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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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LARGE TIME BEHAVIOUR OF THE SOLUTIONS TO SOME NONLINEAR EVOLUTION EQUATIONS

A. HARAUX

Abstract: In this survey paper we describe some recent results about the qualitative behavior of solutions to some equations or systems of nonlinear partial differential equations in a bounded open domain of R^n

- Semi-linear heat-equation and reaction-diffusion systems
- Wave equation with dissipation and almost-periodic forcing term
- Semi-linear wave equation of conservative type and vibrating string with an obstacle.

Key words: Nonlinear equations, heat equation, wave equation, hyperbolic systems, global behaviour, periodic and almost periodic solutions.

Classification: 35B10, 35B15, 35B40, 35K05, 35L05

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0. Introduction. In this survey paper, we describe some recent advances concerning the behavior for large time of the solutions to some classes of partial differential equations represented by a dynamical system in a Banach space of functions defined on a bounded, open domain Ω of $\mathbb{R}^n$ with smooth boundary $\Gamma = \partial\Omega$.

The two basic models which will be studied are the non-linear heat equation

$$u_t - \Delta u = f(t,x,u) \text{ on } \mathbb{R}^+ \times \Omega$$

with Dirichlet or Neumann boundary conditions and the non-linear

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wave equation

$$u_{tt} - \Delta u = f(t,x,u,u_t) \text{ on } \mathbb{R}^+ \times \Omega$$
$$u = 0 \qquad \text{on } \mathbb{R}^+ \times \Gamma$$

As a rule, we shall not try to place our results in the most general framework available. On the contrary, we will try to be as specific as possible in each of the four following situations

1) Semi-linear heat equations.

2) Autonomous hyperbolic systems with a weak damping.

3) Periodic or almost-periodic quasi-autonomous hyperbolic equations with non-linear, local damping term.

4) Autonomous semi-linear wave equations of conservative type and vibrating string with obstacle.

1. C^1 estimates for semi-linear parabolic problems. We report here on a joint work with M. Kirane ([13]). Since a lot of papers have been written on nonlinear heat equations, there will be no attempt here to present a survey of the relevant literature and we focus on a very specific point which is: the obtention of bounds in $C^1(\bar{\Omega})$, uniform for $t \longrightarrow +\infty$, for the solutions of an equation

(1.1) $\qquad u_t - \Delta u = f(t,x,u(t,x)), \; t \geq 0, \; x \in \Omega$

with homogeneous boundary conditions (Dirichlet or Neumann).

For these kinds of semi-linear problems it is generally natural to study the existence of solutions in $X = C(\bar{\Omega})$. It is well-known that the dynamical system generated by (1.1) has very strong "smoothing properties", and the smoothing effect has been studied by many authors in various contexts.

A natural field of applications for the idea of smoothing effect is the study of global behavior for the solutions … action-diffusion systems arising in Chemistry and Biology. For such systems it currently happens that global existence and L^p-bounds are known for all the "interesting" solutions (like nonnegative solutions when the unknowns are supposed to represent the relative concentrations of chemical components).

The interesting question to be solved is then the behavior of the solutions as $t \rightarrow +\infty$. When the structure of the system is such that some "local" Liapunov functionals of the type

$$\Phi(u_1, u_2, \ldots, u_k) = \int_\Omega \varphi(u_1, u_2, \ldots, u_k)\,dx$$

exist at least for the "nonnegative solutions", then asymptotic behavior is attainable through "La Salle's invariance principle" provided that we can establish <u>precompactness of positive trajectories</u> in $X = C(\bar{\Omega})$.

As a consequence of Ascoli's Theorem, this will be satisfied if the trajectories are, for example, <u>uniformly bounded in</u> $[C^1(\bar{\Omega})]^k$ <u>for</u> $t \geq 0$.

The point we want to emphasize here is that such a result is easy to derive from L^∞-bounds (and even L^p-bounds with p big enough) under almost no smoothness assumption on the function f. Moreover, the method that we shall describe is somewhat <u>independent of the dimension</u> of Ω , in contrast with the methods pertaining to energy estimates which are used, for example, in the study of Navier-Stokes equation.

In fact, the idea is very simple and consists in using the smoothing properties of the <u>linear part</u> of (1.1) together with the variation of parameters formula.

