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Claudio H. Morales

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SURJECTIVITY THEOREMS FOR MULTI-VALUED MAPPINGS OF ACCRETIVE TYPE

Claudio H. MORALES

ABSTRACT. Let X be a Banach space and T a m-accretive mapping defined on a subset D of X which takes values in 2^X. Suppose the dual space X^* is uniformly convex and suppose, in addition, T is ϕ-expansive on D(i.e., $\|u-v\| \geq \phi(\|x-y\|)$ for all $x, y \in D$, $u \in T(x)$ and $v \in T(y)$). Then it is shown that T maps D onto X. A number of related surjectivity results are obtained for a more general class of Banach space by assuming, among other conditions, that T is continuous. Also included is an extension of Deimling's domain invariance theorem to multi-valued mappings.

Key words and phrases: m-accretive mapping, surjectivity

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Let X be a (real) Banach space and let $\mathcal{B}(X)$ denote the family of all nonempty, bounded and closed subsets of X supplied with the Hausdorff metric H. Let $J:X\to 2^{X^*}$ be the duality mapping defined by

$$J(x)=\{j\in X^*:\langle x,j\rangle=\|j\|^2=\|x\|^2\}.$$

A mapping $T:D\subset X\to\mathcal{B}(X)$ is said to be <u>strongly accretive</u> if there exists a constant $c\in(0,1)$ such that if $x,y\in D$, $u\in T(x)$, $v\in T(y)$:

$$(1)\qquad \langle u-v,j\rangle\geq c\|x-y\|^2$$

for some $j \in J(x-y)$. This is a well-known class of mappings which has been studied in various contexts by several authors (e.g., [3], [5], [6], [9], [12]). Particularly, we note that any mapping of the form $I-T$, where I is the identity and T a single valued contraction mapping (i.e., a mapping with Lipschitz constant $1-c$) trivially satisfies (1). If the condition (1) holds for $c=0$, then T is said to be accretive and, if in addition the range of $I+rT$ is precisely X for all $r>0$, then T is said to be m-accretive.

Following Kato [8], we may formulate (1) in a more geometric fashion. A mapping T from D to $\mathcal{B}(X)$ is strongly accretive if and only if for some constant $k<1$ and for each $x,y \in D$, $u \in T(x)$, $v \in T(y)$:

(2) $$(\lambda-k)\|x-y\| \leq \|(\lambda-1)(x-y)+u-v\|$$

for all $\lambda>k$; while T is accretive if and only if (2) holds for $k=1$.

The purpose of this paper is to obtain a number of results involving accretive operators which are intimately connected with the theory of ordinary differential equations in Banach spaces. In fact we are able to present new surjectivity theorems for multivalued mappings which are defined in a portion of the Banach space X with no explicit assumption on the continuity of the operator T. Among our results we show that within the framework of spaces X whose dual spaces X^* are uniformly convex, if D is a subset of X and $T:D \to 2^X$ is m-accretive and ϕ-expansive (in the sense described below), then T is surjective. This fact represents a substantial generalization of corollary 3 and Theorem 4 of Kartsatos [7], who assumes that T is a single-valued mapping, and in the first instance that T is defined in the whole space, while in the second T satisfies the assumption where $T-p$ does not attain its infimum on the boundary of an open subset of D for each $p \in X$. Also, in contrast to our approach, Kartsatos derives his results from an existence theory for dif-

Along with the surjectivity problems, we derive some domain invariance theorems which represent extensions of known results to the multi-valued case. Finally, we derive a new theorem concerning the existence of zeros for continuous and ϕ-accretive mappings (in the sense of [7]) under a standard boundary condition.

Throughout this paper we use $\bar{D}$ and ∂D to donote, respectively, the closure and boundary of a subset D of the space X. We also use $|A|$ to denote $\inf\{\|x\|:x\in A\}$, $A\subset X$ and $B(x_0;r)$ to denote the open ball of radius r about x_0.

