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BOUNDARY VALUE PROBLEMS WITH NONLINEARITIES HAVING INFINITE JUMPS

Jean MAWHIN

Dedicated to the memory of Svatopluk FUČÍK

Abstract: We extend some results of Ward for nonlinear perturbations of linear operators whose kernel is made of constant functions to the case where the kernel is spanned by a positive function. Applications are given which extend earlier results of Aguinaldo-Schmitt and Castro.

Key words: Boundary value problems for ordinary differential equations, jumping nonlinearities, Leray-Schauder method.

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1. Introduction. In his fundamental work on nonlinear noncoercive equations, Fučík has emphasized the important concept of "jumping nonlinearity" and has given in [6] the first systematic study of the Dirichlet problem for second order ordinary differential equations with jumping nonlinearities, namely

$$x''(t) + g(x(t)) = h(t),$$

$$x(0) = x(\pi) = 0$$

with $\lim_{x \to -\infty} g(x)/x \neq \lim_{x \to +\infty} g(x)/x$. As most of Fučík's papers, [6] not only contains significant results but also a number of interesting open questions. One of them was solved by Aguinaldo and Schmitt [1] who proved that the problem

$$
\begin{aligned}
&(1.1) \qquad x''(t) + x(t) - \alpha x^-(t) = h(t) \\
&\qquad\qquad x(0) = x(\pi) = 0
\end{aligned}
$$

with $\alpha > 0$, $x^- = \max(-x,0)$ and h continuous has a solution if and only if

$$(1.2) \qquad \int_0^{\pi} h(s) \sin s \, ds \leq 0.$$

Aguinaldo and Schmitt use a continuation theorem due to the author (see e.g. [8]) and obtain the required a priori bounds by a delicate argument linked to the special nature of the nonlinear term in (1.1). Their result was generalized by Castro [5] who proved the sufficient condition (1.2) for the more general problem

$$
\begin{aligned}
&(1.3) \qquad x''(t) + x(t) + g(x(t)) = h(t), \\
&\qquad\qquad x(0) = x(\pi) = 0
\end{aligned}
$$

with $g: R \longrightarrow R$ continuous, $g(x) = 0$ for $x \geq 0$ and $g(x)/x \longrightarrow \alpha > 0$ when $x \longrightarrow -\infty$. Castro's proof uses a rather sophisticated variational argument which strongly uses the sublinear character of g.

The aim of this paper is to provide a partial extension of the method initiated by Ward [9] for the study of periodic solutions of semi-linear ordinary differential equations whose linear part only admits constant periodic solutions. This extension allows the kernel of the linear part to be spanned by a positive function and provides generalizations of the results of Aguinaldo-Schmitt and Castro to ordinary differential equations of arbitrary order and to some classes of nonlinearities which do not have necessarily a linear growth. Finally, the underlying abstract tool is simply a continuation theorem of Leray-Schauder type [8] and the corresponding a priori bounds are obtained in a rather simple way.

In the case of (1.3), our theorem implies the existence of a solution when (1.2) holds when $h \in L^1(0,\pi)$, $g(x) = 0$ for $x \geq 0$ and

$$\lim_{x \to -\infty} \sup g(x) = -\infty .$$

Another easy consequence of our results is that the problem, with $h \in L^1(0,\pi)$

$$x''(t) + x(t) + \alpha \exp x(t) = h(t)$$
$$x(0) = x(\pi) = 0$$

with $\alpha > 0$ has a solution if and only if

$$(1.4) \qquad \int_0^\pi h(t) \sin t \, dt > 0.$$

Finally, our method easily shows that (1.4) is also sufficient for the existence of one solution for the problem

$$x'''(t) + x'(t) + \alpha \exp[x(t) + \sin x'(t)] = h(t),$$
$$x(0) = x'(0) = x'(\pi) = 0.$$

2. <u>Preliminary results on linear operators.</u> Let $I = [a,b]$, $k \geq 0$ an integer, $C^k(I)$ the Banach space of real functions of class C^k on I, with the usual norm $|u|_{C^k} = \sum_{j=0}^{k} \max_{t \in I} |u^{(j)}(t)|$, $L^1(I)$ the Banach space of real functions L-integrable on I with the usual norm

$$|u|_{L^1} = \int_I |u(t)| \, dt.$$

Let $L: D(L) \subset C^k(I) \longrightarrow L^1(I)$ be a closed linear operator having the following properties.

