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Michael H. G. Geisler; Hans Triebel
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CHARACTERIZATIONS OF BESOV SPACES VIA VARIABLE DIFFERENCES

M. GEISLER, H. TRIEBEL

Dedicated to the memory of Svatopluk FUČÍK

Abstract: The paper deals with equivalent quasi-norms in the Besov spaces $B^s_{p,q}(R_n)$ with $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > n\left(\frac{1}{\min(p,1)} - 1\right)$.

Key words: Function spaces, Besov spaces, variable differences

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1. Introduction

In recent times several authors studied function spaces of Besov-Lipschitz-Sobolev type on manifolds, in particular on Lie groups. An approach to Sobolev-Besov spaces on compact Lie groups via (non-commutative) interpolation will be given in [1,2] . As far as Lie groups are concerned one would try to give intrinsic descriptions of Sobolev-Besov spaces, e. g. on the basis of (left or right) invariant vector fields and related flows. However it is convenient (maybe even necessary) to reduce some problems for function spaces on Lie groups to corresponding problems on R_n. We recall that norms of functions $f(x)$ in Besov spaces $B^s_{p,q}(R_n)$ with $s > 0$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$ can be characterized via M-th differences $(\Delta^M_h f)(x)$, where $x \in R_n$ and $h \in R_n$. The above mentioned reduction of Besov spaces on Lie groups on the corresponding spaces $B^s_{p,q}(R_n)$ on R_n yields norms where the "constant" differences $(\Delta^M_h f)(x)$ are replaced by "variable" differences $(\Delta^M_{h+\varepsilon(x,h)} f)(x)$

where $\varepsilon(x,h)$ is a smooth perturbation of h. There are two possible interpretations of these variable differences: Let $\rho(x) = h + \varepsilon(x,h)$ (where we assume that h is fixed at this moment).

(i) Let x and M be fixed, and let

$$(1) \quad (\Delta^M_{\rho(x)} f)(x) = (\Delta^M_{\rho(x)} f)(y)\Big|_{y=x}$$

i. e. only y is considered as a variable and one has the usual M-th differences with respect to the fixed step-length $\rho(x)$. Afterwards one specializes y by y = x.

(ii) $\Delta_{\rho(x)}$ is considered as an operator which maps f(x) into $f(x + \rho(x)) - f(x)$. We denote this operator by Δ_ρ , i. e.

$$(2) \quad (\Delta_\rho f)(x) = f(x+\rho(x)) - f(x).$$

Then Δ^M_ρ is the M-th power of Δ_ρ . For example,

$$(3) \quad (\Delta^2_\rho f)(x) = (\Delta_\rho f)(x+\rho(x)) - (\Delta_\rho f)(x)$$

$$= f(x+\rho(x)+\rho(x+\rho(x))) - 2f(x+\rho(x)) + f(x).$$

The plan of the paper is as follows. The necessary preliminaries are given in Section 2: Definition and properties of the spaces $B^s_{p,q}(R_n)$ (inclusively the case $0 < p \leqq 1$), discussion of the general assumptions for the above vector-function $\varepsilon(x,h)$. Section 3 and Section 4 deal with the characterization of the considered Besov spaces $B^s_{p,q}(R_n)$ via variable differences in the sense of the first and the second interpretation, respectively.

We use the notations from [4] . A modified version of Section 3 of this paper will be incorporated in the Russian edition of [4] (as Subsection 2.5.14). Section 4 (and some modifications) are the basis of the studies in [1,2] .

By the usual abuse of notations c, c', c_j etc. stand for positive constants which may differ from formula to formula.

2. Preliminaries

2.1. Besov Spaces

We use the notations from [4] . As usual R_n stands for the real n-dimensional Euclidean space. $S = S(R_n)$ denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on R_n. Let S' be its dual, i. e. the space of tempered distributions. (We omit "R_n" because all spaces under consideration are defined on R_n). Let Φ be the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset S$ with the following properties:

(i) supp $\varphi_0 \subset \{x \mid |x| \leq 2\}$

(4) supp $\varphi_j \subset \{x \mid 2^{j-1} \leq |x| \leq 2^{j+1}\}$ if $j = 1, 2, 3, \dots$

(ii) For every multi-index α there exists a positive number c_α such that

(5) $2^{j|\alpha|}|D^\alpha \varphi_j(x)| \leq c_\alpha$ for all $j = 0,1,2,\dots$ and all $x \in R_n$,