Let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be a function which is continuous on $\mathbb{R}^+$ with $\phi(0)=0$ and $\phi(r)>0$ for $r>0$. A mapping $T:D\subset X\to 2^X$ is said to be ϕ-expansive on D if for every $x,y\in D$, $u\in T(x)$ and $v\in T(y)$:

$$\|u-v\|\geq\phi(\|x-y\|). \tag{3}$$

Theorem 1. Let X be a Banach space whose dual space X^* is uniformly convex and let D be a subset of X. Suppose $T:D\to 2^X$ is a m-accretive and ϕ-expansive mapping on D for which $\liminf_{r\to\infty}\phi(r)>0$. Then T maps D onto X.

Before proving Theorem 1, we need the following lemma which is an extension of Lemma 2.5 of Kato [8] to multi-valued mappings and include its proof for the sake of completeness.

Lemma. Let X^* be uniformly convex and let $T:D\subset X\to 2^X$ be m-accretive on D. Suppose there exists a sequence $\{x_n\}$ in D such that $x_n\to x\in X$ and a bounded sequence $\{u_n\}$ in X for which $u_n\in T(x_n)$. Then $x\in D$ and a subsequence of $\{u_n\}$ converges weakly to $u\in T(x)$.

Proof. From the fact that X^* is uniformly convex, we may derive that the mapping J is single-valued and uniformly continuous on bounded subsets of X (see Lemma 1.2 of [8]).

For every $y \in D$, the accretiveness of T implies that

$$(4) \qquad \langle v - u_n, J(y - x_n)\rangle \geq 0$$

for all $v \in T(y)$. Since X is reflexive, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ so that $u_{n_k} \rightharpoonup u \in X$ as $k \to \infty$ (" $\rightharpoonup$ " denotes weak convergence). Since $y - x_{n_k} \to y - x$ as $k \to \infty$, $J(y - x_{n_k}) \to J(y - x)$ and by (4) we obtain

$$\langle v - u, J(y - x)\rangle \geq 0.$$

By choosing $\alpha = 1$ in Lemma 1.1 of [8], we yield

$$(5) \qquad \|y - x\| \leq \|y - x + v - u\|.$$

Since the mapping $(I + T)^{-1}$ is single-valued and defined from X onto D, we select $y \in D$ for which $x + u \in (I + T)(y)$ and for a suitable $v \in T(y)$ we have $x + u = y + v$, which implies with (5) that $x = y$ and $u \in T(x)$.

Proof of Theorem 1. Let $0 < \eta < \liminf_{r \to \infty} \phi(r)$ and let $u_0 \in X$. Now, we choose a bounded neighborhood N such that $u_0 \in N \subset X$ and $\|u - u_0\| \leq \eta/2$ for all $u \in N$. Let $x \in T^{-1}(N)$. Then there exists $u \in N$ such that $u \in T(x)$. By choosing $x_0 \in T^{-1}(u_0)$, (3) implies

$$\phi(\|x - x_0\|) \leq \|u - u_0\| \leq \eta/2.$$

Therefore the assumptions on ϕ imply that the set $\{\|x - x_0\| : x \in T^{-1}(N)\}$ is bounded, i.e., $T^{-1}(N)$ is bounded. On the other hand, the family $\{T + \lambda I : \lambda > 0\}$ converges uniformly in the sense of definition 5.3 of Browder [4]. Therefore, Theorem 5.1 of [4], implies that $R(T)$, the range of T is dense in X. To complete the proof, observe that if $u_n \in R(T)$ so that $u_n \to u$, then $u_n \in T(x_n)$ for some $x_n \in D$. The fact that T is ϕ-expansive implies $x_n \to x \in X$. Therefore, by the previous lemma, $x \in D$ and $u \in T(x)$, proving $R(T)$ is closed.

Next, we prove a surjectivity theorem for a general Banach space under the restriction that the operator T has to be defined in the whole space. We first need the following result which appears to be new in the context of multi-valued mappings.

__Theorem 2__. Let X be a Banach space and let $T:X \to B(X)$ be a continuous (relative to H) and accretive mapping. Then T is m-accretive.

