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HERCULES and HYDRA, A GAME ON ROOTED FINITE TREES

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Abstract: A combinatorial analysis of the Hercules and Hydra game is given. The existence of the unique maximal strategy is proved.

Key words: Tree, undecidability, combinatorial game.

Classification: 05C05, 90D99, 03B25

The following game was introduced by Paris and Kirby [1]:

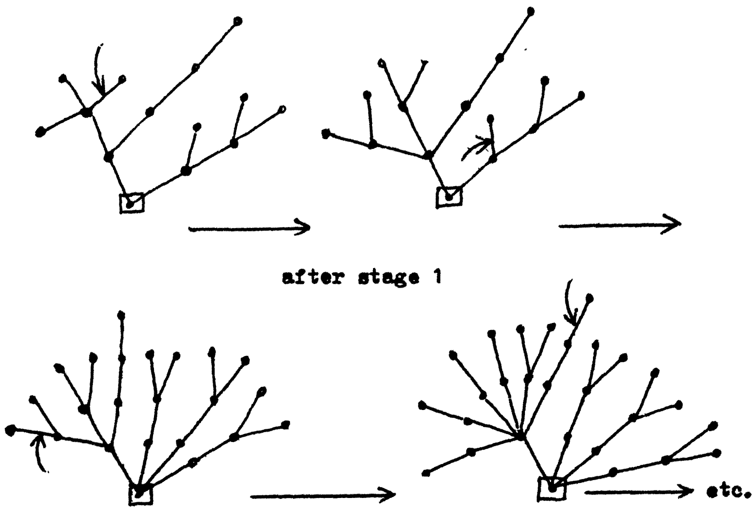
A hydra is a rooted finite tree.

A head of a hydra is a vertex of only one edge which is not the root.

A top of a hydra is a head together with its adjacent edge.

A battle between Hercules and a given hydra proceeds as follows: At stage n $(n \geq 1)$, Hercules chops off one top from the hydra. The hydra then grows n "new throats" in the following manner:

From the vertex that used to be attached to the top which was just chopped off, traverse one edge towards the root until the next vertex is reached. From this vertex sprout n replicas of that part of the hydra (after decapitation) which is "above" the edge just traversed, i.e., those vertices and edges from which, in order to reach the root, this edge would have to be traversed. If the top just chopped off had the root as one of its vertices, then nothing is grown. This is best illustrated by means of an example (Hercules decides to chop off the top marked with an arrow):



after stage 1

after stage 2

after stage 3

Hercules wins if after some finite number of stages nothing is left of the hydra but its root.

Theorems of Paris and Kirby:

i) Every strategy of Hercules is a winning strategy.

ii) The statement "every recursive strategy is a winning strategy" is not provable from Peano Arithmetic (PA).

We considered variations of this game and we succeeded to strengthen the results of Paris and Kirby. Some of these results were announced in [3].

First we need a few definitions:

i) A (finite rooted) tree is a triplet (V,E,v_0), where (V,E) is a finite tree, V is a set of vertices, E is a set of edges and v_0 is a fixed vertex of V (the root).

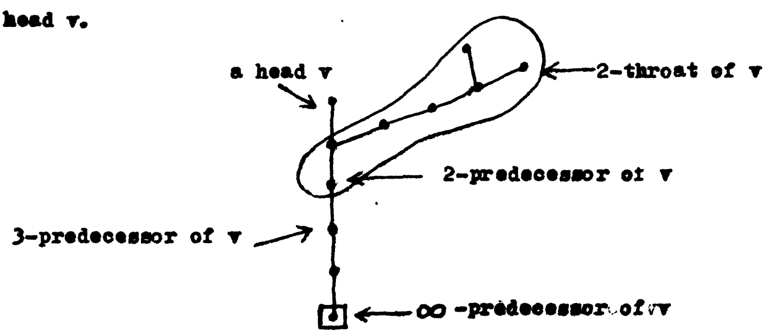
ii) P_n is a (rooted) path of n edges.

(iii) <u>A head</u> of a tree T is a vertex of T of degree one, which is not a root.

(iv) Let i be a positive integer.
<u>i-predecessor</u> of a head v of a tree T is the vertex of the unique path in T from v to the root of T which has distance i from v.

(v) ∞ <u>-predecessor</u> of a head v of a tree T is the root of T.

(vi) Let i be a positive integer or i equals ∞ . Let us have the maximal subtree of a given tree, at which both v and i-predecessor of v have degree one.
<u>i-throat</u> of the head v is then this maximal subtree without the head v.