(L_1) $\ker L = \operatorname{span}\{\varphi\}$, with $\varphi \in D(L)$ such that $\varphi(t) > 0$ for a.e. $t \in I$ and $\int_I \varphi(t)dt = 1$.

(L_2) $\operatorname{Im} L = \{y \in L^1(I): \int_I y(t)\psi(t)dt = 0\}$ for some $\psi \in L^\infty(I)$ such that $\int_I \varphi(t)\psi(t)dt = 1$ and $\psi(t) > 0$ for a.e. $t \in I$.

Let us denote by $L^1_\psi(I)$ the Banach space of real functions L-integrable on I with the norm

$$|u|_\psi = \int_I |u(t)|\, \psi(t)dt,$$

and let us introduce the following closed subspaces of $C^k(I)$, $L^1(I)$, $L^1_\psi(I)$,

$\tilde{C}^k(I)$ is a topological direct summand of ker L in $C^k(I)$,

$$\tilde{L}^1(I) = \{y \in L^1(I):\ \int_I y(t)\ \psi(t)dt = 0\},$$

$$\tilde{L}^1_\psi(I) = \{y \in L^1_\psi(I):\ \int_I y(t)\ \psi(t)dt = 0\}.$$

We introduce another assumption upon L.

(L_3) There exists a continuous linear operator $A: C^k(I) \to L^1(I)$ such that $L - A: D(L) \subset C^k(I) \to L^1(I)$ is one-to-one and onto and such that for some $M \geq 0$ and all $y \in L^1_\psi(I)$, one has

$$|(L - A)^{-1} y|_{C^k} \leq M |y|_\psi .$$

Proposition 1. If conditions (L_1) to (L_3) hold, there exists $\Lambda \geq 0$ such that, for each $x = \bar{x} + \tilde{x} \in D(L)$, with $\bar{x} \in \ker L$, $\tilde{x} \in \tilde{C}^k(I)$, one has

$$|\tilde{x}|_{C^k} \leq \Lambda\, |L\tilde{x}|_\psi = \Lambda\, |Lx|_\psi .$$

Proof. The restriction of L to $D(L) \cap \tilde{C}^k(I)$ being one-to-one and onto $\tilde{L}^1_\psi(I)$, it suffices, by the closed graph theorem, to show that this restriction is a closed operator. By condition (L_3), $(L - A)^{-1}: L^1_\psi(I) \to C^k(I)$ is continuous and hence $L - A: D(L) \subset C^k(I) \to L^1_\psi(I)$ is closed. Let $(\tilde{x}_n)$ be a sequence in $D(L) \cap \tilde{C}^k(I)$ such that $\tilde{x}_n \to \tilde{x} \in \tilde{C}^k(I)$ and $L\tilde{x}_n \to \tilde{y} \in \tilde{L}^1_\psi(I)$ in $L^1_\psi(I)$. Then $A\tilde{x}_n \to A\tilde{x}$ in $L^1(I)$ and hence in $L^1_\psi(I)$ so that $(L - A)\tilde{x}_n \to \tilde{y} - A\tilde{x}$ in $L^1_\psi(I)$. By the closedness of $L - A$ as a mapping between $D(L) \subset C^k(I)$ and $L^1_\psi(I)$, we have $\tilde{x} \in D(L)$ and

$\tilde{y} - A\tilde{x} = L\tilde{x} - A\tilde{x}$, so that $\tilde{y} = L\tilde{x}$. Thus $L: D(L) \subset \tilde{C}^k(I) \to \tilde{L}^1_\psi(I)$ is closed, and the proof is complete.

