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ARMA MODELS WITH NONSTATIONARY WHITE NOISE

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Abstract: The stationarity of ARMA (p,q) processes is investigated when the corresponding white noise is nonstationary with a general covariance structure. The considered ARMA processes are generated from initial random variables $X_1,\dots,X_{\max(p,q)}$ (i.e. they start in a given time point denoted as t = 1) and in this framework the processes $X_1,\dots,X_T$ of the finite length and the processes $X_1,X_2,\dots$ of the infinite length are distinguished. In the latter case with the infinite length the paper confines itself to ARMA processes with so called "almost white noise".

Key words: Nonstationary white noise, ARMA process, stationarity, almost white noise.

Classification: 62M10, 60G10, 60G20

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1. Introduction. The linear models with nonstationary white noise have been studied in several works recently. E.G. Niemi [5] dealt with stationarity and some statistical properties of ARMA processes in which the white noise $\{\varepsilon_t\}$ could have nonconstant bounded variances (i.e. $E\,\varepsilon_t = 0$, $0 < m \leq \operatorname{var}\,\varepsilon_t \leq M$ for some constants m and M, $E\,\varepsilon_t\varepsilon_u = 0$ for $t \neq u$). Statistical treatment of AR processes of this type is suggested in [8].

If the nonstationary zero mean white noise may have a general covariance structure $E\,\varepsilon_t\varepsilon_u$, the situation is more complicated, of course, and many open problems appear in this framework. E.g. it is interesting to investigate under which conditions the linear process using such nonstationary white noise is stationary. The following example shows that the existence of a stationary process of this type is possible.

<u>Example</u>. Let ε be a random variable with $E\,\varepsilon = 0$ and $\operatorname{var} \varepsilon = \sigma^2 > 0$. Let us define $\varepsilon_{3t} = \varepsilon$, $\varepsilon_{3t+1} = \varepsilon$, $\varepsilon_{3t+2} = -\varepsilon$ for all t. The process $\{\varepsilon_t\}$ has the nonstationary covariance structure since $E\,\varepsilon_{3t}\varepsilon_{2t+1} = \sigma^2 \neq E\,\varepsilon_{3t+1}\varepsilon_{3t+2} = -\sigma^2$. On the other hand, MA(3) process $\{X_t\}$ defined as $X_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$ is stationary since $X_t = \varepsilon$ for all t.

In this paper, the stationarity of such processes is investigated which are generated from initial random variables $X_1, \ldots, X_{\max(p,q)}$. Moreover, they can have a finite length (see Section 3) or they can be infinite (see Section 4). In the latter case with the infinite length we confine the general nonstationary white noise to so called almost white noise introduced in [7]. However, at first it is necessary to derive the conditions of stationarity of the process ARMA generated from initial random variables in the classical case with a stationary white noise. It is done in Section 2 generalizing Anděl s results [1] derived for the autoregressive case. The stationarity always means the weak stationarity in this paper.

2. <u>ARMA processes generated from initial random variables and stationary white noise</u>. Let $X_1, \ldots, X_r$ ($r = \max(p,q)$) be random variables with zero mean values and a variance matrix V and $\varepsilon_{s+1}, \ldots, \varepsilon_T$ ($s = r - q$) be random variables with zero mean values. Let random variables $X_{r+1}, \ldots, X_T$ be defined by means of the formula

(1) $X_t = a_1 X_{t-1} + \ldots + a_p X_{t-p} + \varepsilon_t + b_1 \varepsilon_{t-1} + \ldots + b_q \varepsilon_{t-q}$,
$r + 1 \leq t \leq T$,

where $a_1,\dots,a_p$, $b_1,\dots,b_q$ are given numbers. Moreover, let the following assumptions be fulfilled:

(2) $\operatorname{var} \varepsilon_t = \sigma^2 > 0,\ s+1 \le t \le T,$

(3) $\operatorname{cov}(\varepsilon_t, \varepsilon_u) = 0,\ s+1 \le t < u \le T$

(4) $\operatorname{cov}(X_t, \varepsilon_u) = 0,\ 1 \le t \le r,\ s+1 \le u \le T,\ t < u,$