(iii)

(6) $$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for every } x \in R_n$$

We may assume that

(7) $$\varphi_k(x) = \varphi_1(2^{-k+1}x), \qquad k = 1, 2, \dots$$

holds. F and F^{-1} stand for the Fourier transform and its inverse on S', respectively. If $f \in S'$ then $F^{-1}\varphi_j Ff = F^{-1}[\varphi_j Ff]$ makes sense, and by the Paley-Wiener-Schwartz theorem it is an analytic function which we denote by $(F^{-1}\varphi_j Ff)(x)$. Finally if $0 < p \leq \infty$ then

$$\|g \mid L_p\| = \left(\int_{R_n} |g(x)|^p dx\right)^{\frac{1}{p}}$$

has the usual meaning (modification if $p = \infty$). Now we are in the position to define the Besov spaces $B^s_{p,q} = B^s_{p,q}(R_n)$: Let

$-\infty < s < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $\varphi \in \Phi$. Then

$$(8)\quad B^s_{p,q} = \{f \mid f \in S',\ \|f \mid B^s_{p,q}\|^\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|F^{-1}\varphi_j Ff \mid L_p\|^q\Big)^{\frac{1}{q}} < \infty\}$$

(usual modification if $q = \infty$), cf. [4, 2.3.1] . This is a quasi-Banach space (Banach space if $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$). It is independent of the choice of $\varphi \in \Phi$ (in the sense of eq̂uivalent quasi-norms). In this sense we write $\|f \mid B^s_{p,q}\|$ instead of $\|f \mid B^s_{p,q}\|^\varphi$ in the sequel. We mention that $B^s_{p,q}$ with $s > 0$, $1 < p < \infty$ and $1 \leq q \leq \infty$ coincides with the classical Besov spaces. Furthermore, $\mathcal{C}^s = B^s_{\infty,\infty}$ with $s > 0$ are the well-known Hölder-Zygmund spaces. Details may be found in [4] .

Next we formulate a crucial assertion which we need in the sequel. First we recall that

$$(9)\quad (\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad \Delta^M_h = \Delta^1_h \Delta^{M-1}_h$$

with $h \in R_n$, $x \in R_n$ and $M = 2,3,\ldots$ are the usual differences. Let

$$(10)\quad \tilde{\sigma}_p = n\Big(\frac{1}{\min(p,1)} - 1\Big),$$

cf. [4, (2.5.3/8)] . In [4, Theorem 2.5.12] the following assertion is proved: Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s > \tilde{\sigma}_p$, $M > s$ (where M is a natural number) and $\lambda > 0$. Then

$$(11)\quad \|f \mid B^s_{p,q}\|^0_{M,\lambda} = \|f \mid L_p\| + \Big(\int_{|h| \leq \lambda} |h|^{-sq} \|\Delta^M_h f \mid L_p\|^q \frac{dh}{|h|^n}\Big)^{\frac{1}{q}}$$

(modification if $q = \infty$) is an equivalent quasi-norm (norm if $p \geq 1$ and $q \geq 1$) in $B^s_{p,q}$.

2.2. The Perturbation $\varepsilon(x,h)$

We formulate some general assumptions for the vector function $\varepsilon(x,h) \in R_n$ from the Introduction. Let $\lambda > 0$ and M be a natural number, and let $\varepsilon(x,h)$ be a continuous mapping from

$$\{(x,h) \mid x \in R_n,\ h \in R_n,\quad |h| \leq \lambda\}$$

into R_n such that the components of $\mathcal{E}(x,h)$ have continuous first derivatives with respect to the variables $x_1,\ldots,x_n$. It is assumed that for fixed $h \in R_n$ with $|h| \leq \lambda$ and $\mu = 1,\ldots,M$,

$$y = y^{h,\mu}(x) = x + \mu\ \mathcal{E}(x,h), \qquad x \in R_n \tag{12}$$

yields an one-to-one mapping from R_n onto itself and that $\left|\det \frac{\partial(y^{h,\mu}(x))}{\partial(x)}\right|$ (the absolute value of the Jacobian of (12)) can be estimated from below by a positive number which is independent of x, h, and μ. Let $x = x^{h,\mu}(y)$ be the inverse mapping of (12). Then it follows from the Inverse Function Theorem (cf. e. g. [3, p. 35]) that $\left|\det \frac{\partial(x^{h,\mu}(y))}{\partial(y)}\right|$ is uniformly bounded from above with respect to $y \in R_n$, $h \in R_n$ with $|h| \leq \lambda$, and $\mu = 1,\ldots,M$.

