Commentationes Mathematicae Universitatis Carolinae

Miloslav Feistauer; Jan Mandel; Jindřich Nečas
Entropy regularization of the transonic potential flow problem

Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 3, 431--443

Persistent URL: `http://dml.cz/dmlcz/106318`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

25(3) 1984

ENTROPY REGULARIZATION OF THE TRANSONIC POTENTIAL FLOW PROBLEM

Miloslav FEISTAUER, Jan MANDEL, Jindřich NEČAS

Dedicated to the memory of Svatopluk FUČÍK

Abstract: A bound on the velocity of the flow and an entropy condition define a compact subset S of $W^{1,2}$, in which a weak solution of a variational inequality is sought. This inequality replaces the continuity equation and it has solutions which solve the transonic flow problem provided they lie in the interior of S.

Key words: Transonic flow, variational inequality, entropy condition.

Classification: 35A15, 35M05, 49A29, 76H05

---

1. Introduction. This paper is a further development of the approach of Feistauer and Nečas [4], where a weak solution to the transonic potential flow problem is found as a limit of a generic sequence under some à posteriori assumptions on it involving the entropy condition. For more details and further references, see that paper, Glowinski and Pironneau [7], and Glowinski [6].

Let Ω be a bounded, simply connected domain in R^N, $N=2$ or 3, with a Lipschitz boundary $\partial\Omega$. The irrotational, steady, adiabatic, and isentropic flow of a non-viscous, compressible fluid in Ω is modelled by the continuity equation

(1.1) $$- \operatorname{div} (\rho(|\nabla u|^2) \nabla u) = 0 \text{ in } \Omega,$$

where u is the flow potential, ∇u is the velocity, and ρ is the density of the fluid given by

(1.2) $$\rho(s) = \rho_0 \left(1 - \frac{\varkappa-1}{2a_0^2} s\right)^{\frac{1}{\varkappa-1}}, \quad 0 \leq s \leq \frac{2a_0^2}{\varkappa - 1}$$

The constants ρ_0, a_0 are the density and the speed of sound, respectively, at zero velocity, and $1 < \varkappa < 2$ is the adiabatic constant ($\varkappa = 1.4$ for air).

Equation (1.1) is elliptic for $|\nabla u|^2 < \frac{2a_0^2}{\varkappa + 1}$ and hyperbolic for $|\nabla u|^2 > \frac{2a_0^2}{\varkappa + 1}$. The flow is subsonic in the former case and supersonic in the latter case. The boundary between the subsonic and supersonic regions is not known in advance. Moreover, there are in general discontinuities of parameters of the flow on this boundary, the so-called shocks. Physical flows satisfy the entropy condition:

(1.3) There cannot be an increase of the velocity in the direction of the flow through a shock.

Let $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \tilde{R}$ with Γ_1 and Γ_2 open in $\partial\Omega$ and the surface measure $\mu_{N-1}(\tilde{R}) = 0$. Consider the boundary conditions

(1.4) $$u = 0 \text{ on } \Gamma_1,$$

(1.5) $$\rho \frac{\partial u}{\partial n} = g \text{ on } \Gamma_2.$$

If $\Gamma_1 \neq \emptyset$, define

$$V = \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_1\}$$

and for simplicity of notation let $g = 0$ on Γ_1.

If $\Gamma_1 = \emptyset$, define

$$V = \{u \in W^{1,2}(\Omega): \int_\Omega u \, dx = 0\}$$

and assume

$$\int_{\partial\Omega} g \, dS = 0.$$

We consider two formulations of the entropy condition. The natural form, derived in [4], is

$$(1.6) \quad \int_\Omega \rho'(|\nabla u|^2) \, |\nabla u|^2 \, \nabla u \, \nabla h \, dx \leq M \int_\Omega h \, dx \quad \forall\, h \in \mathcal{D}_+(\Omega),$$

where

$$\mathcal{D}_+(\Omega) = \{h \in \mathcal{D}(\Omega): h \geq 0 \text{ in } \Omega\}$$