More precisely, we have the following

__Theorem 1.1.__ Let $T(t)$ be the semi-group generated by the equation

(1.2) $\quad u_t - \Delta u = 0,\ t \geq 0,\ x \in \Omega$

with either Dirichlet or Neumann's boundary condition in $H = L^2(\Omega)$.

Then for any $u_o \in L^\infty(\Omega)$, we have

(1.3) $\quad \forall t > 0,\ T(t)u_o \in C^1(\bar{\Omega})$

$\forall \varepsilon > 0,\ \exists D(\varepsilon) \in \mathbb{R}^+$ such that $\forall t \in]0,1]$,

(1.4) $\quad \| T(t)u_o \|_{C^1(\bar{\Omega})} \leq D(\varepsilon) t^{-\frac{1}{2}-\varepsilon} \| u_o \|_{L^\infty(\Omega)}$

__Proof.__ This result is a simple consequence of the fact that $T(t)$ is __analytic in__ $L^p(\Omega)$ for all $p \in [2,+\infty[$, together with Gagliardo-Nirenberg interpolation-embedding inequalities. For details, cf. [13].

Now we consider equation (1.1) and we assume that f satisfies the following conditions

$f \in C(\mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R})$ with: $\forall (t,x,u,v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R}$

(1.5) $\quad |f(t,x,u) - f(t,x,v)| \leq k(t)\ C(|u|,|v|)|u-v|$

with $k \in L^1_{loc}(\mathbb{R}^+)$, C being bounded on bounded subsets of $\mathbb{R}^+ \times \mathbb{R}^+$

(1.6) $\quad \forall (t,x,u) \in \mathbb{R}^+ \times \bar{\Omega} \times \mathbb{R},\ |f(t,x,u)| \leq C_1(|u|)$

with C_1 bounded on bounded subsets of $\mathbb{R}^+$.

Then we have the following result.

__Theorem 1.2.__ Assume that f satisfies (1.5) and (1.6) and let u be a solution of (1.1) on $\mathbb{R}^+ \times \Omega$ (with either Dirichlet or Neumann boundary conditions). Assume that u satisfies

(1.7) $\quad u \in L^\infty(\mathbb{R}^+, L^\infty(\Omega))$

Then for any $\delta > 0$, we have

(1.8) $u \in C_B(\delta, +\infty ; C^1(\bar{\Omega}))$.

Proof: We give only a formal "sketch". The rigourous derivation of Theorem 1.2 can be found in [13].

Assume for simplicity $\delta = 1$. We have the formula

$$\forall t \geq 0,\ u(t+1) = T(1)u(t) + \int_0^1 T(\sigma) f(t+1-\sigma, x, u(t+1-\sigma, x))\, d\sigma$$

We set $M = \operatorname{Sup\,ess}_{t \geq 0} \|u(t,\cdot)\|_\infty$. Then for almost all $\sigma \in]0,1[$ we have $f(t+1-\sigma, \cdot, u(t+1-\sigma)) \in L^\infty(\Omega)$ and $\|f(t+1-\sigma, \cdot, u(t+1-\sigma, \cdot))\| \leq C_1(M)$

By using (1.4) with for example $\varepsilon = \frac{1}{4}$ we find that $u(t+1) \in C^1(\bar{\Omega})$ and

$$\|u(t+1)\|_{C^1(\bar{\Omega})} \leq D(\tfrac{1}{4}) \left[M + \int_0^1 \frac{C_1(M)}{\sigma^{\frac{3}{4}}}\, d\sigma\right]$$

$$\Rightarrow \operatorname{Sup}_{t \geq 0} \|u(t+1)\|_{C^1(\bar{\Omega})} \leq K(M).$$

Remarks 1.3. a) If in addition to (1.5)-(1.6), f satisfies the following "coerciveness" property

$$\exists C \in \mathbb{R}^+,\ |u| \geq C \Rightarrow \forall (t,x) \in \mathbb{R}^+ \times \Omega,\ f(t,x,u)u \leq 0,$$

then (1.7) is automatically satisfied for any solution of (1.1). This remark is very effective in practice.

b) If in (1.6) we have $C_1(\rho) = C_1 [1+\rho]$ (growth like an affine function), then (1.7) and (1.8) can be deduced under the following very weak assumption:

$u \in L^\infty(\mathbb{R}^+, L^1(\Omega))$ (cf. [13] for proof).

c) The combined use of Theorem 1.2 and the remarks above permits, as a simple application, to study the asymptotic behavior, as $t \to +\infty$, of the solutions to the system

$$(1.9)\quad \begin{cases} u_t - a\,\Delta u + r\,u\,v = 0, \ t \geq 0, \ x \in \Omega \\ v_t - b\,\Delta v - r\,u\,v + \lambda v = 0, \ t \geq 0, \ x \in \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \ t \geq 0, \ x \in \Gamma \end{cases}$$

where a, b, r, λ are positive constants and $u_0(x)$, $v_0(x)$ the initial data are assumed to be ≥ 0.

