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ON THE 1-GENERATED S-NEAR-FIELDS

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Abstract: We continue here the work [5] and study the s-near-fields N that are not the sum of near-fields: they are 1-generated. Making the appropriate assumptions on the additive group we characterize the zero-symmetric case, give some examples and conclude with a characterization of the constant 1-generated s-near-fields.

Key words: Near-rings, subnear-rings, near-fields, s-near-fields, En-generated.

Classification: 12K05, 16A46

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1. Introduction. In [5] we define s-near-fields, the near-rings whose proper subnear-rings are near-fields and discuss the cases in which such structures are the sum of near-fields. Here we study the s-near-fields N that are not the sum of near-fields: they are 1-generated and making the appropriate assumption on the additive group, we show that, in the zero-symmetric case, N is an s-near-field if and only if: Np is a near-field whose subnear-rings are near-fields, N is abelian having p^2 exponent and each element $x \in N \setminus Np$ generates N, or Np is the torsion subgroup of N^+ and is a near-field whose subnear-rings are near-fields and each element $x \in N \setminus Np$ generates N.
We give some examples concerning the first case and in the second we prove that the additive group N^+ is a semi-direct sum of Np^+ and a divisible group. We conclude with a characteri-

zation of the constant 1-generated s-near-fields.

2. Preliminaries. We will indicate with N a left near-ring; for the definitions and the fundamental notations we refer to [6] without an express recall.

Definition A. We call s-near-field a near-ring whose proper subnear-rings are near-fields.

In the following, we consider of course near-rings with proper subnear-rings.

Later on, we will say a near-ring N is n-generated if it can be generated by n elements; we will say a near-ring exactly n-generated (and we will write En-generated) if it has a system of n generators, but it cannot be generated by a system of n-1 elements. Moreover, for $M \subseteq N$ we will indicate with $\langle M \rangle$ the subnear-ring of N generated by M. Remember that here the general results on the s-near-fields (see [5]) exist; however, we repeat:

Proposition 1. An s-near-field is at most E2-generated.

Proof: see [5] Prop. 1

Proposition 2. The N-subgroups and the left ideals of an s-near-field are maximal.

Proof: see [5] Prop. 2.

Prop. 2 allows us to extend the results of [2, 3] to our case. We start with the zero-symmetric case and examine the cases excluded in [5].

3. Zero-symmetric case

Proposition 3. A zero-symmetric s-near-field, if it has nilpotent elements, is 1-generated and it is generated by each nilpotent element, which has index $n = 2$ and is left annihilator of N.

Proof: a nilpotent element of N cannot generate a proper subnear-ring because, according to our hypotheses, it must be a near-field, thus it generates N that is in this way 1-generated (see Prop. 1 of [5]). Moreover let x be a nilpotent element of N and thus $x^n = 0$ for an integer n. The set $A_d(x) = \{y \in N | xy = 0\}$ has at least the element x^{n-1}, therefore it is not null and it is a proper subnear-ring of N. In our hypotheses it is a near-field, but this is excluded because it has the nilpotent element x^{n-1}. Therefore $A_d(x) = N$, $x^2 = 0$ and $xN = \{0\}$.

Besides:

Proposition 4. A zero-symmetric s-near-field N with nilpotent elements is:

a. without proper N-subgroups;

b. $nN = N \Longleftrightarrow A_d(n) = \{y \in N \mid ny = 0 \ = \{0\}$.

Proof a: we will show that $\forall n \in N$ is $nN = N$ or $nN = 0$ and the thesis will follow from this. Let us suppose $\{0\} \neq nN \subset N$; in this case nN being a proper subnear-ring is a near-field. If N has a nilpotent element x, x generates N (see Prop. 3), $x^2 = 0$ and it is a left annihilator of N, therefore nN cannot be a near-field ($\forall n \in N$) because the element $nx \in nN$, if $nx \neq 0$, cannot belong to a near-field, but nx cannot equal 0 because this would give $nN = \{0\}$ as x generates N.

Proof b: let n be an element such that $nN = N$; if $A_d(n)$ is proper, it is an N-subgroup of N and this is excluded from a. Viceversa, if $A_d(n) = \{0\}$, obviously $nN = N$, thus the Proposition is proved.

Corollary 1. A zero-symmetric s-near-field, if it has nilpotent elements is:

a. N-simple, strongly monogenic, faithful and 2-primitive; moreover:

b. the semigroup (N,·) is the union of a right group and of $A_s(N) = \{x \in N \mid xN = \{0\}\}$.