__Proof__. Let $z \in X$ and $c \in (0,1)$. Define the mapping $T_z:X \to B(X)$ by $T_z(x) = cx + T(z)-z$. Then T_z is, clearly, strongly accretive on X (with $k = 1 - c$ in (2)). We shall now show that the set

$$E(z) = \{x \in X: tx \in T_z(x) \text{ for some } t < 0\}$$

is bounded. Let $tx \in T_z(x)$ for some $t < 0$ and select $u \in T_z(x)$ such that $tx = u$. Then by choosing $\lambda = 1 - t$ in (2) we have

$$\begin{aligned}(c - t)\|x\| &\le \|-tx + u - v\| \\ &= \|v\|\end{aligned}$$

for all $v \in T_z(0)$. Since $T_z(0)$ is bounded and $t < 0$, it follows that

$$\|x\| \le |T_z(0)|/c.$$

Therefore $E(z)$ is bounded. Now, we choose $r > 0$ so that the closure of $E(z)$ is contained in the open ball $B(0;r)$. This means that the mapping T_z satisfies the following condition:

$$tx \notin T_z(x) \quad \text{for } x \in \partial B(0;r) \text{ and } t < 0.$$

Therefore, Theorem 1 of [14], implies the existence of $x \in D$ such that

$0 \in T_z(x)$ i.e., $z \in (cI + T)(x)$.

Theorem 3. *Let X be a Banach space and let $T:X \to B(X)$ be a continuous and accretive mapping, which is also ϕ-expansive on X for which $\liminf_{r \to \infty} \phi(r) > 0$. Then T maps X onto X.*

Proof. Following the argument given in the proof of Theorem 1, it can be shown that for each $u_0 \in X$ there exists a neighborhood N of u_0 such that $T^{-1}(N)$ is bounded, and since T is m-accretive, by Theorem 2, we can once again apply Theorem 5.1 of Browder [4] to conclude that $R(T)$ is dense in X.

We now prove that $R(T)$ is a closed set. Let $u_n \in R(T)$ such that $u_n \to u$ as $n \to \infty$. Choose $x_n \in X$ for which $u_n \in T(x_n)$. Since T is ϕ-expansive we have

$$\|u_n - u_m\| \geq (\|x_n - x_m\|),$$

which implies that $\{x_n\}$ is a Cauchy sequence and thus $x_n \to x$ as $n \to \infty$. Since T is continuous, $\lim_{n \to \infty} H(T(x_n), T(x)) = 0$ and therefore Lemma 2 of [i] (see also [14]) implies that $u \in T(x)$. Hence $R(T)$ is closed, proving that $R(T) = X$.

Corollary 1. *Let X be a Banach space and let $T:X \to B(X)$ be a continuous and strongly accretive mapping. Then T maps X onto X.*

Proof. It is easily seen that (1) implies that T is ϕ-expansive on X, and hence Theorem 3 completes the proof.

In the following two results, we restricted the Banach space X, while we relax the assumptions of boundedness and closedness of $T(x)$ for each x.

we first begin with a domain invariance theorem for m-accretive mappings.

Theorem 4. Let X be a Banach space for which each nonempty bounded closed convex subset has the fixed point property for nonexpansive self-mappings. Suppose $T:D \subset X \to 2^X$ is m-accretive and ϕ-expansive on D. Then $T(G)$ is open whenever $G \subset D$ is open in X.

Proof. Let $B(x_0;r) \subset D$ for some $x_0 \in D$ and $r > 0$. Select $v_0 \in T(x_0)$ and define $\tilde{T}:D-x_0 \to 2^X$ by $\tilde{T}(x) = T(x + x_0) - v_0$. Then $|\tilde{T}(0)| = 0$ and if $x \in \partial B(0;r)$,

$$\begin{aligned} 0 = |\tilde{T}(0)| &< \phi(r) \\ &= \phi(\|x + x_0 - x_0\|) \\ &\le \|u - v\| \end{aligned}$$

for all $u \in T(x + x_0)$ and $v \in T(x_0)$. In particular, if we choose $v = v_0$ we have

$$\phi(r) \le \|u - v_0\| \quad \text{for all} \quad u \in T(x + x_0).$$

Therefore,

$$|\tilde{T}(0)| < \phi\ (n) \le |\tilde{T}(x)|\ .$$

Since $\tilde{T}$ is also m-accretive, we may apply Theorem 2 of [13] to conclude that $B(0;\phi(r)) \subset R(\tilde{T})$, i.e, $B(v_0;\phi(r)) \subset T(B(x_0;r))$. The openness of $T(G)$ is an immediate consequence of the latter conclusion.

