

Namely, let τ be the weakest fuzzy topology on K(X) which contains τ^o and all constants.

(6.9) Denotation. The fuzzy topological space $(K(X), \tau)$ will be denoted just K(X).

Since the preimage of a constant fuzzy set is the same constant fuzzy set and since preimages preserve suprema and infima of fuzzy sets, from (6.14)^o we can now obtain the following

(6.14) Theorem. Let X be a linearly ordered connected space. Then the mapping $\varphi : F(X) \to K(X)$ defined by the equality $\varphi[z] = (z)$ is a fuzzy homeomorphism.

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(Oblatum 25.1. 1985)