<u>Remark 1.</u> These are our general assumptions for the vector-function $\mathcal{E}(x,h)$. Sometimes it is sufficient to have the above informations for $y^{h,\mu}$ only for special values of μ. For example, in Section 4 the above assumptions with $\mu = 1$ are sufficient. But on the other hand in the same section we need that the components of $\mathcal{E}(x,h)$ have higher derivatives which are uniformly bounded in R_n.

<u>Remark 2.</u> It is easy to formulate sufficient conditons which ensure the above general assumptions. For example, let

$$\left|\frac{\partial \mathcal{E}}{\partial x_j}(x,h)\right| \leq \varkappa \qquad \text{for } x \in R_n \text{ and } h \in R_n \text{ with } |h| \leq \lambda \tag{13}$$

where $j = 1,\ldots,n$. We claim that the above assumptions are satisfied if $\varkappa > 0$ is small: This is clear as far as the assertions for $\left|\det \frac{\partial(y^{h,\mu}(x))}{\partial(x)}\right|$ are concerned, which shows that the mapping (12) is locally one-to-one. Furthermore, for small values of $\varkappa$ we have $|\mu|\,|\mathcal{E}(x^1,h) - \mathcal{E}(x^2,h)| \leq \frac{1}{2}|x^1 - x^2|$ and consequently,

(14) $\quad |y^{h,\mu}(x^1) - y^{h,\mu}(x^2)| \geq \frac{1}{2} | x^1 - x^2 |$.

This shows that the mapping (12) is globally one-to-one. By (14) it follows also that $y^{h,\mu}(x)$ maps R_n onto a set of R_n which is both open and closed and which coincides, consequently, with R_n.

3. Characterizations via $\Delta^M_{\varrho^h}$ with $\varrho^h(x) = h + \varepsilon(x,h)$

3.1. The Basic Proposition

Let the general assumptions for $\varepsilon(x,h)$ from Subsection 2.2 be satisfied. Let $\varrho^h(x) = h + \varepsilon(x,h)$. In Section 3, $\Delta^M_{\varepsilon(x,h)}$ and $\Delta^M_{\varrho^h(x)}$ have the meaning of (1) with ε and ϱ^h instead of ϱ, respectively.

Proposition. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > \tilde{\sigma}_p$, cf. (10). Let $\lambda > 0$ and let M be a natural number with $M > s$. Let the above hypotheses for $\varepsilon(x,h)$ be satisfied. Let $\eta > 0$ and let M_1 and M_2 be non-negative integers with $M_1 \geq 1$ and $M_1 + M_2 = M$. Then there exists a positive number $\delta = \delta(\eta)$ with the following property: If

(15) $\quad |\varepsilon(x,h)| \leq \delta |h|$ for all $x \in R_n$ and all $h \in R_n$ with $|h| \leq \lambda$,

then

(16) $$\left(\int_{|h| \leq \lambda} |h|^{-sq} \| (\Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h f)(x) | L_p \|^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \leq \eta \| f | B^s_{p,q} \|$$

holds for all $f \in B^s_{p,q}$ (modification if $q = \infty$).

Proof. Without essential restriction of generality we always assume that $q < \infty$ and $\lambda = 2^{-K}$, where K is an integer. Let $\{\varphi_k(x)\}_{k=0}^{\infty} \in \Phi$. For sake of convenience we put $\varphi_k(x) = 0$ if $k = -1,-2,\ldots$ Let $2^{-j-1} \leq |h| \leq 2^{-j}$ with $j = K, K+1,\ldots$ If $f \in B^s_{p,q}$ then we have