A treatment of (1.9) independent of $\dim(\Omega)$ was our original motivation for developing the method described above. This system is one of the possible models for describing the propagation of epidemics (cf. [16]).

d) It is possible to show that for example if $f \in C^\infty$, then $u(t,\cdot) \in C^\infty(\bar{\Omega})$ for $t > 0$ if $\partial\Omega$ is smooth. For this type of results, cf. [14].

2. Some wave equations with a "weak" nonlinear damping term. Let $g: \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a nondecreasing function. To such a function we associate a "multivalued mapping" $\beta : \mathbb{R} \to \mathcal{T}(\mathbb{R})$ defined by

$$\forall u \in \mathbb{R}, \ \beta(u) = [g^-(u), g^+(u)] \cap \mathbb{R} \text{ where}$$

$$g^-(u) = \operatorname*{Sup}_{w<u} [g(w)], \ g^+(u) = \operatorname*{Inf}_{w>u} [g(w)].$$

We say that β is the maximal monotone graph generated by g and we set

$$D(\beta) = \{u \in \mathbb{R}, \ \beta(u) \neq \emptyset\}.$$

It is clear that in the case where g is continuous:

$$\mathbb{R} \to \mathbb{R}, \text{ then } \beta = g \text{ and } D(\beta) = \mathbb{R}.$$

On the other hand, if $g: \mathbb{R} \to \mathbb{R}$ is any nonde easing function such that

$$g(w) = -1 \text{ for } w < 0, \ g(w) = +1 \text{ for } w > 0,$$

then the "graph" β is defined by

$$\begin{cases} \beta(u) = -1 \text{ for } w < 0 \\ \beta(0) = [-1,+1] \\ \beta(u) = +1 \text{ for } w > 0 \end{cases}$$

In this example, we can see clearly that <u>for any choice</u> $g(0) = \xi$, $-1 \leq \xi \leq +1$, the <u>graph</u> of the map $u \to g(u)$ in $\mathbb{R} \times \mathbb{R}$ is never closed. The interest of replacing g by the "multivalued mapping" β precisely consists in the fact that the graph $G(\beta) = \{(u,f) \in \mathbb{R} \times \mathbb{R},\ f \in \beta(u)\}$ is <u>always closed</u>.

In this section, we are interested in nonlinear multivalued partial differential equations of the form

$$(2.1) \qquad \begin{cases} u_{tt} - \Delta u + a(x)\beta(u_t) \ni 0, & t \geq 0,\ x \in \Omega \\ u = 0 & t \geq 0,\ x \in \partial\Omega \end{cases}$$

where β is the maximal monotone graph generated by some nondecreasing function g and $a(x) \geq 0$. We assume from now on that $0 \in \beta(0)$.

The usual method for solving (2.1) is to write it under the form of a system:

$$(2.2) \qquad \begin{cases} u \in C(\mathbb{R}^+, H_0^1(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega)) \\ u_t = v \\ v_t = \Delta u - a(x)\beta(v) \end{cases}$$

Then if we set $U(t) = (u(t,\cdot),v(t,\cdot))$ this system can be viewed as a (multivalued) evolution equation in the <u>energy space</u> $H = H_0^1(\Omega) \times L^2(\Omega)$.

If for example $a(x) \equiv 1$, the classical theory of maximal monotone operators (cf. [2],[3]) provides the existence of a <u>weak</u> solution of (2.2) for any initial data $U_0 = (u_0,v_0) \in H_0^1(\Omega) \times \mathcal{C}$ where

$$\mathcal{C} = \{v \in L^2(\Omega),\ v(x) \in \overline{D(\beta)} \text{ a.e. in } \Omega\}.$$

Our point here is not the study of the initial value problem associated to (2.2), but the analysis of asymptotic behavior of solutions as $t \to +\infty$. In fact, (2.2) generates a nonlinear semi-group of <u>contractive mappings</u> $S(t): C \to C$ with $C = H_0^1(\Omega) \times \mathscr{C}$ a closed convex subset of H.

