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A GENERALIZATION OF THE THEOREM OF MAULDIN

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Abstract: For a perfect Polish space X and a σ-ideal $\mathcal{J}$ of subsets of X, let $\Phi(X,\mathcal{J})$ denote the family of all real-valued functions on X continuous almost everywhere with respect to $\mathcal{J}$. We shall prove that the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for a general class of σ-ideals $\mathcal{J}$, thus generalizing the Mauldin's result for X = [0,1] and the sets of Lebesgue measure zero for $\mathcal{J}$.

Key words: Baire classes of functions, σ-ideals of sets.

Classification: 26A21

Let X be a perfect Polish space. We consider σ-ideals of subsets of X. It is assumed that each σ-ideal contains all singletons $\{x\}$ and does not contain any nonempty open subset of X. For a fixed σ-ideal $\mathcal{J}$, let $\Phi(X,\mathcal{J})$ denote the family of all real-valued functions defined on X which are continuous almost everywhere with respect to $\mathcal{J}$. Suppose that a σ-ideal $\mathcal{J}_0$ is such that the following conditions hold:

(I) there is a compact subset X_0 of X which does not belong to $\mathcal{J}_0$;

(II) for each countable subset A of X, there is a G_δ-set belonging to $\mathcal{J}_0$ such that $A \subseteq B$.

It is proved that the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_0$. Mauldin [8] obtained this result in the case when X is the unit interval and $\mathcal{J} = \mathcal{J}_0$ is the σ-ideal of all sets of the Lebesgue measure zero. Our proof is based on the method presented in [8]. We also use topological properties concerning σ-ideals (for instance, a generalization of the Cantor-Bendixson Theorem is proved). The main result of this note can be applied to the σ-ideal constructed by Mycielski in [10].

Let X be a set and let Φ be a family of real-valued functions defined on X. We define $\Phi_0 = \Phi$ and, for each ordinal $\alpha > 0$, let Φ be the family of all pointwise limits of sequences taken from $\bigcup_{\gamma<\alpha} \Phi_\gamma$. The first uncountable ordinal will be denoted by ω_1. Observe that $\Phi_{\omega_1} = \Phi_{\omega_1+1}$ and Φ_{ω_1} is the smallest subfamily of $\mathbb{R}^X$ which contains Φ and which is closed with respect to pointwise limits of sequences. The Baire order of Φ is a first ordinal α such that $\Phi_\alpha = \Phi_{\alpha+1}$. For example, if Φ denotes the family of all real-valued functions defined on the unit interval, then the Baire order of Φ is ω_1 [11].

Now, let X be a perfect Polish space. Consider those σ-ideals of subsets of X which contain all singletons $\{x\}$ and do not contain any nonempty open subset of X. For a fixed σ-ideal $\mathcal{J}$, let $\Phi = \Phi(X,\mathcal{J})$ be the family of all real-valued functions on X whose set of points of discontinuity belongs to $\mathcal{J}$. Notice that the Baire order of $\Phi(X,\mathcal{J})$ is always positive because the characteristic function of any countable dense subset of X belongs to $\Phi_1(X,\mathcal{J}) \setminus \Phi_0(X,\mathcal{J})$ (we write $\Phi_\alpha(X,\mathcal{J})$ instead of $(\Phi(X,\mathcal{J}))_\alpha$). The problems connected with the Baire order of $\Phi(X,\mathcal{J})$ were studied by Mauldin in [6],[7],[8],[9]. It is known that the order of $\Phi(X,\mathcal{J})$ equals 1 if $\mathcal{J}$ denotes the σ-ideal of all sets of the first category [2]. Mauldin in [8] proved that if X is the unit interval and $\mathcal{J}$ denotes the σ-ideal of all sets of the Lebesgue measure zero. then the order of $\Phi(X,\mathcal{J})$

is ω_1. Several generalizations of this result were obtained in [7]. Another generalization will be presented in this paper.