The simplified form, used in [6, 7], is

$$(1.7) \qquad -\int_\Omega \nabla u \, \nabla h \, dx \leq M \int_\Omega h \, dx \quad \forall\, h \in \mathcal{D}_+(\Omega).$$

Here $M > 0$ is some constant. Define

$$S_{nat} = \{u \in V: \text{ u satisfies (1.6) and } |\nabla u|^2 \leq s_1 < \frac{6a_o^2}{1+\varkappa} \text{ a.e. in } \Omega\}$$

and

$$S_{sim} = \{u \in V: \text{ u satisfies (1.7) and } |\nabla u|^2 \leq s_2 < \frac{2a_o^2}{\varkappa - 1} \text{ a.e. in } \Omega\}.$$

Here s_1 and s_2 are some constants.

Put

$$(1.8) \qquad \Phi(u) = \frac{1}{2} \int_\Omega \Big(\int_0^{|\nabla u|^2} \rho(t) \, dt \Big) \, dx.$$

Then the Gâteaux differential of Φ is

$$D\Phi(u,h) = B(u;u,h),$$

where

$$(1.9) \qquad B(u;v,w) = \int_\Omega \rho(|\nabla u|^2) \, \nabla v \, \nabla w \, dx.$$

The problem (1.1) with the boundary conditions (1.4) and (1.5) has now the weak formulation

$$(1.10) \qquad u \in V; \ B(u;u,v) = \int_{\partial\Omega} g\ v\ dS \quad \forall\ v \in V$$

and we look for physically meaningful solutions which should lie in the set S_{nat} or S_{sim} according to the form of the entropy condition. So, consider the following regularized problems:

1.11. Problem. Minimize $\Phi(u) - \int_{\partial\Omega} g\ u\ dS$ over S_{nat}.

1.12. Problem. Find a solution of the variational inequality

$$u \in S_{sim};\ B(u;u,v-u) \geq \int_{\partial\Omega} g(v-u)\ dS \quad \forall\ v \in S_{sim}.$$

It will be proved that these regularized problems have always solutions. A solution u of (1.11) or (1.12) is a solution of the transonic flow problem (1.10) if

$$\forall\ v \in \tilde{V}\ \exists\ \varepsilon > 0\ \forall\ t \in (0,\varepsilon):\ u + tv \in S_e$$

where $S_e = S_{nat}$ or $S_e = S_{sim}$, respectively, and $\tilde{V}$ is a dense subset of V.

2. Auxiliary propositions. Compactness results which follow are based on Theorem 1 of Murat [10]. We present direct proofs here.

2.1. Lemma. Let Ω be a bounded domain in R^N with a Lipschitz boundary and let $G_n \longrightarrow G$ weakly in $(W^{1,2}(\Omega))^*$. Let $G_n(h) \geq 0$ for all $h \in \mathcal{D}_+(\Omega)$. Then $G_n \longrightarrow G$ strongly in $(W^{1,p}(\Omega))$ for each $p > 2$.

Proof. Let $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega$ be a subdomain of Ω. There exists $\psi \in \mathcal{D}_+(\Omega)$ such that $\psi(x) = 1$ on Ω_1. For $h \in \mathcal{D}(\Omega)$ with supp $h \subset \Omega_1$ we have

$$-\|h\|_{L^\infty(\Omega)}\psi \leq h \leq \|h\|_{L^\infty(\Omega)}\psi ,$$

hence

$$(2.2) \qquad |G_n(h)| \leq G_n(\psi)\,\|h\|_{L^\infty(\Omega)} \leq c(\Omega_1)\,\|h\|_{L^\infty(\Omega)} .$$

Define u_n, $u \in W_o^{1,2}(\Omega)$ by

$$\int_\Omega \nabla u_n \nabla h\, dx = G_n(h), \int_\Omega \nabla u \nabla h\, dx = G(h) \quad \forall\, h \in W_o^{1,2}(\Omega).$$