Proof a: it follows immediately from Prop. 4 if we recall the definitions of N-simple, strongly monogenic, faithful and 2-primitive near-ring (see [6]).

Proof b: it follows from the Theor. 4.3 of [8], if we recall the definition of a right group (see for instance [1]).

Similar results exist for the integer 1-generated s-near-fields, as we can see in Prop. 5 of [5]. In order to characterize the zero-symmetric 1-generated s-near-fields we shall first show the following:

Lemma 1. If N is a zero-symmetric strongly monogenic s-near-field, such that each of its elements has odd order, it is abelian.

Proof: if each element $z \in N$ has odd order q, each equation $x + x = z$ has the solution $((q+1)/2)z$. This solution is the only one: if $x + x = y + y = z$ we have $2x = 2y = z$. If r (odd) is the order of x, $(r-1)x = (r-1)y$ (because r-1 is even) and then $(r-1)x + 2x = (r-1)y + 2y$ and $x = (r+1)y$. Therefore $(r+1)y + (r+1)y = 2y$ and thus $2ry = 0$. The order of y is also odd, thus $ry = 0$, but $x = (r+1)y$. It follows that $x = y$. From the hypothesis, at least one proper subnear-ring and thus a near-field exist in N, and its identity $\underline{e}$ generates a field isomorphic to Z_p. Moreover from Prop. 4 it follows that such an identity $\underline{e}$ is a left identity of N. In fact $\underline{e}N = N$ implies $\underline{e}\underline{e}x = \underline{e}x$ and $\underline{e}(\underline{e}x-x)=0$, so $\underline{e}x = x \ \forall x \in N$. Let $\underline{e}$ be a such left identity and $f: N \longrightarrow N$ the map so defined $f(y) = (-\underline{e})y \ \forall y \in N$. This map is an automorphism of N^+ because it is obviously a homomorphism, moreover, it is a monomorphism because $f(y) = f(y') \Longrightarrow (-\underline{e})y = (-\underline{e})y' \Longrightarrow$ $\Longrightarrow (-\underline{e})y - ((-\underline{e})y') = 0 \Longrightarrow (-\underline{e})y + (-\underline{e})(-y') = 0 \Longrightarrow$ $\Longrightarrow (-\underline{e})(y-y') = 0 \Longrightarrow y = y'$ as $A_d(-\underline{e}) = \{0\}$ (see Prop. 4). Lastly it is an epimorphism because $(-\underline{e})N = N$ and therefore $\forall z \in N \ \exists x \in N \mid (-\underline{e})x = z$. This automorphism is fixed point-free: if it is $(-\underline{e})x = x$ and $xN = N$, it is $(-\underline{e})xy = xy \ \forall y \in N$, and $-\underline{e}$ is a left identity of N because the product xy, while y varies in N, describes N and thus $\underline{e} = -\underline{e}$ and this is to be excluded because $\underline{e}$ has an odd order. If $(-\underline{e})x = x$ and $xN = \{0\}$, x is a generator of N and again $(-\underline{e})z = z \ \forall z \in N$ and this again is absurd. In this way the hypotheses of the theorem of [4] hold and N is abelian.

We know that the identity of a proper subnear-ring of an s-near-field generates a finite field isomorphic to Z_p. Therefore, fron now on, we will consider as non trivial, the set of the elements of order p and more generally we will indicate $Np^n = \{x \in N \mid p^n x = 0\}$ where n is an integer.

Lemma 2. If N is a zero-symmetric 1-generated s-near-field and if the group generated by the elements of order p, $\langle Np \rangle^+$, is a proper subgroup of N^+, then $\langle Np \rangle^+$ is a left ideal of N and coincides with the set of the elements of order p, Np.

Proof: the group $\langle Np \rangle^+$ is a normal subgroup of N^+: $\forall x \in \langle Np \rangle^+$ it is $x = x_1 + x_2 + \ldots + x_n$ with $px_i = 0 \ \forall i \in \{1,2,\ldots,n\}$ and then $\forall z \in N$, $z + x - z = z + x_1 + x_2 + \ldots + x_n - z = (z + x_1 - z) + (z + x_2 - z) + \ldots + (z + x_n - z)$ and this sum belongs to $\langle Np\rangle^+$. Moreover, $\langle Np\rangle^+$ is a left ideal of N: $\forall\, z\in N$ and $\forall\, x\in\langle Np\rangle^+$, $zx\in\langle Np\rangle^+$. Thus $\langle Np\rangle^+$ is a near-field and therefore is abelian, and $\langle Np\rangle^+ = Np$.