Thus the (2-predecessor) Hercules - hydra game (shortly 2-HH game) considered above is just the following game:

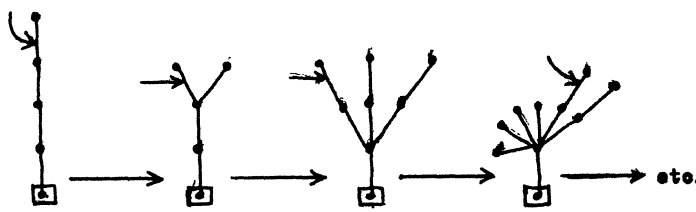
In n-th move (n is a positive integer) Hercules chops off a head v (or a respective top) of the hydra and on return the hydra grows n new replicas of 2-throat of v from the 2-predecessor of v.
If such 2-predecessor does not exist then nothing is grown. More

generally, define i-HH game (i is a positive integer or i equals ∞):

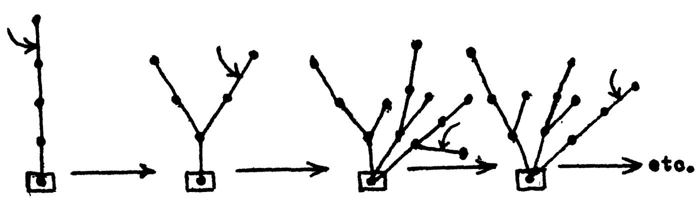
In n-th move Hercules chops off a head v (or a respective top) of the hydra and on return the hydra grows n new replicas of i-throat of v from the i-predecessor of v.
If such i-predecessor does not exist then nothing is grown.

Examples:

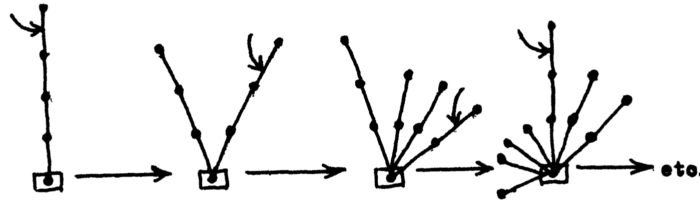
2-HH game:



3-HH game:



∞ -HH game:



Let us define a trajectory of 1-HH game $\mathcal{H}$ as a succession of pairs $(T_n(\mathcal{H}), v_n(\mathcal{H}))_{n=0}^{\infty}$, where $T_n(\mathcal{H})$ is the hydra after the n-th Hercules move and $v_n(\mathcal{H})$ is a head of $T_n(\mathcal{H})$ which Hercules will chop off next.
A size of this trajectory $\mathcal{H}$ is the smallest n such that the tree $T_n(\mathcal{H})$ has no heads. If we cannot prove that such n exists (we cannot prove the finity in PA) then we say $\mathcal{H}$ has not a finite size.
We strengthen the results of Paris and Kirby in PA as follows:

Theorem 0: [PA] $\forall k \geq 2$ all trajectories of k-HH game have a finite size iff all trajectories of 2-HH game have a finite size.

Theorem 1: [PA] All trajectories of ∞-HH game have a finite size.

Theorem 2: [PA] Let $\mathcal{H}$ be a trajectory where the starting hydra $T_0(\mathcal{H})$ has n vertices. Then there exist a trajectory $\mathcal{H}'$ with $T_0(\mathcal{H}') = P_{n-1}$ such that the size of $\mathcal{H}'$ is at least the size of $\mathcal{H}$.

Sketch of the proof of Theorem 0: Theorem 0 follows from the following two statements:

(i) If all trajectories of k-HH game, $k \geq 2$, have a finite size, then all trajectories of (k+1)-HH game have a finite size.

(ii) If all trajectories of 2k-HH game, $k \geq 1$, have a finite size, then all trajectories of 2-HH game have a finite size.

To prove (i) we construct by induction, for a given trajectory of (k+1)-HH game $\mathcal{H}$, a trajectory of k-HH game $\mathcal{H}'$ such that $\mathcal{H}'$ contains $\mathcal{H}$.

The proof of (ii) is similar to (i).

Let us define two strategies MAX, MIN for such i-HH games which have a hydra equals P_n on their beginning. Both these strategies of Hercules are based on suitable relabelling of heads of a hydra after each Hercules move. First we introduce an informal description:
In the first Hercules move the unique head of P_n is labelled by 1. Before the n-th Hercules move let the hydra have all heads labelled by different positive integers.

In the strategy MAX Hercules chops off the head of the maximal label.