(17) $$f(x) = \sum_{m=-\infty}^{\infty} (F^{-1} \varphi_{j+m} F f)(x)$$

and

$$(18)\quad (\Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h f)(x) = \sum_{m=-\infty}^{\infty} (\Delta^{M_1}_{\varepsilon(x,h)} F^{-1}\varphi_{j+m} F \Delta^{M_2}_h f)(x) = \sum_{m=-\infty}^{N} \cdots + \sum_{m=N+1}^{\infty} \cdots$$

where the natural number N will be chosen later on. Let $m \leq N$. Then we have

$$(19)\quad |(\Delta^{M_1}_{\varepsilon(x,h)} F^{-1}\varphi_{j+m} F \Delta^{M_2}_h f)(x)| \leq \sup_{\substack{\varrho \in R_n \\ |\varrho| \leq \delta 2^{-j}}} |(\Delta^{M_1}_{\varrho} F^{-1}\varphi_{j+m} F \Delta^{M_2}_h f)(x)|$$

$$\leq c_1 \delta^{M_1} 2^{-jM_1} \sup_{|x-y| \leq c_2 \delta 2^{-j}} \sum_{|\beta| = M_1} |(D^\beta F^{-1}\varphi_{j+m} F \Delta^{M_2}_h f)(y)|$$

$$\leq c_3 \delta^{M_1} 2^{-jM} \sup_{|x-y| \leq c_4 2^{-j}} \sum_{|\alpha| = M} |(D^\alpha F^{-1}\varphi_{j+m} F f)(y)|,$$

where all the above c's (and also the following ones) are independent of δ, N, j, m etc. We use a maximal inequality and a Nikol's-kij inequality, cf. the scalar case of [4,(1.6.2/1)] (which works also for $p = \infty$) and [4,(1.3.2/5)]. With $|\alpha| = M$ then we have

$$\| \sup_{|x-y| \leq c 2^{-j}} |(D^\alpha F^{-1}\varphi_{j+m} F f)(y)| \,|\, L_p \| \leq c' (1 + 2^{ma}) \| D^\alpha F^{-1}\varphi_{j+m} F f \,|\, L_p \|$$

$$(20)\qquad \leq c'' (1 + 2^{ma})\, 2^{M(j+m)} \| F^{-1}\varphi_{j+m} F f \,|\, L_p \|,$$

where a is a number with $a > \frac{n}{p}$. By (19) and (20) it follows that

$$(21)\quad \| (\Delta^{M_1}_{\varepsilon(x,h)} F^{-1}\varphi_{j+m} F \Delta^{M_2}_h f)(x) | L_p \| \leq c\, \delta^{M_1} 2^{Na} 2^{Mm} \| F^{-1}\varphi_{j+m} F f \,|\, L_p \|.$$

Let $0 < p \leq 1$. Then (21) yields

$$\| \sum_{m=-\infty}^{N} \Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h F^{-1}\varphi_{j+m} F f \,|\, L_p \|^p$$

$$(22)\qquad \leq \sum_{m=-}^{N} \| \Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h F^{-1}\varphi_{j+m} F f \,|\, L_p \|^p$$

$$\leq c\,\delta^{M_1 p} 2^{N a p} 2^{-jsp} \sum_{m=-\infty}^{N} 2^{m(M-s)p} 2^{s(j+m)p} \| F^{-1} \varphi_{j+m} Ff \mid L_p \|^p .$$

If $1 \leq p \leq \infty$, then the counterpart of (22) reads as follows,

$$(23) \quad \Big\| \sum_{m=-\infty}^{N} \Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_{h} F^{-1} \varphi_{j+m} Ff \mid L_p \Big\|$$

$$\leq c\,\delta^{M_1} 2^{Na} 2^{-js} \sum_{m=-\infty}^{N} 2^{m(M-s)} 2^{s(j+m)} \| F^{-1} \varphi_{j+m} Ff \mid L_p \| .$$

Let $m \geq N+1$. If $0 < p \leq 1$ then we have

$$(24) \quad \Big\| \sum_{m=N+1}^{\infty} \Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_{h} F^{-1} \varphi_{j+m} Ff \mid L_p \Big\|^p$$

$$\leq c \sum_{m=N+1}^{\infty} \| \Delta^{M_1}_{\varepsilon(x,h)} F^{-1} \varphi_{j+m} Ff \mid L_p \|^p$$

$$\leq c' \sum_{m=N+1}^{\infty} \sum_{r=0}^{M_1} \| (F^{-1} \varphi_{j+m} Ff)(x + r\,\varepsilon(x,h)) \mid L_p \|^p$$

$$\leq c'' \sum_{m=N+1}^{\infty} \| F^{-1} \varphi_{j+m} Ff \mid L_p \|^p ,$$

where in the last estimate we used our assumption about the mappin properties of $x \longrightarrow x + \mu\, \varepsilon(x,h)$, cf. (12). If $1 \leq p \leq \infty$ then the counterpart of (24) reads as follows,