(5) $\operatorname{cov}(X_t, \varepsilon_u) = d_{t-u}^{(t-u)} \sigma^2,\ s+1 \le u \le t \le r,$

where $d_j^{(k)}$, $k \ge 0$, $0 \le j \le k+q$, are constants uniquely determined by the numbers $a_1,\dots,a_p$, $b_1,\dots,b_q$ such that

(6) $d_0^{(k)} = 1,\ k \ge 0,$

$d_j^{(0)} = b_j,\ 1 \le j \le q,$

$$X_{t+k+1} = \varepsilon_{t+k+1} + d_1^{(k)} \varepsilon_{t+k} + \dots + d_{k+q}^{(k)} \varepsilon_{t+1-q} + c_1^{(k)} X_t + c_2^{(k)} X_{t-2} + \dots + c_p^{(k)} X_{t+1-p},\ k \ge 0$$

($c_i^{(k)}$, $k \ge 0$, $1 \le i \le p$ are other constants also determined uniquely by $a_1,\dots,a_p$, $b_1,\dots,b_q$). The system (1)-(5) can be considered as the system of prescriptions which one uses generating the given process. Although it may look complicated it has simple forms in special cases (see Remark 2).

<u>Remark 1</u>. The constants $c_i^{(k)}$ and $d_j^{(k)}$ can be calculated recursively from (1). It is not difficult to show that

$$\begin{aligned}
c_i^{(0)} &= a_i,\ 1 \le i \le p,\\
c_i^{(k)} &= c_{i+1}^{(k-1)} + c_1^{(k-1)} a_i,\ 1 \le i \le p-1,\\
c_p^{(k)} &= c_1^{(k-1)} a_p,\\
d_j^{(k)} &= d_j^{(k-1)},\ 0 \le j \le k-1,\\
d_k^{(k)} &= d_k^{(k-1)} + c_1^{(k-1)},\\
d_j^{(k)} &= d_j^{(k-1)} + c_1^{(k-1)} b_{j-k},\ k+1 \le j \le k+q-1,\\
d_{k+q}^{(k)} &= c_1^{(k-1)} b_q.
\end{aligned}$$

If $a_p \ne 0$ and $b_q \ne 0$ then the previous sequence $X_1,\dots,X_T$ forms so called process ARMA(p,q) of the finite length generated from the initial random variables $X_1,\dots,X_r$ and the stationary white noise.

Remark 2. Let us consider some special cases of the previous model:

(i) AR(p): the variables $X_1,\dots,X_p$, $\varepsilon_{p+1},\dots,\varepsilon_T$ are given fulfilling (2)-(4) (the assumption (5) has no sense in this case) so that the situation is equivalent to the one considered in [1] for the autoregressive case;

(ii) MA(q): the variables $X_1,\dots,X_q$, $\varepsilon_1,\dots,\varepsilon_T$ are given fulfilling (2)-(4) and

$\mathrm{cov}(X_t, \varepsilon_t) = \sigma^2,$

$\mathrm{cov}(X_t, \varepsilon_u) = b_{t-u}\sigma^2, \quad 1 \le u < t \le q;$

(iii) ARMA(1,1): the variables X_1, $\varepsilon_1,\dots,\varepsilon_T$ are given fulfilling (2)-(4) and

$\mathrm{cov}(X_1, \varepsilon_1) = \sigma^2.$

One can prove the following extension of (4) and (5):

Lemma 1. It holds

(7) $\mathrm{cov}(X_t, \varepsilon_u) = 0, \quad 1 \le t \le T, \ s+1 \le u \le T, \ t < u,$

(8) $\mathrm{cov}(X_t, \varepsilon_u) = d_{t-u}^{(t-u)}\sigma^2, \quad 1 \le t \le T, \ s+1 \le u \le T, \ t \ge u.$

Proof. The formula (7) is obvious. As the formula (8) is concerned, it is obvious for $u > r$ since then one can write

(9) $$X_t = \varepsilon_t + d_1^{(t-u)}\varepsilon_{t-1} + \dots + d_{t+q-u}^{(t-u)}\varepsilon_{u-q} + c_1^{(t-u)}X_{u-1} + \dots + c_p^{(t-u)}X_{u-p}.$$

Generally (8) can be proved by means of the induction with respect to t. For $t \le r$ (8) follows directly from the assumption (5). Let (8) hold for some $t \ge r$. We shall show that then it holds also for t+1.