Let $q > N$. Since the imbedding $W_o^{1,q}(\Omega_1) \subset C(\Omega_1)$ is compact, it follows from interior estimates by Agmon, Douglis, and Nirenberg [1] and from (2.2) that for every subdomain $\Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1$, $\{u_n\}$ is compact in $W^{1,q'}(\Omega_2)$, $\frac{1}{q'} + \frac{1}{q} = 1$. So the interpolation inequality

$$\left(\int_{\Omega_2} |m|^{p'} dx\right)^{\frac{1}{p'}} \leq \left(\int_{\Omega_2} |m|^2 dx\right)^{\frac{\theta}{2}} \left(\int_{\Omega_2} |m|^{q'} dx\right)^{\frac{1-\theta}{q'}}$$

with $\frac{1}{p'} = \frac{\theta}{2} + \frac{1-\theta}{q'}$ gives that the same is true in all $W^{1,\tilde{p}}(\Omega_2)$, $1 < \tilde{p} < 2$. Let $h \in W_o^{1,p}(\Omega)$. Then

$$\int_\Omega \nabla (u_n - u) \nabla h\, dx = \int_{\Omega_2} \nabla (u_n - u) \nabla h\, dx + \int_{\Omega \setminus \Omega_2} \nabla (u_n - u) \nabla h\, dx$$

and we have

$$\left| \int_{\Omega\setminus\Omega_2} \nabla (u_n - u) \nabla h\, dx \right| \leq \left(\int_{\Omega\setminus\Omega_2} |\nabla (u_n - u)|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\Omega_2} |\nabla h|^p\, dx\right)^{\frac{1}{p}} |\Omega \setminus \Omega_2|^{\frac{1}{2}-\frac{1}{p}}.$$

Since $|\Omega \setminus \Omega_2|$ can be made arbitrarily small, we obtain

$$G_n \longrightarrow G \text{ strongly in } (W_o^{1,p}(\Omega))^* .$$

This is another proof of Theorem 1 in Murat [10], cf. also the remark by Brezis [3] to that paper. Note that we did not use the assumption that Ω has a Lipschitz boundary up to now.

For $h \in W^{1,p}(\Omega)$, $p>2$, put

$h = h^1 + h^2$, where $\Delta h^1 = 0$ in Ω and $h^2 \in W_0^{1,2}(\Omega)$.

By Meyers [9], there exists $p_1 > 2$ such that the mappings $h \mapsto h^i$, $i = 1,2$, are continuous from $W^{1,r}(\Omega)$ to itself if $2 \le r \le p_1$. Without loss of generality assume that $p \le p_1$. Since the imbedding $W^{1-\frac{1}{p},p}(\partial\Omega) \subset W^{1/2,2}(\partial\Omega)$ is compact, it holds that the set $\{h^1 : \|h\|_{W^{1,p}(\Omega)} \le 1\}$ is compact, and, consequently,

$$\sup_{\|h\|_{W^{1,p}(\Omega)} \le 1} |(G_n - G)(h^1)| \longrightarrow 0.$$

Since $G_n \longrightarrow G$ strongly in $(W_0^{1,p}(\Omega))^*$, we get the result. $\square$

Lemma 2.1 is a direct extension of Theorem 6 in Murat [10], where the stronger positivity condition

$$G_n(h) \ge 0 \quad \forall\, h \in \mathcal{E}(\bar{\Omega}),\ h \ge 0,$$

has been assumed.

We are now in the position to prove the fundamental properties of the sets S_{nat} and S_{sim}. Let us begin with the simpler case.