Theorem 4 represents the multi-valued version of Theorem 3 of the author [13].

Theorem 5. *Lex X be as in Theorem 4 and let D be an unbounded subset of X for which $T:D \to 2^X$ is m-accretive and ϕ-expansive on D with $\phi(r) \to \infty$. Then T maps D onto X.*

Proof. Let $B(x_0;r)$ be a closed ball for some fixed $x_0 \in D$ and $r > 0$. As before, we choose $v_0 \in T(x_0)$ and we define $\tilde{T}:D - x_0 \to 2^X$ by $\tilde{T}(x) = T(x + x_0) - v_0$. Then

$$|\tilde{T}(0)| < \phi(r) \leq |\tilde{T}(x)|$$

for all $x \in \partial B(0;r)$. Therefore, Theorem 2 of [13] implies that $B(v_0;\phi(r)) \subset T(D)$ for each $r > 0$. Since $\phi(r) \to \infty$ as $r \to \infty$ and v_0 is a fixed element of X, $T(D) = X$.

Theorem 6 below improves Theorem 10.5 of Browder [4], who assumes (for single value T) that T is locally uniformly continuous.

Theorem 6. *Let X be a Banach space and let $T:X \to B(X)$ be a continuous accretive mapping on X. Suppose T^{-1} is locally bounded, i.e., each point x_0 of X has a neighborhood N such that $T^{-1}(N)$ is bounded in X. Then the range of T is dense in X.*

We first show a proposition that will be used in the proof of Theorem 6.

Proposition. *Let X be a Banach space, D an open subset of X and $T:\bar{D} \to B(X)$ a continuous and strongly accretive mapping on $\bar{D}$. Suppose there exist $x_0 \in D$ and $r > 0$ such that*

(6) $$|T(x_0)| < r \leq |T(x)| \quad \textit{for} \quad x \in \partial D.$$

Then $B(0;r) \subset R(T)$.

<u>Proof</u>. Without loss of generality we may assume that $x_0 = 0$ in (6). We first consider a $z \in B(0;(r - |T(0)|)/2$ and define $T_z:\bar{D} \to B(X)$ by $T_z(x) = T(x) - z$. We shall show that $tx \notin T_z(x)$ for $x \in \partial D$ and $t < 0$. To see this, suppose $tx \in T_z(x)$ where $x \in \partial D$ and $t < 0$. By using $\lambda = 1 - t$ in (2) we have

$$\begin{aligned}(1 - t - k)\|x\| &\leq |-tx + T_z(x) - T_z(0)| \\ &\leq |T_z(0)|.\end{aligned}$$

Since $1 - k > 0$, it follows that $\|tx\| < |T_z(0)|$. On the other hand, since $x \in \partial D$ and $\|z\| < (r - |T(0)|)/2$, (6) implies

$$\begin{aligned}|T_z(0)| &\leq |T(0)| + \|z\| \\ &< \|tx + z\| - \|z\| \\ &\leq \|tx\|,\end{aligned}$$

which is a contradiction. Hence, by Theorem 1 of [14], we derive that $z \in T(D)$. To complete the proof, we fix $\|z\| < r$ and let

$$E = \{t \in [0,1]: tz \in R(T)\}$$

Since by the above argument $E \neq \emptyset$, we may follow the proof of Theorem 3 of Kirk-Schöneberg [10] to show that $1 \in E$, i.e., $z \in R(T)$ (see also [13], Theorem 2).