In the following we will indicate with $\overline{N}$ a zero-symmetric 1-generated strongly monogenic near-ring without elements of order 2 whose additive group $\overline{N}^+$ is not perfect $^{(o)}$.

Lemma 3. If $\overline{N}$ is an s-near-field in which the set $\overline{N}p$ of the elements of order p is a proper subgroup of $\overline{N}^+$, then it does not have elements of order $q\neq p$.

Proof: from Lemma 2 we know that if $\langle\overline{N}p\rangle^+$ is proper, it is a left ideal of $\overline{N}$. Let us suppose that elements of prime order q exist in $\overline{N}$ and let $\langle\overline{N}q\rangle^+$ be the subgroup generated by the elements of order q. There are two cases:

1. $\langle\overline{N}q\rangle^+\subset\overline{N}^+$; 2. $\langle\overline{N}q\rangle^+ = \overline{N}^+$.

both will be shown to be absurd.

1. Let $\langle\overline{N}q\rangle^+\subset\overline{N}^+$. Thus $\langle\overline{N}q\rangle^+$ is a left ideal - see Lemma 2 - and $\langle\overline{N}q\rangle^+ = \overline{N}q$ (set of the elements of order q). Let us consider now the near-rings generated by the identities $\underline{e}$ and $\underline{e}'$ of $\overline{N}p$ and $\overline{N}q$. We have $\langle\underline{e}\rangle\cong Z_p$ and $\langle\underline{e}'\rangle\cong Z_q$. If $p<q$, let f be the following map: $f:(\overline{N}p,.)\longrightarrow(\overline{N}q,.)$ so defined: $f(n\underline{e}) = (n\underline{e})\underline{e}'$, for $n = 1,2,\dots,p-1$. This map is a homomorphism because $f((n\underline{e})(n'\underline{e})) = (n\underline{e})(n'\underline{e})\underline{e}' = (n\underline{e})\underline{e}'(n'\underline{e})\underline{e}' = f(n\underline{e})f(n'\underline{e})$. Therefore, in $(\overline{N}q,.)$, a subgroup $f(Z_p,.)$ isomorphic to cyclic group of order p-1, exists. Let us take now the map $\varphi: f(Z_p,.)\rightarrow\longrightarrow(Z_q,.)$ defined by $\varphi:(n\underline{e})\underline{e}'\longrightarrow n\underline{e}'$ for $n = 1,2,\dots,p-1$.

(o) A group is said perfect if it coincides with the derived group.

This map is a homomorphism as well, thus a subgroup of $(Z_q,\cdot)$ exists but this is impossible for a prime p where $p\neq 2$. Elements of order 2 do not belong to $\bar{N}$ and thus case 1. can be excluded.

2. Let $\langle \bar{N}q\rangle^+ = \bar{N}^+$. Each element $g\in\bar{N}$ is $g = g_1 + g_2 + \dots + g_n$ with $qg_i = 0$ $\forall\, i\in\{1,2,\dots,n\}$. We know that $\bar{N}^+$ is non perfect. Thus $\bar{N}' = \{0\}$ or $\bar{N}'$ is a proper subgroup of $\bar{N}^+$. We can easily verify that if $\bar{N}'$ is a proper subgroup of $\bar{N}^+$, it is a left ideal of $\bar{N}$ and thus $\bar{N}' = \bar{N}p$ (see case 1). Therefore $\bar{N}^+/\bar{N}'$ is abelian and $\forall\, g\in\bar{N}$, $qg + \bar{N}' = q(g_1 + g_2 + \dots + g_n) + \bar{N}' = (qg_1 + \bar{N}') + (qg_2 + \bar{N}') + \dots (qg_n + \bar{N}') = \bar{N}'$. Then $\forall\, g\in\bar{N}$, $qg\in\bar{N}'$, and $pqg = 0$. In this way we have shown that each element of N has odd order and so $\bar{N}$ must be abelian (see Lemma 1). This is absurd, it cannot be the case that $\bar{N}'\neq\{0\}$. Anyway $\bar{N}$ is abelian and it is a group of exponent q which cannot have elements of order p. Therefore the case 2 is excluded.