2.3. Theorem. The set S_{sim} is convex and compact (in the $W^{1,2}(\Omega)$ norm).

Proof. Clearly S_{sim} is bounded, convex, and closed. Let $u_n \in S_{sim}$ and without loss of generality

$$u_n \longrightarrow u \text{ weakly in } W^{1,2}(\Omega).$$

Define G_n, $G \in (W^{1,2}(\Omega))^*$ by

$$G_n(h) = \int_\Omega \nabla u_n \nabla h \, dx + M \int_\Omega h \, dx,$$

$$G(h) = \int_\Omega \nabla u \nabla h \, dx + M \int_\Omega h \, dx.$$

We have $G_n \rightharpoonup G$ weakly in $(W^{1,2}(\Omega))^*$ and $G_n(h) \geq 0$ for all $h \in \mathcal{D}_+(\Omega)$, so from Lemma 2.1,

$$G_n \longrightarrow G \text{ strongly in } (W^{1,\infty}(\Omega))^* .$$

Now

$$\int_\Omega \nabla (u_n - u) \nabla (u_n - u) \, dx$$

$$= (G_n - G)(u_n - u) \longrightarrow 0,$$

hence $u_n \longrightarrow u$ strongly in $W^{1,2}(\Omega)$. □

2.4. Theorem. The set S_{nat} is compact (in the $W^{1,2}(\Omega)$ norm).

Proof. Clearly S_{nat} is bounded and closed. Let $u_n \in S_{nat}$ and without loss of generality

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega).$$

Define $G_n \in (W^{1,2}(\Omega))^*$ by

$$G_n = M \int_\Omega h \, dx - \int_\Omega \wp'(|\nabla u_n|^2) |\nabla u_n|^2 \nabla u_n \nabla h \, dx.$$

We can suppose $G_n \rightharpoonup G$ weakly in $(W^{1,2}(\Omega))^*$. Since $G_n(h) \geq 0$ for all $h \in \mathcal{D}_+(\Omega)$, Lemma 2.1 implies

$$G_n \longrightarrow G \text{ strongly in } (W^{1,\infty}(\Omega))^* .$$

Now

$$- \int_\Omega \wp'(|\nabla u_n|^2) |\nabla u_n|^2 \nabla u_n \nabla (u_n - u) \, dx$$

$$+ \int_\Omega \wp'(|\nabla u|^2) |\nabla u|^2 \nabla u \nabla (u_n - u) \, dx$$

$$\overset{\text{def}}{=} \int_\Omega F_n(x) \, dx$$

$$= G_n(u_n - u) - M \int_\Omega (u_n - u)$$

$$+ \int_\Omega \varrho'(|\nabla u|^2)|\nabla u|^2 \nabla u \nabla (u_n - u) \to 0.$$

Because $|\nabla u_n|^2 \leq s_1$, it holds $|\nabla u|^2 \leq s_1$ (the set of v such that $|\nabla v|^2 \leq s_1$ is convex and closed, hence weakly closed) and we get from

$$s_1 < \frac{6a_0^2}{1+\varkappa}$$

that

$$(2.5)\quad -\varrho'(|\xi|^2)|\xi|^2 \xi(\xi - \zeta) +$$
$$+ \varrho'(|\zeta|^2)|\zeta|^2 \zeta(\xi - \zeta) > 0$$

for all ξ, ζ such that $|\xi|^2 \leq s_1$, $|\zeta|^2 \leq s_1$, $\xi \neq \zeta$. Consequently, $F_n(x) \geq 0$ and we can suppose that $F_n(x) \to 0$ in $Z \subset \Omega$ with $|\Omega \setminus Z| = 0$ and ∇u_n, ∇u defined in Z. We claim that

$$\nabla u_n(x) \to \nabla u(x) \quad \forall\, x \in Z.$$

Take $x \in Z$. There exists a subsequence such that $\nabla u_{n_k} \to \xi$. If $\xi \neq \nabla u(x)$, we get by (2.5) a contradiction with $F_n(x) \to 0$. Hence $u_n \to u$ strongly in $W^{1,2}(\Omega)$. $\square$

In the sequel, we shall use the following generalization of the secant modulus inequality taken from Mandel [8], which for $\alpha = 1/2$ and $B(u;u,h) = D\,\Phi(u,h)$ was proved by Nečas and Hlaváček [12] and by Fučík, Kratochvíl, and Nečas [5] in the case of equations (K = H).