$$(25) \quad \Big\| \sum_{m=N+1}^{\infty} \Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_{h} F^{-1} \varphi_{j+m} Ff \mid L_p \Big\|$$

$$\leq c \sum_{m=N+1}^{\infty} \| F^{-1} \varphi_{j+m} Ff \mid L_p \| .$$

We summarize our estimates: If h with $2^{-j-1} \leq |h| \leq 2^{-j}$ is given, then we have (22), (24) for $0 < p \leq 1$ and (23), (25) for $1 < p \leq \infty$. Let again $0 < p \leq 1$. Then (18), (22) and (24) yield

$$(26)\quad \int_{|h|\leq 2^{-K}} |h|^{-sq} \| (\Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h f)(x) | L_p \|^q \frac{dh}{|h|^n}$$

$$\leq c \sum_{j=K}^{\infty} 2^{jsq} \sup_{2^{-j-1}\leq |h| \leq 2^{-j}} \| (\Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h f)(x) | L_p \|^q$$

$$\leq c' \delta^{M_1 q} 2^{Naq} \sum_{j=K}^{\infty} \sum_{m=-\infty}^{N} 2^{m(M-s)q} 2^{(N-m)\sigma q} 2^{s(j+m)q} \| F^{-1} \varphi_{j+m} Ff | L_p \|^q$$

$$+ c' \sum_{j=K}^{\infty} \sum_{m=N+1}^{\infty} 2^{-msq} 2^{(m-N)\sigma q} 2^{s(j+m)q} \| F^{-1} \varphi_{j+m} Ff | L_p \|^q$$

where $\sigma > 0$ is an arbitrary number. We choose σ such that $0 < \sigma < s < s+\sigma < M$ holds. Then we have

$$(27)\quad \int_{|h|\leq 2^{-K}} |h|^{-sq} \| (\Delta^{M_1}_{\varepsilon(x,h)} \Delta^{M_2}_h f)(x) | L_p \|^q \frac{dh}{|h|^n}$$

$$\leq c \left(\delta^{M_1} 2^{Na} 2^{N(M-s)} + 2^{-Ns} \right)^q \| f | B^s_{p,q} \|^q .$$

If we choose N large and afterwards δ small, then we obtain (16). If $1 < p \leq \infty$, then (27) follows in the same way from (23) and (25). The proof is complete.

Remark 3. The above Proposition is the basis for our considerations in Subsection 3.2 (Theorem 1). However in Section 4 we need a modification of this proposition which we describe now. We shall obtain a modified estimate (19) where we have on the right-hand side of (19) the additional term

$$(19') \qquad c_3 2^{-jM} \sup_{|x-y|\leq c_4 2^{-j}} \sum_{0<|\alpha|<M} |(D^\alpha F^{-1}\varphi_{j+m} Ff)(y)| .$$

Via (20) and (21) it follows that the factor $\delta^{M_1 p} \| F^{-1}\varphi_{j+m} Ff | L_p \|^p$ on the right-hand side of (22) must be replaced by

$$(22') \qquad (\delta^{M_1 p} + 2^{-\varkappa(j+m)p}) \| F^{-1}\varphi_{j+m} Ff | L_p \|^p$$

with some $\varkappa > 0$ (and similarly in (23)). This shows that we have the additional term

$$(27') \qquad c\, 2^{Naq}\, 2^{N(M-s)q} \| f \mid B^{s-\varkappa}_{p,q} \|^q$$

on the right-hand side of (27). Let $s-\varkappa > \tilde{\sigma}_p$. By (11) (with $s-\varkappa$ instead of s) we have

$$\| f \mid B^{s-\varkappa}_{p,q} \| \leq \zeta \| f \mid B^s_{p,q} \| + c_\zeta \| f \mid L_p \|$$

where $\zeta > 0$ is a given positive number. In other words, if we have the additional terms (19') on the right-hand side of (19), then we obtain an estimate of type (16) with the additional term $c_\eta \| f | L_p \|$ on the right-hand side.

3.2. Theorem

We recall that in the following theorem $\Delta^M_{h+\varepsilon(x,h)}$ must be understood in the sense of the interpretation from (1).