If $r < u \leq t+1$ then $\mathrm{cov}(X_{t+1}, \varepsilon_u) = d_{t+1-u}^{(t+1-u)} \sigma^2$ according to the previous discussion. Therefore let $s+1 \leq u \leq r$. Then according to the induction assumption

$$\begin{aligned}\mathrm{cov}(X_{t+1}, \varepsilon_u) &= \mathrm{cov}(\varepsilon_{t+1}, \varepsilon_u) + b_1 \mathrm{cov}(\varepsilon_t, \varepsilon_u) + \ldots + \\ &+ b_q \mathrm{cov}(\varepsilon_{t+1-q}, \varepsilon_u) + a_1 \mathrm{cov}(X_t, \varepsilon_u) + \ldots + a_p \mathrm{cov}(X_{t+1-p}, \varepsilon_u) \\ &= \mathrm{cov}(\varepsilon_{t+1}, \varepsilon_u) + b_1 \mathrm{cov}(\varepsilon_t, \varepsilon_u) + \ldots + b_q \mathrm{cov}(\varepsilon_{t+1-q}, \varepsilon_u) + \\ &+ a_1 d_{t-u}^{(t-u)} \sigma^2 + \ldots + a_p d_{t+1-p-u}^{(t+1-p-u)} \sigma^2,\end{aligned}$$

where possibly $d_k^{(k)} = 0$ for $k < 0$. Thanks to the definition of the coefficients $d_j^{(k)}$ and to the properties of ε_t we can further write

$$\begin{aligned}\mathrm{cov}(X_{t+1}, \varepsilon_u) &= \mathrm{cov}(\varepsilon_{t+r+2-u}, \varepsilon_{r+1}) + b_1 \mathrm{cov}(\varepsilon_{t+r+1-u}, \varepsilon_{r+1}) + \ldots \\ &\ldots + b_q \mathrm{cov}(\varepsilon_{t+r+2-q-u}, \varepsilon_{r+1}) + a_1 \mathrm{cov}(X_{t+r+1-u}, \varepsilon_{r+1}) + \ldots + \\ &+ a_p \mathrm{cov}(X_{t+r+2-p-u}, \varepsilon_{r+1}) \\ &= \mathrm{cov}(X_{t+r+2-u}, \varepsilon_{r+1}) = d_{t+1-u}^{(t+1-u)} \sigma^2,\end{aligned}$$

where the last equality follows from the discussion in the beginning of the proof (if $t+r+2-u > T$ then we can imagine that we have infinite sequence $\varepsilon_{s+1}, \ldots, \varepsilon_T, \varepsilon_{T+1}, \ldots$ in our disposal).

Lemma 2. The sequence $X_1, \ldots, X_T$ is stationary if and only if

(10) $\mathrm{var}(X_1, \ldots, X_r) = \mathrm{var}(X_2, \ldots, X_{r+1})$.

Proof. Let (10) hold. We shall show by means of the induction that then

(11) $\mathrm{var}(X_1, \ldots, X_h) = \mathrm{var}(X_2, \ldots, X_{h+1}), \quad r \leq h \leq T-1$

(hence the stationarity will follow for $h = T-1$). The case $h = r$ corresponds directly to (10). Let (11) hold for some $r \leq h \leq T-2$. Then (11) will be proved for $h+1$ if we show that

(12) $\mathrm{cov}(X_{h+1}, X_j) = \mathrm{cov}(X_{h+2}, X_{j+1}), \quad 1 \leq j \leq h+1.$

We have due to (1)

$$\operatorname{cov}(X_{k+1},X_j)=a_1\operatorname{cov}(X_k,X_j)+\ldots+a_p\operatorname{cov}(X_{k+1-p},X_j)+$$
$$+\operatorname{cov}(\varepsilon_{h+1},X_j)+b_1\operatorname{cov}(\varepsilon_h,X_j)+\ldots+b_q\operatorname{cov}(\varepsilon_{h+1-q},X_j),$$
$$\operatorname{cov}(X_{k+2},X_{j+1})=a_1\operatorname{cov}(X_{k+1},X_{j+1})+\ldots+a_p\operatorname{cov}(X_{k+2-p},X_{j+1})+$$
$$+\operatorname{cov}(\varepsilon_{h+2},X_{j+1})+b_1\operatorname{cov}(\varepsilon_{h+1},X_{j+1})+\ldots+b_q\operatorname{cov}(\varepsilon_{h+2-q},X_{j+1}).$$