Mauldin in [6] gave the following characterization of the generalized Baire classes:

Theorem 1. If α is an ordinal, $0 < \alpha < \omega_1$, then a function f is in $\Phi(X,\mathcal{J})$ if and only if there is a function g in the Baire class α such that the set $\{x: f(x) \neq g(x)\}$ is a subset of an F_σ set belonging to $\mathcal{J}$.

The Baire order of $\Phi(X,\mathcal{J})$ treated as a function of $\mathcal{J}$ is monotonic in the following sense:

Proposition 1. If $\mathcal{J}$ and $\mathcal{Y}$ are σ-ideals of subsets of X and $\mathcal{J} \subseteq \mathcal{Y}$, then the order of $\Phi(X,\mathcal{Y})$ is not greater than the order of $\Phi(X,\mathcal{J})$.

Proof. Let α be the order of $\Phi(X,\mathcal{J})$. Observe that it is enough to demonstrate the inclusion

$$\Phi_{\alpha+1}(X,\mathcal{Y}) \subseteq \Phi_{\alpha}(X,\mathcal{Y}).$$

It obviously holds if $\alpha = \omega_1$. Let $\alpha < \omega_1$. If f belongs to $\Phi_{\alpha+1}(X,\mathcal{Y})$, then, by Theorem 1, there exists a function g in the Baire class $\alpha + 1$ such that the set $\{x: f(x) \neq g(x)\}$ is a subset of an F_σ set belonging to $\mathcal{Y}$. Of course, g belongs to $\Phi_{\alpha+1}(X,\mathcal{J})$. Then, from the definition of α it follows that g belongs to $\Phi_{\alpha}(X,\mathcal{J})$. Hence, by Theorem 1, there exists a function h in the Baire class α such that the set $\{x: g(x) \neq h(x)\}$ is a subset of an F_σ set belonging to $\mathcal{J}$. Since $\mathcal{J} \subseteq \mathcal{Y}$, the set $\{x: f(x) \neq h(x)\}$ is a subset of an F_σ set belonging to $\mathcal{Y}$. Hence, by Theorem 1, the function f belongs to $\Phi_{\alpha}(X,\mathcal{Y})$.

The main result of this note is:

Theorem 2. Let $\mathcal{J}_0$ be a σ-ideal of subsets of X such that the following conditions hold:

(I) there is a compact subset X_0 of X which does not belong to $\mathcal{J}_0$;

(II) for each countable subset A of X, there is a G_δ set B belonging to $\mathcal{J}_0$ such that $A \subseteq B$.

Then the Baire order of $\Phi(X,\mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_0$.

Remark. Considering X equal to the unit interval and $\mathcal{J}$, $\mathcal{J}_0$ equal to the σ-ideal of sets of the Lebesgue measure zero, we get the theorem of Mauldin [8].

In virtue of Proposition 1, we shall prove Theorem 2 if we only verify that the order of $\Phi(X,\mathcal{J}_0)$ is ω_1. The argument of this fact will be based on the method presented in [8].

The proof of Mauldin begins with a construction of a family which consists of perfect sets A such that if an open set V intersects A, then the set $V \cap A$ has positive measure. We shall generalize that property.

Let $\mathcal{J}$ be a σ-ideal of subsets of X.

Definition 1 (compare [4]). A closed nonempty subset A of X will be called <u>$\mathcal{J}$-perfect</u> if and only if, for each open set V such that V intersects A, we have $V \cap A \notin \mathcal{J}$.

Remark. Since $\mathcal{J}$ does not contain any nonempty open subset of X, the set X is $\mathcal{J}$-perfect.

Definition 2 (compare [10]). If A is a subset of X, then let $A^{(\mathcal{J})}$ denote the set of all points x of X such that, for each neighbourhood V of x, we have $V \cap A \notin \mathcal{J}$.