Theorem 1. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > \tilde{\sigma}_p$. Let $\lambda > 0$ and let M be a natural number with $M > s$. Let the general assumptions for $\varepsilon(x,h)$ from Subsection 2.2 be satisfied. Then there exists a positive number $\delta = \delta(s,p,q,M,\lambda)$ with the following property: If

(28) $|\varepsilon(x,h)| \leq \delta |h|$ for all $x \in R_n$ and all $h \in R_n$ with $|h| \leq \lambda$

then

$$(29) \qquad \| f \mid B^s_{p,q} \|^\varepsilon_{M,\lambda} = \| f | L_p \| + \Big(\int_{|h|\leq\lambda} |h|^{-sq} \| \Delta^M_{h+\varepsilon(x,h)} f | L_p \|^q \frac{dh}{|h|^n} \Big)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $B^s_{p,q}$ (modification if $q=\infty$).

Proof. If $x\in R_n$ and $h\in R_n$ with $|h|\leq\lambda$ are fixed then we have

$$(\Delta^M_{h+\varepsilon} f)(y) = \left(F^{-1}\left(e^{i\xi(h+\varepsilon)}-1\right)^M Ff\right)(y)$$

$$= \left(F^{-1}\left[e^{i\xi h}-1+e^{i\xi h}\left(e^{i\xi\varepsilon}-1\right)\right]^M Ff\right)(y) \tag{30}$$

with $\varepsilon = \varepsilon(x,h)$. By this formula it follows that

$$(\Delta^M_{h+\varepsilon} f)(y) = (\Delta^M_h f)(y) + \sum_{M_1=1}^{M} c_{M_1}\left(\Delta^{M_1}_{\varepsilon}\Delta^{M-M_1}_h f\right)(y+M_1 h) \tag{31}$$

holds with some coefficients c_{M_1}. We put $y = x$. We assume without restriction of generality that $q<\infty$. Then (16) yields

$$\int_{|h|\leq\lambda} |h|^{-sq}\,\|\Delta^M_{h+\varepsilon(x,h)} f\,|L_p\|^q \frac{dh}{|h|^n} \tag{32}$$

$$\leq c\int_{|h|\leq\lambda} |h|^{-sq}\,\|\Delta^M_h f\,|L_p\|^q \frac{dh}{|h|^n} + c\,\|f\,|B^s_{p,q}\|^q$$

Consequently,

$$\|f\,|B^s_{p,q}\|^{\varepsilon}_{M,\lambda} \leq c\,\|f\,|B^s_{p,q}\|, \tag{33}$$

cf. (11). In order to prove the reverse assertion we put

$$(\Delta^M_h f)(x) = (\Delta^M_{h+\varepsilon} f)(x) - \sum_{M_1=1}^{M} c_{M_1}\left(\Delta^{M_1}_{\varepsilon}\Delta^{M-M_1}_h f\right)(x+M_1 h) \tag{34}$$

(cf. (21)) in (11). We use again (16) and obtain that

$$\|f\,|B^s_{p,q}\| \leq c\,\|f\,|B^s_{p,q}\|^{0}_{M,\lambda} \tag{35}$$

$$\leq c'\,\|f\,|B^s_{p,q}\|^{\varepsilon}_{M,\lambda} + \eta\,\|f\,|B^s_{p,q}\|$$

holds, where $\eta > 0$ is at our disposal. Let η be small. Then we arrive at

$$\|f\,|B^s_{p,q}\| \leq c\,\|f\,|B^s_{p,q}\|^{\varepsilon}_{M,\lambda} \tag{36}$$

The proof is complete.

4. Characterizations via $\Delta^M_{\rho^h}$ with $\rho^h(x) = h + \varepsilon(x,h)$

4.1. Two Preparations

We wish to replace $\Delta^M_{\rho^h} = \Delta^M_{h+\varepsilon(x,h)}$ in Theorem 1 by $\Delta^M_{\rho^h}$, where the latter stands for the second interpretation of iterated variable differences as it has been described in the Introduction, cf. (2) and (3). We wish to use the same ideas as in the proofs of the above Proposition (inclusively Remark 3) and Theorem 1. However there are several technical difficulties. It is the aim of this subsection to handle two of them.