<u>Proof of Theorem 6</u>. Let $w_0 \in \overline{R(T)}$. Then the assumption on T^{-1} implies the existence of an open ball $\mathcal{B}(w_0;\delta)$ so that $T^{-1}(B(w_0;\delta))$ is bounded. Let $v_0 \in R(T)$ such that $\|w_0 - v_0\| < \delta/5$. Then the set

$$F = \{x \in X\colon\ y \in T(x) \ \text{ for some } \ y \in \mathcal{B}(v_0;\ 3\delta/4)\}$$

is bounded. Thus for r sufficiently large F is contained in the open ball $B = B(0;r)$ with $F \cap \partial B = \emptyset$. Since $v_0 \in R(T)$, there exists $x_0 \in F$ such that $v_0 \in T(x_0)$ with

(7) $$0 = |T(x_0) - v_0| < 3\delta/4 \le |T(x) - v_0| \quad \text{for } \ x \in \partial B.$$

We now select $n_0 \in \mathbb{N}$ so that $3r < n_0\delta$ and define a mapping $T_n\colon \bar{B} \to \mathcal{B}(X)$ by $T_n(x) = T(x) - v_0 + (1/n)x$. Then for $n \ge n_0$ and $x \in \partial B$,

$$|T_n(x_0)| \le (1/n)\|x_0\| < 3\delta/4 - r/n_0 = \eta$$

and thus (7) yields

$$|T_n(x_0)| < \eta \le |T_n(x)| \qquad \text{for } \ x \in \partial B$$

which implies, by the previous proposition, that $B(0;\eta) \subset R(T_n)$. This means, if $\|z\| < \eta$ and $n \ge n_0$, there exists $x_n \in T(x_n)$ for which $z = u_n - v_0 + (1/n)x_n$. Therefore $u_n \to z + v_0$ as $n \to \infty$, implying $B(v_0;\eta) \subset \overline{R(T)}$ and since $\eta = 3\delta/4 - r/n_0$, $B(w_0;\delta/5) \subset \overline{R(T)}$. Hence $\overline{R(T)}$ is open in X and thus $R(T)$ is dense in X.

Now, we derive a corollary which is an extension of the density portion of Theorem 3 of Kirk-Schöneberg [9] to multi-valued mapping.

<u>Corollary 2</u>. *Let* X *be a Banach space and* $T:X \to B(X)$ *a continuous accretive mapping such that* $|T(x)| \to \infty$ *as* $\|x\| \to \infty$. *Then the range of* T *is dense in* X.

<u>Proof</u>. Let $w \in X$ and choose δ sufficiently large such that the set

$$E = \{x \in X: \|y\| \le \delta + \|w\| \text{ for some } y \in T(x)\}$$

is nonempty and the fact that $|T(x)| \to \infty$ as $\|x\| \to \infty$, implies E is bounded. Since $T^{-1}(B(w;\delta)) \subset E$, Theorem 6 completes the proof.

A mapping $T:D \subset X \to B(X)$ is said to be closed if $T(C)$ is closed whenever C is closed in D. We also say that T is one-to-one if for every $x,y \in D$ such that $x \neq y$, then $T(x) \cap T(y) = \emptyset$. The closedness (or one-to-oneness) of T holds locally if each $x \in D$ has a neighborhood N such that the restriction of T to N is globally closed (or globally one-to one). Similarly, T is said to be locally accretive if for each $x \in D$ there exists a neighborhood N so that the restriction of T to N is globally accretive.

Our next result represents an extension of the domain invariance theorem of Deimling [6] to the multi-valued case, by following the formulation of Schöneberg [16].

<u>Theorem 7.</u> *Let* X *be a Banach space and* D *an open subset of* X. *Suppose* $T:D \to B(X)$ *is a continuous, locally closed, locally one-to-one and locally accretive mapping. Then* $T(D)$ *is open.*

<u>Proof</u>. Let $x_0 \in D$ and $y_0 \in T(x_0)$. Since T is locally closed, locally one-to-one and locally accretive, there exists a closed ball $B = \bar{B}(x_0;r)$ where T is globally accretive, closed and one-to-one. Then the number

$$\delta = \inf\{|T(x) - y_0| : \|x - x_0\| = r\} > 0.$$

Let $\eta > 0$ so that $\eta(1 + r) < \delta$ and for $0 < c < \eta$ define the mapping $h_t : B \to \mathcal{B}(X)$ by $h_t(x) = c(x - x_0) + t(y_0 - y) - y_0 + T(x)$ for $t \in [0,1]$ and $y \in B(y_0;\eta)$, and also define the set