In the strategy MIN Hercules chops off the head of the minimal label.
The heads of the new hydra (after the n-th Hercules move) are labelled then as follows (let v be the head which was just chopped off):
If v had not i-predecessor, then the new hydra has no new heads. In such case no heads are relabelled.
If v had i-predecessor then the heads of i-throat of v must be relabelled and the heads of new n replicas of i-throat of v must be labelled. Other heads are not relabelled.
Let C be the maximal label of a head of the former hydra. If a head x is the unique head of i-throat of v then the new label of x is equal to the label of v plus C.
If a head x is a head of i-throat of v then the new label of x is equal to the former label of x plus C.
Further the labels of heads of j-th new replica of i-throat of v (j=1,...,n) are equal to the new labels of the samples of these heads in i-throat of v plus (j+1)C.
The procedure of relabelling can be easily formalized: Let us

denote $\lambda(v,n,\mathcal{H})$ the label of a head v of the tree $T_n(\mathcal{H})$ in a trajectory of i-HH game $\mathcal{H}$. Then λ is defined by the following procedure:

BEGIN

$\lambda(v,1,\mathcal{H}) := 1$

WHILE $T_{n+1}(\mathcal{H})$ has heads DO

IF $v_n(\mathcal{H})$ has not i-predecessor THEN

$\lambda(v,n+1,\mathcal{H}) := \lambda(v,n,\mathcal{H})$

ELSE

BEGIN

denote $C := \max\{\lambda(v,n,\mathcal{H});\ v$ is a head of $T_n(\mathcal{H})\}$;

denote $\tau_0, \tau_1, \ldots, \tau_n$ i-throat of $v_n(\mathcal{H})$ and n new replicas of this i-throat of $v_n(\mathcal{H})$;

IF v is a head of τ_j, $j \in \{0,\ldots,n\}$ THEN

IF v is the unique head of τ_j THEN

$\lambda(v,n+1,\mathcal{H}) := \lambda(v_n(\mathcal{H}),n,\mathcal{H}) + (j+1)\,C$

ELSE

$\lambda(v,n+1,\mathcal{H}) := \lambda(\bar{v},n,\mathcal{H}) + (j+1)\,C$,

where $\bar{v}$ is the sample of v in the i-throat of $v_n(\mathcal{H})$

ELSE

$\lambda(v,n+1,\mathcal{H}) := \lambda(v,n,\mathcal{H})$;

END END.

We have the following results in PA:

Theorem 3: [PA] Let $\mathcal{H}$ be a trajectory of i-HH game, $T_0(\mathcal{H}) = P_n$. Then the trajectory $\mathcal{H}'$, $T_0(\mathcal{H}') = T_0(\mathcal{H})$, defined by the strategy MAX (MIN, respectively) has at least (at most, respectively) the same size as $\mathcal{H}$.

Thus, combining Theorem 3 and Theorem 2, the strategy MAX defines the longest trajectory of i-HH game.

a given hydra	2-HH game	3-HH game	∞-HH game
	more than $\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}100$	158	$41 \cdot 2^{39} - 3$

R e f e r e n c e s

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[2] M. LOEBL: Honza a drak; SVOČ 1984.

[3] J. NEŠETŘIL: Some non standard Ramsey-like applications; Theoret. Comp. Science, 1984.

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(Oblatum 15.10. 1984)

Theorem 4: [PA] All trajectories defined by the strategy MIN have a finite size.

<u>Sketch of the proof of Theorem 3</u>: To prove the maximality of MAX, we need two additional definitions:

1. Let m be a positive integer. $\underline{MAX_m(P_n)}$ is a trajectory which satisfies:

a) $T_0(MAX_m(P_n)) = P_n$;

b) if $j < m$ then $\lambda(v_j(MAX_m(P_n), j, MAX_m(P_n)) =$
$= \max \{\lambda(v, j, MAX_m(P_n));\ v \text{ is a head of } T_j(MAX_m(P_n))\}$.

2. Let M, N be trees. Let $j \geq 0$. Then $\underline{M \geq jN}$, if there exists a trajectory of i-HH game $\mathcal{H}$ with the following properties:

a) $T_j(\mathcal{H}) = M$;

b) $\mathcal{H}$ has not a finite size or there exists $i \geq j$ such that $T_i(\mathcal{H}) = N$.

The maximality of MAX follows from

<u>Lemma</u>: Let m be a positive integer. Then
$T_{m+1}(MAX_{m+1}(P_n)) \geq m + 1\ T_{m+1}(MAX_m(P_n))$.

The proof of minimality of MIN is analogous.

Proofs of the results of this paper will appear elsewhere.

Let us show also the concrete sizes of trajectories defined by the strategy MAX:

a given hydra	2-HH game	3-HH game	∞-HH game
	3	2	3
	37	5	13