It is possible to establish that for any $(u_0, v_0) \in C$ the set $\bigcup_{t \geq 0} U(t)$ is <u>precompact</u> in H. Then a theory of Dafermos and Slemrod ([5]) predicts that U(t) must be asymptotic, as $t \to +\infty$, to some <u>almost-periodic</u> solution of (2.2).

Now since $0 \in \beta(0)$, the functional

$$\Phi(U) = |U|_H^2 = \int_\Omega \{|\nabla u|^2 + |v|^2\} dx,$$

is a Liapunov functional for S(t). Hence any solution of (2.2) which is almost-periodic in H must satisfy

$$\Phi(U(t)) \equiv \Phi(U(0)), \quad \forall t \in \mathbb{R}.$$

By using this property together with some more specific properties of (2.1), the following result was obtained in 1978 ([7]).

<u>Theorem 2.1</u>. Assume that $0 \in \beta(0)$ and $a(x) \equiv 1$.

Let $u \in C(\mathbb{R}^+, H_0^1(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega))$ be any (weak) solution of (2.1). Then:

- If $g \equiv 0$ in an open neighbourhood of 0, we have

$$\lim_{t \to +\infty} \int_\Omega \{|\nabla u(t,x) - \nabla \xi(t,x)|^2 + |u_t(t,x) - \xi_t(t,x)|^2\} dx = 0$$

where ξ is a solution of

$$\xi \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)), \quad \xi_{tt} - \Delta \xi = 0 \text{ on } \mathbb{R} \times \Omega$$

such that $\xi_t(t,x) \in \beta^{-1}(\{0\})$ a.e. on $\mathbb{R} \times \Omega$.

- If $g(v)$ is not identically 0 in any neighbourhood of 0, then $u(t,x)$ is asymptotic in $H_0^1(\Omega)$ strong, as $t \to +\infty$, to a function $\varsigma(x)$ such that $\varsigma \in H_0^1(\Omega)$ and $\Delta \varsigma(x) \in \beta(0)$ in the sense of $\mathcal{D}'(\Omega)$, i.e. $g^+(0) - \Delta\varsigma$ and $\Delta\varsigma - g^-(0)$ are nonnegative measures on Ω whenever they are defined (if $g^+(0)$ or $g^-(0)$ is infinite, the corresponding condition disappears).

Remark 2.2. a) Theorem 2.1 is optimal in the sense that any element of the form $\varsigma(t,x)$ or $\varsigma(x)$ satisfying the conditions above represents a particular solution of (2.1). (Respectively a common solution with the wave equation and an "equilibrium solution".)

b) In [6] C.M. Dafermos considered the case where $a(x) \neq$ cte and g is Lipschitz-continuous together with its inverse g^{-1}. Then if for example $a(x) \in C(\Omega)$ and $a(x_0) > 0$ somewhere in Ω, all solutions of (2.1) tend to 0 as $t \to +\infty$.

The following generalization has been obtained in 1983 (cf. [11]).

Theorem 2.3. Assume that $a \in L^\infty(\Omega)$ and moreover

(2.3) $\quad \text{mes}\,\{x \in \Omega, a(x) > 0\} \neq 0$

(2.4) $\quad \exists C \in \mathbb{R}^+, \ \forall v \in \mathbb{R}, \ |g(v)| \leq C(1 + |v|)$

(2.5) $\quad$ g(v) is not identically zero in any neighbourhood of 0.

Then any weak solution of (2.1) converges in $H_0^1(\Omega)$ strong, as $t \to +\infty$, to some $\varsigma(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

(2.6) $\quad \Delta\varsigma(x) \in a(x)\,\beta(0)$ a.e. on Ω.

Proof: Let $E = \{x \in \Omega, a(x) > 0\}$. By differentiation of $\Phi(U) = \int_\Omega \{|\nabla u|^2 + |u_t|^2\}dx$ it is easy to see that any almost-periodic strong solution of (2.1) must satisfy

(2.7) $\quad u_t = 0$ a.e. on $\mathbb{R} \times E$

(here we use (2.5) in an essential manner).