$$M_c = \{t \in [0,1] : 0 \in h_t(x) \text{ for some } x \in B\}.$$

It is clear that for each $c > 0$, M_c is non-empty $(0 \in M_c)$. We shall now show that $\sup M_c = 1$. To see this, let t_c $\quad$ t_c and let $\{t_n\}$ be a sequence of M_c for which $t_n \to t_c$ as $n \to \infty$. Then, for each n, there exists $x_n \in B$ so that $0 \in h_{t_n}(x_n)$. This means, we may select $u_n \in T(x_n)$ so that $c(x_n - x_0) + t_n(y_0 - y) - y_0 + u_n = 0$. Since the mapping h_t is, clearly, strongly accretive on B, we can conclude that

$$\begin{aligned} c\|x_n - x_m\| &\le |h_{t_c}(x_n) - h_{t_c}(x_m)| \\ &\le \|c(x_n - x_m) + u_n - u_m\| \\ &= |t_n - t_m|\|y_0 - y\|. \end{aligned}$$

Therefore $\{x_n\}$ converges, to say to $\bar{x} \in B$, and hence $u_n \to u \in X$. Since T is continuous relative to the Hausdorff metric, Lemma 3 of [14] implies that $0 \in h_{t_c}(\bar{x})$. Now, using the assumptions on c and y we have

$$\begin{aligned}\|u - y_0\| &\le t_c\|y_0 - y\| + c\|\bar{x} - x_0\| \\ &\le \eta + cr \\ &< \delta,\end{aligned}$$

implying that $\|\bar{x} - x_0\| < r$. It follows that $t_c \in M_c$.

Suppose now that $t_c < 1$. Since the point $\bar{x}$ is the unique zero of h_{t_c} in B, there exists a closed ball $B_1 \subset B$ centered at $\bar{x}$ such that

$$\rho = \inf\{|h_{t_c}(x)|: x \in \partial B_1\} > 0.$$

Hence for some $t > t_c$,

$$|h_t(x)| > r/2 \quad \text{for all} \quad x \in \partial B_1,$$

and

$$|h_t(\bar{x})| \le r/2.$$

Then, by Theorem 3.2 of [9] there exists $x \in B_1$ such that $0 \in h_t(\bar{x})$ which contradicts the fact that t_c is the supremum of M_c. Hence $0 \in c(x - x_0) - y + T(x)$ for each $c \in (0,\eta)$. If $c_n \to 0$, then there exist $x_n \in B$ and $u_n \in T(x_n)$ such that $c_n(x_n - x_0) + u_n = y$ and thus $u_n \to y$ as $n \to \infty$. It follows from the closedness of T on B that $y \in T(x)$ for some $x \in B$, which implies that $B(y_0;\eta) \subset T(D)$. Hence $T(D)$ is open in X.

An operator $T:D \subset X \to \mathcal{B}(X)$ is said to be $\underline{\phi\text{-accretive}}$ if for each $x,y \in D$ there exists $j \in J(x - y)$ satisfying

$$(7) \qquad \langle u - v, j\rangle \geq \phi(\|x - y\|)\|x - y\|$$

for $u \in T(x)$ and $v \in T(y)$, where ϕ is a mapping from $\mathbb{R}^+$ into $\mathbb{R}^+$ which is continuous on $\mathbb{R}^+$ with $\phi(0) = 0$ and $\phi(r) > 0$ for $r > 0$. We also say that T is locally ϕ-accretive on D if each $x \in D$ has a neighborhood N such that the restriction of T to N is globally ϕ-accretive.

We should mention that the notion of ϕ-accretive mappings formulated by Browder [2] is not related to the formulation given here. Nevertheless, Ray and Walker [15] discuss, to some extent, a more related version of this concept. In fact, they show a domain invariance theorem (see Theorem 4.1) which can be derived directly from Theorem 7 of this paper.