Theorem 1. A near-ring $\bar{N}$, whose additive group is not an abelian group of exponent p, is an s-near-field if and only if it satisfies one of the following conditions:

1. $\bar{N}p$ is a near-field whose subnear-rings are near-fields, $\bar{N}$ is abelian, having a characteristic p^2 and each $x\in\bar{N}\setminus\bar{N}p$ generates $\bar{N}$.
2. $\bar{N}p$ is the torsion subgroup of $\bar{N}$, is a near-field whose subnear-rings are near-fields and each element $x\in\bar{N}\setminus\bar{N}p$ generates $\bar{N}$.

Proof: let $\bar{N}p$ be the set of the elements of order p. Suppose that $\langle\bar{N}p\rangle^+ = \bar{N}^+$. From the hypotheses $\bar{N}^+$ is not perfect: if $\bar{N}'$ is a proper subgroup, it is also a left ideal, whose elements have prime order q. Thus $\bar{N}^+/\bar{N}'$ is abelian and $\forall\, g \in N \;\; pg + \bar{N}' = p(g_1 + g_2 + \ldots + g_n) + \bar{N}' = pg_1 + \bar{N}' + \ldots + pg_n + \bar{N}' = \bar{N}'$ and $pqg = 0$. Each element of $\bar{N}$ has odd order and so $\bar{N}$ is abelian (see Lemma 1). Therefore $\bar{N}'$ is an abelian group of exponent p and this is excluded by the hypotheses. Thus $\langle \bar{N}p \rangle^+$ is a proper subgroup of $\bar{N}^+$, and $\langle \bar{N}p \rangle^+ = \bar{N}p$ (see Lemma 2) and it is a left ideal, that is, a near-field whose subnear-rings are near-fields. By Lemma 3 we know that elements of prime order different from p do not exist in $\bar{N}$. Suppose that $\bar{N}$ has elements of p-power order: let $\bar{N}p^n = \{x \in N \mid p^n x = 0\}$. The group $\langle \bar{N}p^n \rangle^+$ generated by $\bar{N}p^n$ is normal: in fact $\forall\, z \in \bar{N}$, $\forall\, x \in \langle \bar{N}p^n \rangle^+$, $z + x - z = z + x_1 + x_2 + \ldots x_n - z$ with $p^n x_i = 0 \;\forall\, i \in \{1, 2, \ldots, n\}$. Thus $z + x - z = (z + x_1 - z) + (z + x_2 - z) + \ldots + (z + x_n - z)$ and this sum belongs to $\langle \bar{N}p^n \rangle^+$. Moreover $\langle \bar{N}p^n \rangle^+$ is a left ideal because $\forall\, z \in \bar{N}$ and $\forall\, x \in \langle \bar{N}p^n \rangle^+$, $zx \in \langle \bar{N}p^n \rangle^+$. Then $\langle \bar{N}p^n \rangle = \bar{N}p^n$, but the left ideals of $\bar{N}$ are maximal (see Prop. 2), therefore $\bar{N}p^n = \bar{N}p \neq \bar{N}$, $\bar{N}p$ is a near-field whose subnear-rings are near-fields, $\bar{N}$ is abelian (see Lemma 1) and each element $x \in \bar{N} \setminus \bar{N}p$ generates $\bar{N}$. We are in the case 1. If $\bar{N}p^n = \emptyset$ for $n \geq 2$, the elements of $\bar{N} \setminus \bar{N}p$ are torsion free and each of them generates $\bar{N}$; we are in case 2. Conversely, the proof is trivial.

<u>Examples</u>: As an additive group we consider C_9, cyclic group of order 9 and we define the following products:

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	8	7	6	5	4	3	2	1
3	0	1	2	3	4	5	6	7	8
4	0	8	7	6	5	4	3	2	1
5	0	1	2	3	4	5	6	7	8
6	0	8	7	6	5	4	3	2	1
7	0	1	2	3	4	5	6	7	8
8	0	8	7	6	5	4	3	2	1

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0
3	0	1	2	3	4	5	6	7	8
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	8	7	6	5	4	3	2	1
7	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0

The first is an integer s-near-field, the second has nilpotent elements; they are both examples concerning case 1.

We can characterize the structure of the s-near-fields with torsion-free elements in this way:

<u>Theorem 2</u>. The additive group $\bar{N}^+$, of an s-near field satisfying the conditions of the case 2 of Theor. 1, is the semidirect sum of $\bar{N}p$ and of the torsion free divisible group.