Thus $-\Delta u(t,x) \equiv h(x)$ a.e. on $\mathbb{R}\times E$.

Let

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x\in E \\ 0 & \text{if } x\notin E \end{cases}$$

and define $\tilde{v}$ such that $\tilde{v}\in H_0^1(\Omega)\cap H^2(\Omega)$, $-\Delta\tilde{v} = \tilde{h}$.

Then $\tilde{u} = u - \tilde{v}$ and we have

$$(2.8) \qquad \begin{cases} \Box\tilde{u} = 0 \text{ on } \mathbb{R}\times\Omega \\ \tilde{u}_t = 0 \text{ on } \mathbb{R}\times E. \end{cases}$$

Finally a Fourier analysis (cf. [6]) shows that any solution of (2.8) is in fact identically zero. Hence any <u>strong</u> almost-periodic solution of (2.1) is in fact an equilibrium solution. Thus the result of Theorem 2.3 is true if $(u_0,v_0)\in H^2(\Omega)\cap H_0^1(\Omega)\times \times H_0^1(\Omega)$. The general case follows by a standard completion argument.

<u>Remark 2.4</u>. The case $\Omega =]0,\ell[$, with $g(v) = -1$ if $v<0$, $g(v) = +1$ if $v>0$ can be viewed as the free oscillations of a string with fixed ends which is submitted to a "distributed dry friction" acting only <u>locally</u>. In this case, Theorem 2.3 says that the string tends to an <u>equilibrium position</u> which moreover is <u>rectilinear</u> on each component of the <u>unconstrained region</u>.

3. <u>Damped wave equations with periodic or almost-periodic forcing term</u>. In this section, we consider the nonlinear (possibly multivalued) partial differential equation

$$(3.1) \begin{cases} u_{tt} - \Delta u + \beta(u_t) \ni f(t,x), & t\geq 0,\ x\in\Omega \\ u = 0 & t\geq 0,\ x\in\Gamma . \end{cases}$$

The theory of Cauchy problem for (3.1) is classical (cf.[2], [3]) for any $f\in L^1_{loc}(\mathbb{R}^+,L^2(\Omega))$. Existence of periodic or almost periodic solutions of (3.1) has been studied in [8] and [9]. There are still open problems in this direction and we will not discuss this question here.

Throughout this section, we will assume that the following hypotheses are satisfied

(3.2) The function $f(t,\cdot)$: $\mathbb{R} \to L^2(\Omega)$ is S^1-almost periodic.

(3.3) There exists at least one solution u of (3.1) defined for $t \in \mathbb{R}$ and such that $U(t)$: $\mathbb{R} \to H$ is (strongly) almost periodic. (If f is periodic with period $T > 0$ we assume that there is a T-periodic solution of (3.1).)

From now on we shall denote by $\omega(t,x)$ one of the almost periodic (resp. T-periodic) solutions of (3.1) and we consider the two following questions:

- Asymptotic behavior of the other solutions as $t \to +\infty$.
- Uniqueness of almost-periodic solutions.

The two questions are obviously related. The main difficulty in answering the first one is the fact that precompactness of trajectories to equation (3.1) in the energy space is until now unknown except when quite strong hypotheses are done on the function g (cf. [9], Theorem 4.1, p. 206).

The following general results have been stated in this form only in 1981. They somehow generalize Theorem 2.1 to the case of quasi-autonomous equation (3.1).

Theorem 3.1. If g is strictly increasing (i.e.

$$\forall u_1 \in D(\beta),\ \forall u_2 \in D(\beta),\ u_1 < u_2 \Rightarrow g(u_1) < g(u_2))$$

then for any solution u of (3.1) there exists $\xi(x) \in H_0^1(\Omega)$ such that $\omega(t,x) + \xi(x)$ satisfies (3.1) and

(3.4) $$\lim\ (u(t,x) - \omega(t,x) - \xi(x)) = 0 \text{ in } H_0^1(\Omega) \text{ weak.}$$

Theorem 3.2. If g is continuous with $D(\beta) = \mathbb{R}$, then for any solution u of (3.1) there exists $\varsigma(t,x)$ such that $\omega(t,x) + \varsigma(t,x)$ satisfies (3.1) and