Let us quote from [10] a few properties of the operation

$A^{(\mathcal{J})}$:

(i) $A^{(\mathcal{J})}$ is closed and included in the closure of A;

(ii) $(A^{(\mathcal{J})})^{(\mathcal{J})} = A^{(\mathcal{J})}$;

(iii) $A \setminus A^{(\mathcal{J})} \in \mathcal{J}$.

Proposition 2. A nonempty subset A of X is $\mathcal{J}$-perfect if and only if $A = A^{(\mathcal{J})}$.

Proof. Assume that A is $\mathcal{J}$-perfect. Then, immediately from the definitions it follows that $A \subseteq A^{(\mathcal{J})}$. Since A is closed, therefore, by (i), we have $A^{(\mathcal{J})} \subseteq A$. Conversely, assume that $A = A^{(\mathcal{J})}$. Then, by (i), the set A is closed. Let an open set V intersect A. Consider a point which belongs to $V \cap A$. Then it belongs to A and from Definition 2 it follows that $V \cap A \notin \mathcal{J}$. Thus A is $\mathcal{J}$-perfect.

Proposition 3. For each closed subset A of X, there is a unique decomposition $A = B \cup C$ into disjoint sets such that B is empty or $\mathcal{J}$-perfect, and $C \in \mathcal{J}$.

Proof. If $A \in \mathcal{J}$, then we put $B = \emptyset$, $C = A$, and $A = B \cup C$ is the required unique decomposition. If $A \notin \mathcal{J}$, then we put $B = A^{(\mathcal{J})}$, $C = A \setminus B$. In virtue of (iii), we have $C \in \mathcal{J}$. Since $A \notin \mathcal{J}$, therefore $B \notin \mathcal{J}$. Hence B is nonempty and it follows from (ii) that $B^{(\mathcal{J})} = B$. Thus, in virtue of Proposition 2, the set B is $\mathcal{J}$-perfect. Now, assume that $A = B' \cup C'$ where B', C' are disjoint, B' is $\mathcal{J}$-perfect and $C' \in \mathcal{J}$. If $x \in B'$ and V is any neighbourhood of x, then $V \cap B'$ $\mathcal{J}$. Hence $V \cap A \notin \mathcal{J}$ and $x \in A^{(\mathcal{J})}$. Thus $B' \subseteq B$. If $x \in C'$, then there is a neighbourhood V of x such that $V \cap B' = \emptyset$ since B', C' are disjoint and B' is closed. Now, $V \cap B' = \emptyset$ implies $V \cap A = V \cap C'$ and then $V \cap A \in \mathcal{J}$.

Hence $x \in C$. So, we have $B' \subseteq B$, $C' \subseteq C$. Since $B \cup C = B' \cup C'$ and $B \cap C = \emptyset = B' \cap C'$, there must be $B = B'$, $C = C'$.

Remarks. Martin in [5] explored topologies generated by the operation of the derived set. Notice that $A^{(\mathcal{J})}$ is such an operation. Then $A \cup A^{(\mathcal{J})}$ is a closure operation and it generates a topology which we denote by $\mathcal{T}$ (comp. [1],[5],[10]). From [5], Th. 1, it follows that if $x \in A^{(\mathcal{J})}$ implies $x \in (A \setminus \{x\})^{(\mathcal{J})}$, then the derived set of A in the topology $\mathcal{T}$ coincides with $A^{(\mathcal{J})}$. We have assumed that $\{x\} \in \mathcal{J}$ for each $x \in X$, therefore the above-mentioned condition holds. Thus, Proposition 2 means that $\mathcal{J}$-perfect sets are identical with perfect sets in the topology $\mathcal{T}$. Proposition 3 is a kind of generalization of the Cantor-Bendixson Theorem. Similar results were obtained in [1] (Satz II) and [4](Th. 1.3).