First we shall prove (12) for $1\le j\le h$. According to Lemma 1 it holds

(13) $\operatorname{cov}(\varepsilon_k,X_j)=\operatorname{cov}(\varepsilon_{k+1},X_{j+1}),\ h+1-q\le k\le h+1,\ 1\le j\le h.$

Further it holds directly according to the induction assumption

(14) $\operatorname{cov}(X_k,X_j)=\operatorname{cov}(X_{k+1},X_{j+1}),\ h+1-p\le k\le h,\ 1\le j\le h.$

From (13) and (14) it follows (12) for $1\le j\le h$. For $j=h+1$ we can make use of the fact that

(15) $\operatorname{cov}(\varepsilon_k,X_{h+1})=\operatorname{cov}(\varepsilon_k,X_{h+2}),\ h+1-q\le k\le h+1$

(see again Lemma 1) and

(16) $\operatorname{cov}(X_k,X_{h+1})=\operatorname{cov}(X_{k+1},X_{h+2}),\ h+1-p\le k\le h$

(see (12) proved for $1\le j\le h$). From (15) and (16) it follows (12) for $j=h+1$. It concludes the proof since the inverse implication is obvious.

Let us introduce the following three matrices of the type $r\times r$

$$M=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ 0\ldots 0 & a_p & a_{p-1} & \ldots & a_2 & a_1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \sigma^2(1+b_1^2+\dots+b_q^2) \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & \sigma^2 b_q \\ 0 & \dots & 0 & \sigma^2(b_{q-1}+b_q d_1^{(1)}) \\ \vdots \\ 0 & \dots & 0 & \sigma^2(b_1+b_2 d_1^{(1)}+\dots+b_q d_{q-1}^{(q-1)}) \end{pmatrix},$$

where in the last row of M there are the coefficients $a_p,\dots$ $\dots,a_1$ preceded by $r-p$ zeros if $r>p$ and similarly for the last column of Z.

__Lemma 3.__ The sequence $X_1,\dots,X_T$ is stationary if and only if the matrix $V = \mathrm{var}(X_1,\dots,X_r)$ satisfies the equation

(17) $V = MVM' + MZ + Z'M' + \Lambda$.

__Proof.__ One can write

$$\begin{pmatrix} X_2 \\ \vdots \\ X_r \\ X_{r+1} \end{pmatrix} = M \begin{pmatrix} X_1 \\ \vdots \\ X_{r-1} \\ X_r \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varepsilon_{r+1}+b_1\varepsilon_r+\dots+b_q\varepsilon_{r+1-q} \end{pmatrix}.$$

The variance matrices of the random vectors on both sides of this equation must be equal so that

$$\mathrm{var}(X_2,\dots,X_{r+1}) = MVM' + MZ + Z'M' + \Lambda$$

(the covariance matrix Z can be calculated due to Lemma 1). Now the assertion of Lemma 3 is obvious according to Lemma 2.

Let us consider an ARMA process of the form

(18) $Y_t = a_1 Y_{t-1} + \dots + a_p Y_{t-p} + \varepsilon_t + b_1 \varepsilon_{t-1} + \dots + b_q \varepsilon_{t-q},$

$-\infty < t < \infty,$

where ε_t is a stationaty white noise with the variance σ^2. The well known condition of stationarity of (18) has the form: all roots of the polynomial

(19) $z^p - a_1 z^{p-1} - \dots - a_p$

are less than one in the absolute value (let us refer to it as to the condition of AR-regularity).

If the process Y_t is stationary then it is not difficult to show that the equality (17) in which X_t is replaced by Y_t must hold. From this fact several conclusions can be drawn. Firstly, the autocovariances of the stationary ARMA(p,q) process $X_1,\dots,X_T$ of the finite length are equal to the corresponding autocovariances of the stationary ARMA(p,q) process (18). Further under the condition of AR-regularity the equation (17) has the unique solution of the form

(20) $$V = \sum_{k=0}^{\infty} M^k (MZ + Z'M' + \Lambda) M'^k$$

(the convergence of the infinite sum in (20) follows from Perron's formula and from the fact that the matrix M has r-p zero eigenvalues and the remaining p eigenvalues are the roots of the polynomial (19), see [1], Section XIII.1, for the autoregressive case). The solution (20) is positive definite since due to its uniqueness it must be $V = \mathrm{var}(Y_1,\dots,Y_r)$ and

$\mathrm{var}(Y_1,\dots,Y_r)$ is positive definite (the process Y_t is non-deterministic and therefore all finite variance matrices of it are regular).