A Representation Formula. First we look for preparations which at the end substitute the formulas (18) and (19). For this purpose we describe the structure of $(\Delta^M_\rho f)(x)$, where M is a natural number and the vector-function $\rho = \rho(x) = (\rho_1(x), \dots, \rho_n(x))$ has infinitely differentiable components. (As a matter of fact it would be sufficient to know that the components of $\rho(x)$ have continuous derivatives up to the order M-1). For smooth functions f(x) we have

$$(\Delta^1_\rho f)(x) = \int_0^1 \frac{d}{dt} f(x + t\rho(x))\,dt$$

$$(37) \qquad = \int_0^1 \sum_{j=1}^n \left(\frac{\partial f}{\partial y_j}\right)(x + t\rho(x))\,\rho_j(x)\,dt, \qquad x \in R_n$$

and

$$(\Delta^2_\rho f)(x) = \int_0^1 \frac{d}{dt}(\Delta^1_\rho f)(x + t\rho(x))\,dt$$

$$(38) \qquad = \sum_{j=1}^n \int_0^1 \int_0^1 \frac{d}{dt}\left\{\frac{\partial f}{\partial y_j}[x + t\rho(x) + \tau\rho(x + t\rho(x))]\,\rho_j(x + t\rho(x))\right\} d\tau\,dt.$$

However, $\frac{d}{dt}\{\ \}$ is the sum of terms of the type $\frac{\partial^2 f}{\partial y_j \partial y_k}(\cdots)\,\rho_{l_1}(\cdots)\,\rho_{l_2}(\cdots) H$ and $\frac{\partial f}{\partial y_j}(\cdots)\,\rho_l(\cdots)\,H$ where ... indicates appropriate arguments and H stands for a general function which differs from term to term and where first derivatives of the ρ_j's are involved. Iteration yields a corresponding representation formula for $(\Delta^M_\varrho f)(x)$ via terms of the type

(39) $\quad (D^\alpha f)(\cdots)\,\varrho_{l_1}(\cdots)\cdots\varrho_{l_{|\alpha|}}(\cdots)\,H^\alpha_{l_1,\dots,l_{|\alpha|}}$ with $0<|\alpha|\leqq M$,

where $H^\alpha_{l_1,\dots,l_{|\alpha|}}$ is a sum of products of at least $M-|\alpha|$ factors of the components of ϱ and its derivatives up to the order $M-1$ (with appropriate arguments). We discuss the omitted arguments in (39). Let $\varrho=\varrho(x,h)$ where $x\in R_n$ and $h\in R_n$ with, say, $|h|\leqq 1$, and let

(40) $\quad |\varrho(x,h)|\leqq\delta|h|$ and $|D^\alpha_x\varrho(x,h)|\leqq A|h|$ if $|\alpha|\leqq M-1$.

Then the arguments in the involved functions in (39) are represented by points which are contained in the ball $\{y\,|\,|x-y|\leqq c|h|\}$, where c is independent of h. Furthermore we have

(41) $$\left|(\Delta^M_{\varrho(\cdot,h)}f)(x)\right|\leqq c|h|^M\sup_{|x-y|\leqq c'|h|}\sum_{0<|\alpha|\leqq M}|(D^\alpha f)(y)|\,\delta^{|\alpha|}$$

where c and c' are independent of δ and h (but depend on A in (40)).

<u>An Inequality.</u> Next we look for a preparation which replaces (30) and (31). If $h\in R_n$ then T_h denotes the usual translation operator, i. e. $(T_hf)(x)=f(x+h)$. Then (2) yields

(42) $$\Delta_{h+\varepsilon(\cdot,h)}=\Delta_{\varepsilon(\cdot,h)}T_h+\Delta_h$$

and

(43) $$\Delta^M_{h+\varepsilon(\cdot,h)}=\Delta^M_h+\sum[\cdots]\,\Delta_{\varepsilon(\cdot,h)}[\cdots]=\Delta^M_h+R^M_h,$$

where $[\dots]$ indicates products of T_h, Δ_h and $\Delta_{\varepsilon(\cdot,h)}$. We assume that for some $\lambda>0$

(44) $\quad |\varepsilon(x,h)|\leqq\delta|h|$ and $|D^\alpha\varepsilon(x,h)|\leqq A|h|$,

$\qquad x\in R_n$, $h\in R_n$, $|h|\leqq\lambda$ and $|\alpha|\leqq M-1$.

holds. Then it follows in the same way as in (37) - (41) that

(45) $$|(R^M_hf)(x)|\leqq$$

$$c|h|^M\Big(\delta\sup_{|x-y|\leqq c'|h|}\sum_{|\alpha|=M}|(D^\alpha f)(y)|+\sup_{|x-y|\leqq c'|h|}\sum_{0<|\alpha|<M}|(D^\alpha f)(y)|\Big)$$

holds, where c and c' are independent of δ and h (but depend on A in (44)).