Now, suppose that $\mathcal{J}_0$ and X_0 fulfil all the hypotheses of Theorem 2. Since X_0 is closed and $X_0 \notin \mathcal{J}_0$, therefore by Proposition 3, there is an $\mathcal{J}_0$-perfect set $X_* \subseteq X_0$. Of course, X_* is compact. Let

$$\mathcal{J}_0^* = \{A \cap X_* : A \in \mathcal{J}_0\}.$$

Observe that $\mathcal{J}_0^* \subseteq \mathcal{J}_0$ and $\mathcal{J}_0^*$ is a σ-ideal of subsets of the perfect Polish space X_*.

Lemma 1 (compare [9], Th. 2). The Baire order of $\Phi(X_*, \mathcal{J}_0^*)$ is not greater than the Baire order of $\Phi(X, \mathcal{J}_0)$.

Proof. Suppose that the order of $\Phi(X_*, \mathcal{J}_0^*)$ is greater than the order of $\Phi(X, \mathcal{J}_0)$. Thus, the order of $\Phi(X, \mathcal{J}_0)$ equals a countable ordinal α. Let f belong to $\Phi_{\alpha+1}(X_*, \mathcal{J}_0^*)$. Then, by Theorem 1, there is a function g defined on X_* which is in the Baire class $\alpha + 1$, such that the set

$$A = \{x: f(x) \neq g(x)\}$$

is a subset of a set B which is of type F_σ with respect to X_* and belongs to $\mathcal{J}_0^*$. Let $\hat{f}$, $\hat{g}$ be extensions of f, g, respectively, to the whole X, such that $\hat{f}(x) = \hat{g}(x) = 0$ for $x \in X \setminus X^*$. Then $\hat{g}$ belongs to the Baire class $\alpha + 1$ and we have $\{x: \hat{f}(x) \neq \hat{g}(x)\} = A$. As above, $A \subseteq B$ and one can easily check that B is an F_σ set with respect to X, belonging to $\mathcal{J}_0$. Thus, by Theorem 1, $\hat{f}$ belongs to $\Phi_{\alpha+1}(X, \mathcal{J}_0)$. Hence $\hat{f}$ is in $\Phi_\alpha(X, \mathcal{J}_0)$ by the definition of α. It can be shown by transfinite induction that, for all γ, $0 \leq \gamma < \omega_1$, if a function is in $\Phi_\gamma(X, \mathcal{J}_0)$, then its restriction to X_* is in $\Phi_\gamma(X_*, \mathcal{J}_0^*)$. Therefore the function f, which is the restriction of $\hat{f}$ to X_*, belongs to $\Phi_\alpha(X_*, \mathcal{J}_0^*)$. So, it follows that $\Phi_\alpha(X_*, \mathcal{J}_0^*) = \Phi_{\alpha+1}(X_*, \mathcal{J}_0^*)$. This contradicts the assumption that the order of $\Phi(X_*, \mathcal{J}_0^*)$ is greater than α.

Now, in virtue of Lemma 1, it is enough to prove that the Baire order of $\Phi(X_*, \mathcal{J}_0^*)$ equals ω_1. Thus, we shall consider X_*, $\mathcal{J}_0^*$ instead of X, $\mathcal{J}_0$, respectively. For simplicity, we shall preserve the notation X, $\mathcal{J}_0$. We shall only add thé assumption that X is compact. Observe that the condition (II) is still true.

Lemma 2. For each F_σ subset D of X such that $D \notin \mathcal{J}_0$ there is a set D_0 included in D such that D_0 is $\mathcal{J}_0$-perfect and nowhere dense in D.

Proof. Let A be a countable subset of D, dense in D. Since the condition (II) holds, there is a G_δ set $B \in \mathcal{J}_0$ such that $A \subseteq B$. Let $E = D \setminus B$. The set E is of type F_σ, of the first category in D, and $E \notin \mathcal{J}_0$. Let $E = \bigcup_{n=1}^{\infty} E_n$ where E_n are closed and nowhere dense in D. Then there exists $E_{n_0} \notin \mathcal{J}_0$. In virtue of

Proposition 3, there exists a set D_0 which is contained in E_{p_0} and $\mathcal{J}_0$-perfect. The set D_0 just fulfils the conclusion.