(3.5) $$\lim_{t\to+\infty}(u(t,x) - \omega(t,x) - \varsigma(t,x)) = 0 \text{ in } H_0^1(\Omega) \text{ weak.}$$

(3.6) $$\varsigma \in C(\mathbb{R},H_0^1(\Omega))\cap C^1(\mathbb{R},L^2(\Omega)) \text{ and } \varsigma_{tt} - \Delta\varsigma = 0 \text{ on } \mathbb{R}\times\Omega.$$

Remark 3.3. a) The proofs of Theorems 3.1 and 3.2 are very technical and will not be given here (cf. [9] for the details),

b) If g is continuous with $D(\beta) \neq \mathbb{R}$, we do not know whether the result of Theorem 3.2 still holds true.

c) Theorems 3.1 and 3.2 immediately imply the following result.

Corollary 3.4. If g is continuous and strictly increasing with $D(\beta) = \mathbb{R}$, then for any solution u of (3.1) we have

(3.7) $$\lim_{t\to+\infty}(u(t,x) - \omega(t,x)) = 0 \text{ in } H_0^1(\Omega) \text{ weak.}$$

This result can be generalized in two possible ways

Theorem 3.5. Assume that g is strictly increasing with $D(\beta) = \mathbb{R}$ and $\beta(0) = \{0\}$. If in addition we have

(3.8) ω_t is absolutely continuous: $\mathbb{R} \to L^2(\Omega)$,

then (3.7) is satisfied for any solution u of (3.1).

Theorem 3.6. Assume that g is continuous with $D(\beta) = \mathbb{R}$. Assume moreover that we have

(3.9) There exists an open neighbourhood V of 0 in $\mathbb{R}$ such that

$$u_1 \in V,\ u_2 \in V,\ u_1 > u_2 \Rightarrow g(u_1) > g(u_2)$$

(3.10) $\omega_t \in L^\infty(\mathbb{R}, H^1_0(\Omega) \cap C^{0,\alpha}(\Omega)),\ \alpha > 0$

Then (3.7) is satisfied for any solution u of (3.1).

Proof of Theorem 3.5. Let u be a solution of (3.1) of the form $\omega(t,x) + \xi(x)$. From $D(\beta) = \mathbb{R}$ we deduce that $\Box u$ and $\Box\omega$ are in $L^1_{loc}(\mathbb{R}, L^1(\Omega))$. If $\xi = u - \omega \not\equiv 0$, let $A \subset \Omega$ and $\delta > 0$ be such that meas(A) > 0 and $|\Delta\xi(x)| \geq \delta$ a.e. on A. Then $\omega_t(t,x)$ remains in a discrete set not containing 0 for $(t,x) \in \mathbb{R} \times A$. This is easily shown to contradict (3.8).

Proof of Theorem 3.6. Let u(t,x) be a solution of (3.1) of the form $\omega(t,x) + y(t,x)$ where

$$y \in C(\mathbb{R}, H^1_0(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \text{ and } \Box y = 0 \text{ on } \mathbb{R} \times \Omega.$$

We have: a.e. on Ω, $g(u_t(t,x)) = g(\omega_t(t,x))$.

Now let W be a neighbourhood of Γ such that

$$x \in W \Rightarrow \forall t \in \mathbb{R},\ \omega_t(t,x) \in V.$$

Since V is open, we deduce: $u_t(t,x) = \omega_t(t,x)$ a.e. on $\mathbb{R} \times W \Rightarrow$ $\Rightarrow y_t(t,x) = 0$ on $\mathbb{R} \times W$. The end of the proof is identical to that of Theorem 2.3.

Remark 3.7. It is natural to ask whether the regularity conditions in the statements of Theorem 3.5 and 3.6 can be dropped to obtain the same conclusion. The answer turns out to be no: even when $\Omega =]0,1[$ it is possible to construct explicit counter-examples of non-uniqueness (cf. [10] and [11]).

4. Undamped oscillations for some nonlinear wave equations. It is well-known that the solutions of the ordinary wave equation with Dirichlet boundary conditions in a bounded open domain

Ω are almost periodic in the energy space and admit a generalized Fourier representation of the form:

$$u(t,x) \sim \sum_{n=1}^{+\infty} \cos(\sqrt{\lambda_n}\, t + \alpha_n) w_n(x)$$

where $\{\lambda_n\}$ is the sequence of eigenvalues of $(-\Delta)$ in $H_0^1(\Omega)$ and w_n is a solution of $-\Delta w_n = \lambda_n w_n$, $\forall n \in \mathbb{N}$.