4.2. Theorem

After the above preparations we are in the position to prove the counterpart of Theorem 1 with $\Delta^M_{h+\varepsilon(\cdot,h)}$ instead of $\Delta^M_{h+\varepsilon(\cdot,h)}$

Theorem 2. Let $0 < p \leq \infty$, $0 < q \leq \infty$ and $s > \tilde{\sigma}_p$. Let $\lambda > 0$ and let M be a natural number with $M > s$. Let the general assumptions for $\varepsilon(x,h)$ from Subsection 2.2 be satisfied. Let additionally the components of $\varepsilon(x,h)$ be M-1 times continuously differentiable (with respect to $x \in R_n$) with

(46) $|D^\alpha_x \varepsilon(x,h)| \leq A|h|$ if $x \in R_n$, $|h| \leq \lambda$, and $|\alpha| \leq M-1$

for some positive number A. Then there exists a positive number δ (which depends on s,p,q,M, λ and the number A in (46)) with the following property: If

(47) $|\varepsilon(x,h)| \leq \delta |h|$ for all $x \in R_n$ and $h \in R_n$ with $|h| \leq \lambda$

then

(48) $$\|f|B^s_{p,q}\|_{M,\lambda,\varepsilon} = \|f|L_p\| + \left(\int_{|h| \leq \lambda} |h|^{-sq} \|\Delta^M_{h+\varepsilon(\cdot,h)} f|L_p\|^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $B^s_{p,q}$ (modification if $q = \infty$).

Proof. Step 1. Let R^M_h be the remainder term from (43). It is a linear operator. Let $\eta > 0$ be given. Then we claim that

(49) $$\left(\int_{|h| \leq \lambda} |h|^{-sq} \|R^M_h f(x)|L_p\|^q \frac{dh}{|h|^n} \right)^{\frac{1}{q}} \leq \eta \|f|B^s_{p,q}\| + c_\eta \|f|L_p\|$$

holds for all $f \in B^s_{p,q}$ provided that the positive number δ in (47) is sufficiently small (modification if $q = \infty$). The proof is the same as in the above Proposition: Let again $\lambda = 2^{-K}$, $q < \infty$ (without restriction of generality) and $2^{-j-1} \leq |h| \leq 2^{-j}$ with j =

K, K+1,... We use (17), the splitting (18) with R_h^M instead of $\Delta^{M_1}_{\varepsilon(x,h)}\Delta^{M_2}_{h}$ and (45) with $F^{-1}\varphi_{j+m}Ff$ instead of f. Then the counterpart of (19) reads as follows: There exists a constant c such that for integers m (with $m \leq N$)

$$|(R_h^M F^{-1}\varphi_{j+m}Ff)(x)| \leq c\, 2^{-jM}\Big[\delta \sup_{|x-y|\leq c' 2^{-j}} \sum_{|\alpha|=M} |(D^\alpha F^{-1}\varphi_{j+m}Ff)(y)|$$

$$+ \sup_{|x-y|\leq c' 2^{-j}} \sum_{0<|\alpha|<M} |(D^\alpha F^{-1}\varphi_{j+m}Ff)(y)|\Big] \tag{50}$$

holds. This is the modification which we treated in Remark 3, cf. (19'). Let $m \geq N+1$. We recall that R_h^M is the sum of iterated differences, cf. (42), (43), with iterated smooth one-to-one mappings $x \longrightarrow x+h$ and $x \longrightarrow x+\varepsilon(x,h)$ of R_n onto itself. The iterations of the latter mappings are also one-to-one mappings of R_n onto itself. This yields obvious counterparts of (24) and (25). The rest is now the same as in the proof of the Proposition and the considerations in Remark 3. This proves (49).

Step 2. We use (43) and (49). Then the above theorem follows in the same way as at the end of the proof of Theorem 1.

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Sektion Mathematik
Universität Jena
DDR-6900 Jena
Universitäts-Hochhaus

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