Lemma 3. For each $\mathcal{J}_0$-perfect set P, for each nonempty set V open with respect to P, and for each closed set F_0 contained in P and nowhere dense in P, there is a set D_0 included in $V \setminus F_0$ which is $\mathcal{J}_0$-perfect and nowhere dense in P.

Proof. It is enough to apply Lemma 2 to the set $D = V \setminus F_0$.

The following lemma can be proved by using Lemma 3 and repeating Mauldin's construction (see [8], the proof of Lemma 1).

Lemma 4. Let P be an $\mathcal{J}_0$-perfect set. There is a double sequence $\{F_{nk}\}_{n,k=1}^{\infty}$ of disjoint subsets of P such that

(a) each F_{nk} is $\mathcal{J}_0$-perfect and nowhere dense in P;

(b) if n is a natural number and V is a nonempty set open with respect to P, then there is some k such that F_{nk} is a subset of V.

The next part of the proof of Theorem 2 is analogous to that of [8]. Instead of the unit interval one considers the space X; moreover, the notations $\lambda(A) = 0$, $\lambda(A) > 0$ are to be replaced by $A \in \mathcal{J}_0$, $A \notin \mathcal{J}_0$, respectively (here $\lambda(A)$ means the Lebesgue measure of A).

In such a way we obtain the following lemma (compare [8], Lemma 4):

Lemma 5. There is an $F_{\sigma\delta}$ set H included in X and a Borel measurable function f from H onto the set $\mathcal{N}$ of all irrational numbers between 0 and 1, such that if $z \in \mathcal{N}$, then $f^{-1}(\{z\})$ is not a subset of an F_σ set belonging to $\mathcal{J}_0$.

The further two theorems play the same role as Theorems 1 and 2 in [8].

The countable product of identical sets which are all equal to X will be denoted by X^{ω_0}. Assume that X^{ω_0} is equipped with the Tychonoff topology. Notice that X^{ω_0} forms a Polish space.

Theorem 3. There is a Borel measurable mapping h from X onto X^{ω_0} such that if $t \in X^{\omega_0}$, then $h^{-1}(\{t\})$ is not a subset of an F_σ set belonging to $\mathcal{J}_0$.

Proof. Let f be a function described in Lemma 5. Since X^{ω_0} is a Polish space, there exists a continuous mapping g of $\mathcal{N}$ onto X^{ω_0} (see [3], p. 353, Th. 1). Consider $x_0 \in X$ and put

$$h(x) = \begin{cases} g(f(x)) & \text{if } x \in H \\ (x_0, x_0, x_0, \dots) & \text{if } x \in X \setminus H. \end{cases}$$

The mapping h has the required properties.

Theorem 4. There exists a transfinite sequence of "universal functions" $\{U_\alpha\}_{0<\alpha<\omega_1}$ such that, for each α, $0<\alpha<\omega_1$, we have

(1) U_α is a Borel measurable function on $X \times X$ into the unit interval I,

(2) if f is a function in the Baire class α, which maps X into I, then the set of all x, such that $U_\alpha(x,y) = f(y)$ for each y in X, is not a subset of an F_σ set belonging to $\mathcal{J}_0$.

Proof (cf. [11], p. 339). Since X is compact and I is separable, then the space of all continuous functions on X into I with the topology generated by the uniform convergence is separable (see [3], p. 120, Th. 2). Let $\{S_n\}_{n=1}$ be a countable dense subset of this space. Choose an arbitrary sequence $\{x_n\}_{n=1}^{\infty}$ of distinct points of X. For $(x,y) \in X \times X$, let

$$U_0(x,y) = \begin{cases} S_n(y) & \text{if } x = x_n \\ 0 & \text{otherwise.} \end{cases}$$