Remark 3. In the autoregressive case the explicit form of the inverse matrix V^{-1} expressed by means of the numbers $a_1,\dots,a_p$ is given in [1], Section XIII.2. On the other hand, the formula (20) can be recommended thanks to the rapid convergence of the infinite sum in it.

Finally one can extend all preceding considerations to the case of ARMA process $X_1, X_2, \dots$ of the infinite length. If the assumptions (2)-(5) hold for the infinite sequence $\varepsilon_{s+1}, \varepsilon_{s+2}, \dots$ and the initial random variables $X_1,\dots,X_r$ conclusions of this section stay valid for $X_1, X_2, \dots$.

3. ARMA processes of finite length generated from initial random variables and nonstationary white noise. Now we shall deal with the process ARMA of the finite length when the zero mean sequence ε_t, $s+1 \le t \le T$, loses the properties of the stationary white noise for $r < t \le T$. Therefore let the assumptions (2) and (3) be replaced by

(21) $\mathrm{var}\ \varepsilon_t = \sigma^2 > 0$, $s+1 \le t \le r$,

(22) $\mathrm{cov}(\varepsilon_t, \varepsilon_u) = 0$, $s+1 \le t \le r$, $t < u \le T$,

(23) $E = \mathrm{var}(\varepsilon_{r+1},\dots,\varepsilon_T)$ is regular.

Theorem 4. Let the assumptions (4),(5),(21)-(23) be valid. Further let the variance matrix V of $(X_1,\dots,X_r)'$ be the solution of (17). Then the sequence $X_1,\dots,X_T$ generated according to (1) is stationary if and only if $\varepsilon_{s+1},\dots,\varepsilon_T$ is the stationary white noise.

Remark 4. In the autoregressive case ($q=0$) the assumptions (5),(21) and (22) are omitted.

Proof. If $\varepsilon_{s+1},\dots,\varepsilon_T$ is the stationary white noise then the sequence $X_1,\dots,X_T$ is stationary according to Lemma 3.

On the contrary let $X_1,\dots,X_T$ be stationary. Then we want to show that

(24) $E = \sigma^2 I$,

where I is the unit matrix. There exists a lower triangular matrix $F = (f_{ij})$ with positive numbers on its diagonal such that

$$E = \sigma^2 FF'$$

since E is positive definite (so called Cholesky decomposition). Let us define the random variables $\tilde{\varepsilon}_{r+1},\dots,\tilde{\varepsilon}_T$ by means of the formula

(25) $(\tilde{\varepsilon}_{r+1},\dots,\tilde{\varepsilon}_T)' = F^{-1}(\varepsilon_{r+1},\dots,\varepsilon_T)'$.

Then $\mathrm{var}(\tilde{\varepsilon}_{r+1},\dots,\tilde{\varepsilon}_T) = \sigma^2 I$ so that the sequence $\varepsilon_{s+1},\dots,\varepsilon_r,\ \tilde{\varepsilon}_{r+1},\dots,\tilde{\varepsilon}_T$ forms a stationary white noise. Therefore the sequence $X_{s+1},\dots,X_r,\ \tilde{X}_{r+1},\dots,\tilde{X}_T$ generated by means of (1) replacing $\varepsilon_{r+1},\dots,\varepsilon_T$ by $\tilde{\varepsilon}_{r+1},\dots,\tilde{\varepsilon}_T$ is stationary according to Lemma 3.

