Piotr Biler

Convergence of solutions of generalized Korteweg-de Vries-Burgers equations to those of first order equations

Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 1, 173--176

Persistent URL: `http://dml.cz/dmlcz/106354`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,1 (1985)

CONVERGENCE OF SOLUTIONS OF GENERALIZED KORTEWEG-DE VRIES-BURGERS EQUATIONS TO THOSE OF FIRST ORDER EQUATIONS

Piotr BILER

Abstract: We indicate the proof of the convergence of solutions of generalized Korteweg-de Vries-Burgers equations to the solutions of the limit first order equation when the parameters of the equations tend to zero.

Key words: Generalized Korteweg-de Vries-Burgers equation, propagation of nonlinear waves, convergence of solutions depending on parameters.

Classification: 35Q20, 35L60

This note deals with the convergence of solutions of one-dimensional equations describing propagation of the nonlinear waves of the type

(1) $u_t + f(u)_x + \delta(Hu)_x + \varepsilon Bu = 0$

as δ, ε approach zero. These equations - generalizing the KdV-B equation - have been studied in [1] where, under some assumptions on the pseudodifferential operators H, B characterizing dispersive and dissipative properties of the medium and on the nonlinear flux function f, several theorems on existence, uniqueness and regularity of solutions of the Cauchy problem for (1) were proved.

This paper was presented in written form on the International Spring School on Evolution Equations, Dobřichovice by Prague, May 21-25, 1984.

We shall show that if the parameter δ is small compared to ε then there exists a subsequence of the solutions of (1) converging to a solution of the limit conservation law

$$(2) \qquad u_t + f(u)_x = 0.$$

More precisely: we consider (1) where $|f'(u)| \leq c(1 + |u|)$, $u \in \mathbb{R}$, $Bu = -u_{xx}$ (the simplest dissipative term) and $Hu(x) = -au\ \ (x) + \int p(\xi)\hat{u}(\xi)e^{ix\xi}\,d\xi$, $a \geq 0$ and the symbol p satisfies $0 \leq p(\xi) = p(-\xi) \leq C(1 + |\xi|)^{\mu}$ for some $\mu < 2$. Thus (1) is the KdV-B equation with perturbed dispersion operator ($a = 1$) and also (1) includes a class of the model wave equations with low order (<3) dispersion operator ($a = 0$).

Below $|\cdot|_p$ denotes the $L^p(\mathbb{R})$ norm, $\|\cdot\|_m$ the Sobolev space $H^m(\mathbb{R})$ norm and C denotes different inessential positive constants.

Theorem. Let $\Omega = \mathbb{R} \times [0,T]$, $T > 0$, and $u^{\varepsilon}_{\delta} : \Omega \to \mathbb{R}$ be a sequence of solutions of (1) with the initial conditions $u^{\varepsilon}_{0\delta}$ satisfying $\|u^{\varepsilon}_{0\delta}\|_2 + |u^{\varepsilon}_{0\delta}|_4 \leq C$.
If $\delta = o(\varepsilon^3)$, $\varepsilon \to 0^+$, then there exists a subsequence $\{u^k\} = \{u^{\varepsilon_k}_{\delta_k}\}$ converging weakly in $L^4(\Omega)$ to u, $f(u^k) \to f(u)$ (as distributions) and u is a solution of (2).
If in addition $f'' > 0$ then $u^k \to u$ strongly in $L^p(\Omega)$, $1 < p < 4$.

The proof repeats the main arguments in [2], where similar facts have been proved for the classical KdV-B equation (Th. 4.1, Th. 5.1) using Tartar's compensated compactness theory.

Similarly as in [2] it suffices to show that

(3) $\{u^{\varepsilon}_{\delta}\}$ is bounded in $L^4(\Omega)$,

(4) $\{\varepsilon(u^{\varepsilon}_{\delta})_x\}$.

(5) $\{\sigma H u^{\varepsilon}_{\sigma}\}$ are compact in $L^2(\Omega)$,

(6) $\{\varepsilon (u^{\varepsilon}_{\sigma})^2_x\}$,

(7) $\{\sigma (u^{\varepsilon}_{\sigma})_x H u^{\varepsilon}_{\sigma}\}$ are bounded in $L^1(\Omega)$.

The conditions (6) and (4) follow from the energy inequality

$$(8) \quad |u(T)|_2^2 + 2\varepsilon \int_0^T |u_x|_2^2 \leq C$$

obtained by taking the inner product of (1) with u and integrating in t.

Applying the multiplier $u^3 - 2\varepsilon^2 c^{-2} u_{xx}$ to (1) after some integrations by parts we arrive at the inequality

$$(9) \quad \frac{1}{4}|u|_4^4 + \varepsilon^2 c^{-2} |u_x|_2^2 + \varepsilon \int_0^T |uu_x|_2^2 + \varepsilon^3 c^{-2} \int_0^T |u_{xx}|_2^2 \leq$$

$$\leq -3a\sigma \int_0^T \int u_x u_{xx} u^2 + \sigma \int_0^T \left| \int i\xi\, p(\xi) \overline{\hat{u}(\xi)} \widehat{u^3}(\xi)\, d\xi \right|.$$

The second integral on the right hand side of (9) is estimated by $C \cdot \|u\|_2^2$ using Schwarz inequality and some properties of multiplication in Sobolev spaces like Lemma 10 in [1].

If a = 0 then the assumption $\sigma = o(\varepsilon^3)$ immediately implies (3). If $a > 0$ then a supplementary estimate is needed. Multiplying (1) by $\sigma H u + f(u)$ after rearrangements of terms and simple estimates we obtain

$$\frac{a}{2}\sigma |u_x|_2^2 + \frac{\sigma}{2} \int p(\xi) |\hat{u}(\xi)|^2\, d\xi + a\sigma\varepsilon \int_0^T |u_{xx}|_2^2 \leq$$

$$\leq C + \varepsilon \int_0^T \int |f'(u)| u_x^2 + \int |F(u)| \leq C(1 + |u|_\infty)$$

from (8) and assumptions on f, F, $F' = f$, and next $|u|_\infty \leq$ $\leq C \cdot \sigma^{-1/3}$. This allows to estimate the first term on the right hand side of (9) by expressions like

$\frac{\varepsilon^3}{2} c^{-2} \int_0^T |u_{xx}|_2^2$, $\varepsilon \int_0^T |uu_x|_2^2$. Finally (3) in the case $a > 0$ is also a consequence of (9) as

$$(10) \quad \frac{1}{4}|u|_4^4 + \frac{\varepsilon^3 c^{-2}}{2} \int_0^T |u_{xx}|_2^2 \leq C.$$

(5) and then (7) follow from (10) and $\delta = o(\varepsilon^3)$ - observing that $|Hu|_2 \le C \cdot \|u\|_2$.

Remark. A similar result on convergence of solutions of (1) in $L^{2(K+1)}$ with special nonlinearity $u^{2K}u_x$, $K \in \mathbb{N}$, holds if $\delta = O(\varepsilon^2)$. To see this, it suffices to multiply (1) by $\delta Hu + u^{2K+1}/(2K + 1)$, integrate and recall (8).

References

[1] J.C. SAUT: Sur quelques généralisations de l'équation de Korteweg-de Vries, J. Math. pures appl. 58(1979), 21-61.

[2] M.E. SCHONBEK: Convergence of solutions to nonlinear dispersive equations, Comm. P.D.E. 7(1982), 959-1000.

Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

(Oblatum 25.5. 1984)