Let $h = (h_1, h_2, h_3, \dots)$ be a mapping described in Theorem 3. For each ordinal α, $0 \leq \alpha < \omega_1$, and for each $(x,y) \in X \times X$, let

$$U_{\alpha+1}(x,y) = \limsup_{n \to \infty} U_\alpha(h_n(x),y).$$

If α is a limit ordinal, then let $\{\gamma_n\}_{n=1}^{\infty}$ be an increasing sequence of ordinals less than α which converges to α, and let

$$U_\alpha(x,y) = \limsup_{n \to \infty} U_{\gamma_n}(h_n(x),y).$$

Using transfinite induction, one shows that the sequence $\{U_\alpha\}_{0<\alpha<\omega_1}$ has properties (1),(2) (see [8], the proof of Th.2).

Now, the last part of the proof of Theorem 2 can be given. Suppose that the order of $\Phi(X, \mathcal{J}_0)$ is $\alpha < \omega_1$. Let U_α be defined as above and let

$$f(x) = \lim_{n \to \infty} (1 - U_\alpha(x,x))^n, \quad x \in X.$$

Since $0 \leq U_\alpha(x,x) \leq 1$, the equation $f(x) = U_\alpha(x,x)$ never holds. By Theorem 4, (1), the function f is Borel measurable. So, f belongs to $\Phi_\alpha(X, \mathcal{J}_0)$. In virtue of Theorem 1, there is a function g in the Baire class α such that the set A of all x for which $f(x) \neq g(x)$ is a subset of an F_σ set belonging to $\mathcal{J}_0$. In virtue of Theorem 4, (2), the set B of all x, such that $U_\alpha(x,y) = g(y)$ for each y in X, is not a subset of an F_σ set belonging to $\mathcal{J}_0$. Hence there is a point x_0 which belongs to $B \setminus A$. Then we have $U_\alpha(x_0,y) = g(y)$ for each y in X, and $f(x_0) = g(x_0)$. In particular, for $y = x_0$, we obtain $f(x_0) = U_\alpha(x_0,x_0)$. This is a contradiction. The proof of Theorem 2 has been comp-

Example. Consider $X = \{0,1\}^{\omega_0}$ and assume that $\{0,1\}$, X are equipped with the discrete and the Tychonoff topologies, respectively. The space X is homeomorphic to the Cantor set and so, X is a compact and perfect Polish space. Mycielski in [10] defined a σ-ideal $\mathcal{J}_0$ of subsets of X such that the condition (II) is fulfilled. Since X is compact, the condition (I) also holds. Hence, by Theorem 2, the Baire order of $\Phi(X, \mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_0$. Let ν be a measure on $\{0,1\}$ such that $\nu(\{0\}) = \nu(\{1\}) = 1/2$ and let μ denote the product measure on X generated by ν. Mycielski showed that there exists a decomposition of X into two disjoint sets: one of them belongs to $\mathcal{J}_0$ and the other is of the measure μ zero and of the first category. Let

$$\mathcal{J}_\mu = \{A : \mu(A) = 0\}.$$

Since μ is a finite regular Borel measure which has no atoms, the Baire order of $\Phi(X, \mathcal{J}_\mu)$ is ω_1 (see [9], Th. 7). According to Proposition 1, the order of $\Phi(X, \mathcal{J})$ is ω_1 for each σ-ideal $\mathcal{J}$ included in $\mathcal{J}_\mu$.

Problems. Can the condition (I) in Theorem 2 be omitted? Observe that it is possible if we add the assumption that X is locally compact. Indeed, then we put as X_0 a compact set which is a closure of an open nonempty set. The next question is: does the converse of Theorem 2 hold in this case? Saying precisely, let $\mathcal{J}$ be a σ-ideal of a locally compact perfect Polish space X and suppose that the order of $\Phi(X, \mathcal{J})$ is ω_1. We ask whether a σ-ideal $\mathcal{J}_0$ exists such that $\mathcal{J}$ is included in $\mathcal{J}_0$ and the condition (II) holds.

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