We shall prove that $F = I$ (then (24) will hold). First we can write

$$X_{r+1} = \varepsilon_{r+1} + b_1\varepsilon_r + \dots + b_q\varepsilon_{r+1-q} + a_1X_r + \dots + a_pX_{r+1-p}$$
$$= f_{11}\tilde{\varepsilon}_{r+1} + b_1\varepsilon_r + \dots + b_q\varepsilon_{r+1-q} + a_1X_r + \dots + a_pX_{r+1-p},$$
$$\tilde{X}_{r+1} = \tilde{\varepsilon}_{r+1} + b_1\varepsilon_r + \dots + b_q\varepsilon_{r+1-q} + a_1X_r + \dots + a_pX_{r+1-p}.$$

Hence it holds obviously $\mathrm{var}\, X_{r+1} - \mathrm{var}\, \tilde{X}_{r+1} = f_{11}^2\sigma^2 - \sigma^2$. Since $\mathrm{var}\, X_{r+1} = \mathrm{var}\, X_r = \mathrm{var}\, \tilde{X}_{r+1}$ it must be $f_{11} = 1$ (consequently it is $\varepsilon_{r+1} = \tilde{\varepsilon}_{r+1}$ and $X_{r+1} = \tilde{X}_{r+1}$).

Further it is

$$X_{r+2}=\varepsilon_{r+2}+d_1^{(1)}\varepsilon_{r+1}+d_2^{(1)}\varepsilon_r+\dots+d_{q+1}^{(1)}\varepsilon_{r+1-q}+c_1^{(1)}X_r+\dots$$
$$\dots+c_p^{(1)}X_{r+1-p}=f_{22}\tilde{\varepsilon}_{r+2}+(f_{21}+d_1^{(1)})\tilde{\varepsilon}_{r+1}+d_2^{(1)}\varepsilon_r+\dots+$$
$$+d_{q+1}^{(1)}\varepsilon_{r+1-q}+c_1^{(1)}X_r+\dots+c_p^{(1)}X_{r+1-p},$$
$$\tilde{X}_{r+2}=\tilde{\varepsilon}_{r+2}+d_1^{(1)}\tilde{\varepsilon}_{r+1}+d_2^{(1)}\varepsilon_r+\dots+d_{q+1}^{(1)}\varepsilon_{r+1-q}+c_1^{(1)}X_r+\dots$$
$$\dots+c_p^{(1)}X_{r+1-p}.$$

Hence it holds $\operatorname{cov}(X_{r+2},X_{r+1})-\operatorname{cov}(\tilde{X}_{r+2},\tilde{X}_{r+1})=(f_{21}+d_1^{(1)})\sigma^2-$ $-d_1^{(1)}\sigma^2=f_{21}\sigma^2$. Since $\operatorname{cov}(X_{r+2},X_{r+1})=\operatorname{cov}(X_{r+1},X_r)=$ $=\operatorname{cov}(\tilde{X}_{r+1},X_r)=\operatorname{cov}(\tilde{X}_{r+2},\tilde{X}_{r+1})$ it must be $f_{21}=0$. Comparing the variances of X_{r+2} and $\tilde{X}_{r+2}$ one will find analogously that $f_{22}=$ $=1$ etc. (it is possible to proceed by means of the induction).

One of the practical interpretations of Theorem 4 is the following: when we generate an ARMA process on the computer by means of (1) under the condition of AR-regularity using initial random variables with the variance matrix (20) we cannot obtain the stationary process if the random shocks generated by the computer do not form a white noise.

4. ARMA processes of infinite length generated from initial random variables and nonstationary white noise. It is interesting from the theoretical point of view to generalize the considerations from Section 3 to ARMA processes $X_1,X_2,\dots$ of the infinite length. In order that the previous methodology could be used we confine ourselves to such nonstationary white noise process $\varepsilon_{r+1},\varepsilon_{r+2},\dots$ which was called by Tjøstheim and Thomas [7] almost white noise. It means that there exists a (stationary) white noise $\tilde{\varepsilon}_{r+1},\tilde{\varepsilon}_{r+2},\dots$ (i.e. $E\,\tilde{\varepsilon}_t=0$, $\operatorname{var}\tilde{\varepsilon}_t=\sigma^2>0$, $\operatorname{cov}(\tilde{\varepsilon}_t,\tilde{\varepsilon}_u)=0$ for $t\neq u$) such that $\varepsilon_t=B\,\tilde{\varepsilon}_t$, $t=r+1,r+2,\dots$, where B is a linear bounded operator with a bounded inverse B^{-1}

defined on Hilbert space H generated in the usual way by the process $\tilde{\varepsilon}_{r+1}, \tilde{\varepsilon}_{r+2}, \dots$.