<u>*Corollary 3.*</u> *Let X and D as in Theorem 7 and let $T:D \to \mathcal{B}(X)$ be continuous and locally ϕ-accertive on D with $\liminf_{r \to \infty} \phi(r) > 0$. Then $T(D)$ is open.*

<u>Proof</u>. Since T is clearly locally one-to-one and locally accretive, it remains to show that T is locally closed. To see this, let N be a neighborhood of $x \in D$ such that T is ϕ-accretive on N and let C be a closed subset of X contained in N. Since ϕ-accretiveness implies ϕ-expansiveness, we follow the argument given in the proof of Theorem 3 in order to conclude that $T(C)$ is closed. This means T is closed on N and thus Theorem 7 completes the proof.

Finally, we prove a new theorem for ϕ-accretive mapping satisfying the well-known boundary condition (8) below.

Theorem 8. *Let X be a Banach space and D an open subset of X. Suppose $T:\bar{D} \to B(X)$ is a continuous and ϕ-accretive mapping (with $\phi(t) \to \infty$ as $t \to \infty$) which satisfies for some $z \in D$*

$$(8) \qquad t(x - z) \notin T(x) \quad \text{for} \quad x \in \partial D \quad \text{and} \quad t < 0.$$

Then $0 \in R(T)$.

Proof. By translating T and D, we may take $z = 0$ in (8). We begin by showing that the set

$$E = \{x \in D : tx \in T(x) \quad \text{for some} \quad t < 0\}$$

is bounded. Let $x \in E$, $j \in J(x)$ and let fix $u \in T(0)$. Then there exists a $t < 0$ so that

$$\begin{aligned} \|x\|\phi(\|x\|) &\le \langle tx - u, j\rangle \\ &\le t\|x\|^2 + \|u\|\|x\|. \end{aligned}$$

Since $t < 0$,

$$\phi(\|x\|) \le \|u\|$$

and the assumptions on ϕ conclude the boundedness of E. Because of this latter fact, there is no loss in generality in assuming D is bounded.

Following the author's argument given in Theorem 1 of [14], we claim there exist $x \in D$ and $t \in (0,1)$ so that $0 \in h_t(x)$, where the mapping h_t from $\bar{D}$ into $B(X)$ is defined by $h_t(x) = (1 - t)x + tT(x)$ for each $t \in [0,1]$.

Then the set

$$M = \{t \in [0,1]: 0 \in h_t(x) \quad \text{for some} \quad x \in D\}$$

is nonempty with $\sup M > 0$. We shall now show that $t_0 = \sup M$ belongs to M. Let $\{t_n\}$ be a sequence of M with $t_n \to t_0$ as $n \to \infty$. Then, for each n, there exists $x_n \in D$ so that $0 \in h_{t_n}(x_n)$. This means, we may select $u_n \in T(x_n)$ for which $(1 - t_n)x_n + t_n u_n = 0$. By ϕ-accretiveness of T, there exists $j \in J(x_n - x_m)$ such that

$$\begin{aligned}
\phi(\|x_n - x_m\|)\|x_n - x_m\| &\le \langle u_n - u_m, j\rangle \\
&\le \langle (1 - t_n^{-1})x_n - (1 - t_n^{-1})x_m, j\rangle \\
&\le \langle (1 - t_n^{-1})(x_n - x_m) + (t_m^{-1} - t_n^{-1})x_m, j\rangle \\
&\le (1 - t_n^{-1})\|x_n - x_m\|^2 + |t_m^{-1} - t_n^{-1}|\|x_m\|\|x_n - x_m\|.
\end{aligned}$$

Since $1 - t_n^{-1} \le 0$ and $\{x_n\}$ is bounded,

$$\phi(\|x_n - x_m\|) \to 0 \quad \text{as} \quad n, m \to \infty$$

and thus $\{x_n\}$ is a Cauchy sequence. Hence $x_n \to x$ and $u_n \to u$ for some $x \in \bar{D}$ and $u \in X$. The continuity of T implies that $0 \in h_{t_0}(x)$ and by (8), $x \in D$. Therefore $t_0 \in M$. In order to show that $t_0 = 1$, we may invoke details given in the proof of Theorem 1 of [14].

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Department of Mathematics, University of Alabama in Huntsville, Huntsville, Alabama 3589, U.S.A.

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