2.6. Lemma. Let H be a Hilbert space, K a closed convex subset of H, Φ a functional on H with the Gâteaux differential $D\,\Phi(u,h)$, and $B(u;.,.)$ a symmetric, bounded, H-elliptic bilinear form on H for each $u \in H$ and such that with some constant α

$$(2.7) \quad \alpha B(u;u - v,u - v) \geq \Phi(v) - \Phi(u) - D\,\Phi(u,v - u) \quad \forall\, u,\ v \in K.$$

Let $f \in H^*$, $\Psi(u) = \Phi(u) - f(u)$.

If w is defined for a given $u \in K$ as the solution of the variational inequality

$$(2.8) \quad \begin{aligned} &w \in K;\ D\,\Phi(u,\ v - w) + B(u;\ w - u,\ v - w) \\ &\qquad \geq f(v - w) \quad \forall\, v \in K, \end{aligned}$$

then it holds

$$\Psi(u) - \Psi(w) \geq (1 - \alpha)\, B(u;\ u - w,\ u - w).$$

Proof. Add (2.7) with $v = w$ and (2.8) with $v = u$. □

3. Solution of regularized problems. With the theorems of the preceding section at our disposition, the proofs of our main results are quite straightforward.

3.1. Theorem. The problem 1.11 has a solution.

Proof. The functional Φ is continuous and the set S_{nat} is compact. □

3.2. Theorem. The problem 1.12 has a solution.

Proof. The bilinear forms $B(u;.,.)$ defined by (1.9) are uniformly bounded and uniformly V-elliptic for all $u \in S_{sim}$. Hence the variational inequality

$$(3.3) \quad w \in S_{sim};B(u;w,v - w) \geq \int_{\partial\Omega} g(v - w)\,dS \quad \forall\, v \in S_{sim}$$

has a unique solution for any $u \in S_{sim}$ and the mapping $u \longmapsto w(u)$ is continuous. Since solving the problem 1.12 is equivalent to the fixed point problem $u = w(u)$ and S_{sim} is convex and compact, the Schauder fixed point theorem applies. $\square$

As in [4], a very natural approach is to find a solution to the problem 1.12 as a limit of the sequence $u_{n+1} = w(u_n)$, where the mapping $u \longmapsto w(u)$ is defined by (3.3). This is the secant modulus method for variational inequalities [8, 12].

3.4. Theorem. Let $u_0 \in S_{sim}$ arbitrary and $u_{n+1} = w(u_n)$. Then $u_{n+1} - u_n \longrightarrow 0$ strongly in $W^{1,2}(\Omega)$ and any infinite subsequence of $\{u_n\}$ contains a subsequence convergent strongly in $W^{1,2}(\Omega)$. The limit of any convergent subsequence is a solution of the problem 1.12.

Proof. With Φ and B defined by (1.8) and (1.9), respectively, we have (2.7) with $\alpha = 1/2$ (see [4]), and Lemma 2.6 with $f(u) = \int_{\partial\Omega} g\, u\, dS$ yields

$$(3.5) \quad c \| u_{n+1} - u_n \|_V^2 \leq \Psi(u_n) - \Psi(u_{n+1}) \longrightarrow 0$$

using uniform V-ellipticity of the bilinear forms $B(u_n;\cdot,\cdot)$ and the fact that $\Psi = \Phi - f$ is bounded from below on S_{sim}. Compactness of S_{sim} yields immediately the existence of convergent subsequences. From the continuity of the mapping $u \longmapsto w(u)$ and from (3.5), the limit u of any convergent subsequence of $\{u_n\}$ satisfies $u = w(u)$. $\square$

Note that using the particular properties of the problem at hand, we proved in Theorem 3.4 the existence of solutions of the problem 1.12 without recourse to the Schauder theorem. Anyway, existence of solutions of 1.12 also follows from the simple fact that any minimizer of $\Phi(u) - \int_{\partial\Omega} gu\, dS$ in S_{sim} is a solution of 1.12. We choose the fixed point approach, because it leads naturally to the secant modulus method, which is a promising <u>numerical</u> method.