<u>Proof</u>: let $p\bar{N} = \{px \mid x \in \bar{N}\}$. We observe that $p\bar{N} \cap \bar{N}p = \{0\}$ because if $z \in p\bar{N} \cap \bar{N}p$ there would exist an element $y \in \bar{N}$ such that $py = z$ and $p^2y = 0$. But now in $\bar{N}$, elements of p^2 order do not exist. Therefore $p\bar{N}$ is a proper subset of $\bar{N}$, and it is contained in $\bar{N} \setminus \bar{N}p$. If $\bar{N}$ was abelian, $p\bar{N}$ would be a left ideal of $\bar{N}$: in fact, $px + py = p(x + y)$ and $z(px) = p(zx) \in p\bar{N}$ and so this is absurd. Thus $\bar{N}$ is non-abelian. Let us consider now the subgroup generated by $k\bar{N}$, where $k \in \mathbb{N}$. If $\langle k\bar{N}\rangle^+ = \{0\}$, then all the elements of $\bar{N}$ have torsion, moreover if $\langle k\bar{N}\rangle^+$ is a proper subgroup of $\bar{N}^+$, it is a proper subnear-ring and thus a near-field. In this case $\langle k\bar{N}\rangle = \bar{N}p$ and so this is excluded because $\bar{N}$ would have torsion. Lastly if $\langle k\bar{N}\rangle \cap \bar{N}p = \{0\}$, then $\langle k\bar{N}\rangle \subset \bar{N} \setminus \bar{N}p$. Therefore $\langle k\bar{N}\rangle = \bar{N}\ \forall\ k \in \mathbb{N}$ and $\bar{N}$ is semi-divisible.

From the hypotheses $\overline{N}^+$ is non perfect, and $\overline{N}$ is non abelian, so $\overline{N}'$ is a proper subgroup of $\overline{N}^+$. In particular $\overline{N}'$ is a left ideal of $\overline{N}$ and therefore $\overline{N}' = \overline{N}p$. The factor $\overline{N}^+/\overline{N}'$ is abelian, thus it is divisible and torsion-free, therefore $\overline{N}^+$ is a semi-direct sum of $\overline{N}p$ with a torsion-free divisible group (see [7] pag. 68).

At last we give a characterization of the constant 1-generated s-near-fields.

4. Constant case

Theorem 3. A constant near-ring N is a 1-generated s-near-field if and only if the additive group N^+ is cyclic of order 4.

Proof: from Prop. 6 of [5] we know that if N is a constant s-near-field with two ideals, it is E2-generated, so N has only one ideal I. Let $\underline{a}$ be the non null element of the ideal I isomorphic to $M_c(Z_2)$ - we recall that $M_c(Z_2) = \{f: Z_2 \longrightarrow Z_2 / f$ constant$\}$ (see [6] 1.4.a) - and $\underline{x}$ a generator of N. An integer p such that $p\underline{x} = \underline{a}$ exist, but $2\underline{a} = 0$, then $|N| = |N^+| = 2p$. Moreover, the factor near-ring N/I must be simple because I is maximal and thus p is prime. Finally, if $p \neq 2$, the group N^+, cyclic of order 2p, has two proper subgroups and N has two ideals, but this is excluded. Thus p = 2 and N^+ is cyclic of order 4. Conversely, the proof is trivial.

R e f e r e n c e s

[1] A.H. CLIFFORD and G.B. PRESTON: The algebraic theory of semigroups, Math. Sur. 7, Amer. Math. Soc. Providence 1961.

[2] C. FERRERO COTTI e M.G. RINALDI: Sugli stems i cui ideali

propri sono massimali, Riv. Mat. Univ. Parma (4) 6(1980), 73-79.

[3] C. FERRERO COTTI e M.G. RINALDI: Sugli stems i cui ideali sinistri (destri) propri sono massimali, Riv. Mat. Univ. Parma (4) 7(1981), 23-33.

[4] B.H. NEUMANN: On the commutativity of addition, J. London Math. Soc. 15(1940), 203-208.

[5] S. PELLEGRINI MANARA: On a class of near-rings sum of near-fields, Acta Univ. Car. Ser. Math. Phys. Vol. 25 NO. 1(1984), 19-27.

[6] G. PILZ: Near-rings, North Holland N.Y. (1977).

[7] E. SCHENKMANN: Group Theory, D. Van Nostrand Co.,Inc.(1965).

[8] G. SZETO: Planar and strongly uniform near-rings, Proc. Austral. Math. Soc. 44(1974), 269-274.

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