On the basis of the general theory given in [2] one can obtain various necessary and sufficient conditions for the process $\varepsilon_{r+1}, \varepsilon_{r+2}, \dots$ with zero mean values to be the almost white noise, e.g.

(I) there exist constants k_1 and k_2 such that it holds for an arbitrary natural n and arbitrary numbers $\gamma_1, \dots, \gamma_n$

$$(26) \qquad k_1 \sum_{j=1}^{n} \gamma_j^2 \le E\Big(\sum_{j=1}^{n} \gamma_j \varepsilon_{r+j}\Big)^2 \le k_2 \sum_{j=1}^{n} \gamma_j^2 ;$$

or another condition is following

(II) the infinite matrix (so called Gramm matrix)

$$(27) \qquad A = (\mathrm{cov}(\varepsilon_{r+j}, \varepsilon_{r+k})), \ j,k = 1,2,\dots$$

forms the bounded linear operator with the bounded inverse in the space ℓ_2.

The simple example of the almost white noise is the process of uncorrelated random variables with zero means and variances lying between two positive constants (i.e. $0 < m \le \mathrm{var}\, \varepsilon_t \le M$) mentioned in Section 1. The verification of the condition (I) is trivial in this case. More complicated example is the zero mean process $\varepsilon_{r+1}, \varepsilon_{r+2}, \dots$ such that the variance matrix of each of its finite parts has all eigenvalues lying between two positive constants $0 < m < M$. Then according to [6, 1f.2.1] it is

$$m \sum_{j=1}^{n} \gamma_j^2 \le \lambda_{\min} \sum_{j=1}^{n} \gamma_j^2 \le \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i \gamma_j \mathrm{cov}(\varepsilon_{r+i}, \varepsilon_{r+j}) \le$$
$$\le \lambda_{\max} \sum_{j=1}^{n} \gamma_j^2 \le M \sum_{j=1}^{n} \gamma_j^2 ;$$

where $\lambda_{\min}$ (resp. $\lambda_{\max}$) is the minimal (resp. maximal) eigenvalue of $\mathrm{var}(\varepsilon_{r+1}, \dots, \varepsilon_{r+u})$ so that the condition (I) is fulfilled.

Tjøstheim and Thomas [7] showed that the almost white noise is the special case of so called UBLS (<u>uniformly bounded linearly stationary</u>) processes which present the natural generalization of the stationary processes (see also [4]). These authors also give some simple examples of the almost white noise processes. According to [3] the operator from the definition of the almost white noise can be determined by means of the infinite lower triangular matrix $B = (b_{ij})$ such that

$$A = \sigma^2 BB', \tag{28}$$

the process $\tilde{\varepsilon}_{r+1}, \tilde{\varepsilon}_{r+2}, \dots$ defined explicitly by means of

$$\begin{aligned} \varepsilon_{r+1} &= b_{11}\,\tilde{\varepsilon}_{r+1}, \\ \varepsilon_{r+2} &= b_{21}\,\tilde{\varepsilon}_{r+1} + b_{22}\,\tilde{\varepsilon}_{r+2}, \\ &\;\vdots \end{aligned} \tag{29}$$

is the corresponding stationary white noise with the variance σ^2.

Now Theorem 4 can be generalized in the following way.

<u>Theorem 5</u>. Let $\varepsilon_{s+1}, \varepsilon_{s+2}, \dots$ be a zero mean process such that the assumptions (5),(21) and

$$\operatorname{cov}(\varepsilon_t, \varepsilon_u) = 0, \quad s+1 \leq t \leq r, \quad t < u, \tag{30}$$

$$\operatorname{cov}(X_t, \varepsilon_u) = 0, \quad 1 \leq t \leq r, \quad s+1 \leq u, \quad t < u \tag{31}$$

are fulfilled and $\varepsilon_{r+1}, \varepsilon_{r+2}, \dots$ is the almost white noise. Further let the variance matrix V of $(X_1, \dots, X_r)'$ be the solution of (17). Then the sequence $X_1, X_2, \dots$ generated according to (1) is stationary if and only if $\varepsilon_{s+1}, \varepsilon_{s+2}, \dots$ is the stationary white noise.

<u>Proof</u> is analogous to the one of Theorem 4 using (29) instead of (25).

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