4. <u>Extensions</u>. All propositions and proofs remain valid when the entropy condition (1.7) is replaced by its abstract form

$$- E(u,h) \leq M(h) \quad \forall\ h \in \mathcal{D}_+(\Omega),$$

where E is a V-elliptic bounded bilinear form and $M \in (W_0^{1,2}(\Omega))^*$.

Symmetry and V-ellipticity of the forms $B(u;.,.)$ is in fact not needed in Theorem 3.2. It is sufficient that $B(u;.,.)$ be uniformly bounded and have the continuity property

$$\forall\ v \in V:\ B(u_n;v,.) \longrightarrow B(u;v,.) \text{ in } V^* \text{ if } u_n \longrightarrow u \text{ in } V.$$

The proof follows by an application of the Schauder theorem to the continuous mapping $u \longmapsto w(u)$ defined by

$$\bar{w} \in S_{sim};\ (\bar{w}, v - \bar{w})_V \geq (u, v - \bar{w})_V - B(u;u, v - \bar{w})$$

$$+ \int_{\partial\Omega} g(v - \bar{w})\ dS \quad \forall\ v \in S_{sim}.$$

Theorem 3.4 remains valid for more general problems as long as the assumptions of Lemma 2.6 are satisfied with some $\alpha < 1$ and $D\,\Phi\,(u,h) = B(u;u,h)$. For conditions implying (2.7), see Mandel [6].

The compactness of the sets S_{sim} and S_{nat} makes it possible to use the concept of discrete compactness (see Anselone and Ansorge [21]), which yields strong convergence of subsequences of solutions of suitable finite dimensional approximate problems to a solution of the regularized problem. This will be studied in following papers.

R e f e r e n c e s

[1] S. AGMON, A. DOUGLIS, and L. NIRENBERG: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, II, Comm. Pure Appl. Math. 12(1959), 623-727, 17(1964), 35-92.

[2] P.M. ANSELONE and R. ANSORGE: Compactness principles in nonlinear operator approximation theory, Numer. Funct. Anal. Optimiz. 1(1979), 589-618.

[3] H. BREZIS: Remarque sur l'article précedent de F. Murat, J. Math. pures et appl. 60(1981), 321-322.

[4] M. FEISTAUER and J. NEČAS: On the solvability of transonic potential flow problems, Z. für Analysis und ihre Anwendungen (to appear).

[5] S. FUČÍK, A. KRATOCHVÍL, and J. NEČAS: Kačanov-Galerkin method, Comment. Math. Univ. Carolinae 14(1973), 651-659.

[6] R. GLOWINSKI: Lectures on Numerical Methods for Nonlinear Variational Problems, Springer-Verlag Heidelberg 1980.

[7] R. GLOWINSKI and O. PIRONNEAU: On the computation of transonic flows. In: H. Fujita (Ed.): Functional Analysis and Numerical Analysis, Japan Society for the Promotion of Science, 1978.

[8] J. MANDEL: On an iterative method for nonlinear variational inequalities, Numer. Funct. Anal. Optimiz. (Submitted).

[9] N.G. MEYERS: An L^p estimate for the gradient of solutions of second order elliptic divergence equations, Ann. S.N.S. Pisa 17(1963), 189-206.

[10] F. MURAT: L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q<2$, J. Math. pures et appl. 60(1981), 309-321.

[11] J. NEČAS: Les méthodes directes en théorie des équations elliptiques, Academia, Praha 1967.

[12] J. NEČAS and I. HLAVÁČEK: Solution of Signorini's contact problem in the deformation theory of plasticity by secant modulus method, Apl. Mat. 28(1983), 199-214.

Katedra aplikované matematiky MFF UK (M. Feistauer and J. Nečas), Výpočetní centrum UK při MFF UK (J. Mandel), Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia

(Oblatum 9.7. 1984)