This representation is useful to study the oscillatory character of the solutions and also the transmission of oscillations (cf. for example the argument in the proofs of Theorems 2.3 and 3.6).

In this section we describe some recent results concerning the oscillatory properties of the solutions to two different kinds of "nonlinear perturbations" of the wave equation.

<u>Example 1</u>: <u>semilinear perturbations</u>. We report on a recent joint work with T. Cazenave ([4]).

Let $g: \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ be sufficiently regular with respect to the third variable u. We consider the nonlinear problem

$$(4.1) \qquad \begin{cases} u \in C(J, H_0^1(\Omega)) \cap C^1(J; L^2(\Omega)) \\ u_{tt} - \Delta u + g(t,x,u(t,x)) = 0 \text{ on } J \times \Omega \end{cases}$$

where Ω is (for simplicity) assumed to be connected.

We start with a general result.

<u>Proposition 4.1</u>. Let λ_1 be the smallest eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$ and assume that $g \in C^1(\mathbb{R} \times \overline{\Omega} \times \mathbb{R})$ is such that

$$\exists\, \gamma \geq 0,\ (n-2)\gamma \leq 4,\ |g_u(t,x,u)| \leq C(1 + |u|^{\gamma}) \text{ on } \mathbb{R} \times \overline{\Omega} \times \mathbb{R}$$

$$\forall (t,x,u) \in \mathbb{R} \times \overline{\Omega} \times \mathbb{R},\ g(t,x,u)\, u \geq 0.$$

Then if u is a solution of (4.1) such that $u(t,x) \geq 0$ a.e. on

$J \times \Omega$, we must have either $u \equiv 0$, or $|J| \leq \frac{\pi}{\sqrt{\lambda_1}}$.

<u>Idea of the proof</u>: let $\varphi(x) \in C^{\infty}(\Omega)$ be such that

$$\varphi \in H_0^1(\Omega), \quad -\Delta\varphi = \lambda_1\varphi \quad , \quad \int_\Omega \varphi(x)\,dx = 1.$$

On multiplying the equation by $\varphi(x)$ and integrating over Ω , we obtain that if $u(t,x) \geq 0$ a.e. on $J \times \Omega$, then

$$\frac{d^2}{dt^2}\left(\int_\Omega u(t,x)\,\varphi(x)\,dx\right) \leq -\lambda_1 \int_\Omega u(t,x)\,\varphi(x)\,dx \text{ on } J.$$

This inequality is easily shown to imply either $|J| \leq \frac{\pi}{\sqrt{\lambda_1}}$, or

$$\int_\Omega u(t,x)\,\varphi(x)\,dx \equiv 0 \text{ on } J.$$

Since $\varphi > 0$ in Ω , proposition 4.1 is proved.

When $\Omega =]0,1[$, much more precise results are available. We can for example state the following

<u>Theorem 4.2</u>. Let $g \in C^1(\mathbb{R})$ be nondecreasing and such that $g(-u) = -g(u)$, $\forall u \in \mathbb{R}$. Let u be a solution of

$$(4.2) \qquad \begin{aligned} &u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega)) \\ &u_{tt} - u_{xx} + g(u) = 0 \text{ on } \mathbb{R} \times \Omega \end{aligned}$$

Then we have the following results

1) If $u \not\equiv 0$ on $\mathbb{R} \times \Omega$ and $x_0 \in \Omega \setminus \mathbb{Q}$, then for any $\alpha \in \mathbb{R}$ there exists t_1 and t_2 in $[\alpha, \alpha+2]$ such that $u(t_1,x_0) > 0$ and $u(t_2,x_0) < 0$.

2) As $t \to +\infty$ we have the following alternative

- Either $\lim_{t\to+\infty} \|u(t)\|_{L^\infty(\Omega)} = 0$
- Or for any $x_0 \in \Omega \setminus \mathbb{Q}$

$$\liminf_{t\to+\infty} u(t,x_0) < 0 < \limsup_{t\to+\infty} u(t,x_0).$$

Remark 4.3. a) In fact the function $t \longrightarrow u(t,x)$ must take both positive and negative values on any interval of the form $[\alpha, \alpha+2]$ except for the points x of a finite set (containing $\{0,1\}$).

b) If for some point $x_0 \in]0,1[$, $x_0 \notin \{0,1\}$ the property above is not satisfied, then $u(t,x_0) \equiv 0$ on $\mathbb{R}$ and $u(t,2x_0-x) = -u(t,x)$ for all $x \in]0,1[$ such that $2x_0 - 1 \leq x \leq 2x_0$.

c) We do not know whether some nontrivial solutions of (4.2) effectively tend to 0 as $t \longrightarrow +\infty$.

In view of the energy conservation for (4.2) if this happens it must correspond to rather complicated oscillation phenomena.

For the proof of Theorem 4.2, cf. [4].

Example 2: vibrating string with an obstacle. In the plane Oxu, we consider the oscillations of a vibrating string with fixed ends $\pm \frac{1}{2}$ in presence of a fixed obstacle $\{u = -h\}$ against which the string is subject to rebound without energy loss.

The correct formulation of this problem (in case of "small oscillations") has been given in 1980 by M. Schatzman ([15]). The displacement u(t,x) is a solution of the "singular hamiltonian system" (with $\Omega =]-\frac{1}{2}, +\frac{1}{2}[$):

$$(4.3) \begin{cases} u \in C(\mathbb{R}, H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}, L^2(\Omega)) \\ u \geq -h \text{ in } \mathbb{R} \times \Omega \\ u_{tt} - u_{xx} \geq 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \Omega) \\ \mathrm{Supp}(u_{tt}-u_{xx}) \subset \{(t,x),\ u(t,x) = -h\} \\ \frac{\partial}{\partial x}\{-2u_t u_x\} + \frac{\partial}{\partial t}\{|u_t|^2 + |u_x|^2\} = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \Omega) \end{cases}$$

These conditions allow to prove an existence and uniqueness Theorem for the initial value problem under the "compatibility hypotheses" $u_0 \geq -h$ a.e. in $\mathbb{R} \times \Omega$ and $u_t(0,x) = v_0(x) = 0$ a.e. on $\{u_0 = -h\}$. The solution conserves the energy in a relevant sense and it is therefore natural to wonder whether the motion is almost periodic in some natural sense.

The compatibility conditions are automatically satisfied if we assume for example

(4.4) $\quad u_0 \geq 0$ a.e. in Ω , $v_0 = 0$ a.e. in Ω .

In such a case the solution is even as a function of t.

We now report on the results obtained in our joint work with H. Cabannes ([12]).

<u>Theorem 4.4.</u> If $u_0 \in H_0^1(\Omega)$ satisfies the following assumptions for some $a \in \Omega$

u_0 is non-decreasing and $u_0(x) < 1$ on $[-\frac{1}{2}, a[$

$u_0(a) = 1$

u_0 is non-increasing and $u_0(x) < 1$ on $]a, \frac{1}{2}]$

then the solution of (4.3) with initial data $(u_0, 0)$ is such that the function $t \longrightarrow u(t,x)$ is (strongly) almost periodic from $\mathbb{R}$ to $H_0^1(\Omega)$.

<u>Theorem 4.5.</u> a) If $h = \frac{p}{q}$, $p \in \mathbb{N}^*$, $q \in \mathbb{N}^*$, $p \cap q = 1$, the motion is periodic with as a period the integer $p + q$ if $p + q$ is even, $2(p+q)$ if $p + q$ is odd.

b) If h is irrational, the motion is <u>not periodic</u>, except in the <u>single case</u> $u_0(x) = 1 - 2|x|$. In that case the motion is periodic with period $1 + h$.

<u>Remark 4.6.</u> a) The proofs of Theorems 4.4 and 4.5 consists in almost "computing" the solution u(t,x): hence they are not

very instructive and will not be given here.

b) The solution $u(t,x)$ starting from an initial datum $(u_0,0)$ as above must oscillate "at least as fast as" the solutions of the free wave equation $u_{tt} - u_{xx} = 0$ in a global sense. Indeed, none of the inequalities

$$u(t,x) \geq 0 \text{ on } \Omega \quad \text{or} \quad u(t,x) \leq 0 \text{ on } \Omega$$

can be satisfied on a time interval of length > 1 except if $u(t,x) \equiv 0$ on $\mathbb{R} \times \Omega$. The proof is analogous to that of Proposition 4.1.

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