In our applications, L will be a differential operator and condition (L_3) can often be deduced from the following more concrete assumption.

(L_3') $k = 0$ and there exists $A: C^0(I) \to L^1(I)$ linear continuous and $G_A \in C^0(I \times I)$ such that $L - A: D(L) \subset C^0(I) \to L^1(I)$ is one-to-one and onto,

$$(2.1) \qquad (L - A)^{-1} y(t) = \int_I G_A(t,s) y(s) ds, \quad t \in I$$

and $G_A/\psi \in L^\infty(I \times I)$.

<u>Proposition 2</u>. If conditions (L_1), (L_2) and (L_3') hold, then the conclusion of Proposition 1 is valid with $k = 0$.

<u>Proof</u>. By (2.1), we have, for each $y \in L^1(I)$ and $t \in I$,

$$|(L-A)^{-1} y(t)| = \left| \int_I [G_A(t,s)/\psi(s)] \, y(s) \psi(s) ds \right| \leq$$

$$\leq |G_A/\psi|_{L^\infty} \int_I |y(s)| \psi(s) ds = |G_A/\psi|_{L^\infty} |y|_\psi .$$

Hence condition (L_3) with $k = 0$ holds and the result follows from Proposition 1.

<u>Example 1</u>. As a first example, let L be defined by $D(L) = \{x \in C^0[0,\pi] :$ x is of class C^1 on $I = [0,\pi]$, x' is absolutely continuous on I and $x(0) = x(\pi) = 0\}$, $L: D(L) \subset C^0(I) \to L^1(I)$, $x \mapsto -x'' - x$, so that L is closed, $\ker L = \operatorname{span}\{\sin(\cdot)\}$, $\operatorname{Im} L = \{y \in L^1(I): \int_I y(t) \sin t \, dt = 0\}$ and we can take $\tilde{C}^0(I) = \{x \in C^0(I): \int_I x(t) \sin t \, dt = 0\}$.
Moreover, for $A = -\mathrm{Id}$, $G_A = G$, the usual Green function of $-d^2/dt^2$ with the Dirichlet boundary conditions on $[0,\pi]$, namely

$$G(t,s) = \begin{cases} (s/\pi)(\pi - t) & \text{if } 0 \le s \le t \le \pi, \\ (t/\pi)(\pi - s) & \text{if } 0 \le t \le s \le \pi. \end{cases}$$

Therefore, if $0 < s \le t < \pi$, we have $0 < \pi - t \le \pi - s$ and hence

$$0 \le G(t,s)/\sin s \le s(\pi - s)/\pi \sin s \le C$$

as $\lim_{s \to 0} s/\sin s = \lim_{s \to \pi} (\pi - s)/\sin s = 1$, and similarly for $0 < t \le s < \pi$. Thus, all the conditions of Proposition 2 are satisfied and hence

$$|\tilde{x}|_{C^0} \le \Lambda \int_I |x''(t) + x(t)| \sin t \, dt = \Lambda \, |x'' + x|_{\sin}$$

for all $x = \bar{x} + \tilde{x} \in D(L)$ with $\bar{x} = c \sin(\cdot)$ and $\int_0^\pi \tilde{x}(t) \sin t \, dt = 0$.

<u>Example 2</u>. For a less direct application of the above result, let L_1 be defined by $D(L_1) = \{x \in C^1(I) : I = [0, \pi],\ x \in C^1(I),\ x''$ is absolutely continuous on I and $x(0) = x'(0) = x'(\pi) = 0\}$, $L_1 : D(L_1) \subset C^1(I) \to L^1(I)$, $x \mapsto -x''' - x'$, so that L_1 is closed,

$$\ker L_1 = \operatorname{span}\{1 - \cos(\cdot)\}$$

$$\operatorname{Im} L_1 = \{y \in L^1(I) : \int_I y(t) \sin t \, dt = 0\}.$$

Let $y = x'$, so that, as $x(0) = 0$, $x(t) = \int_0^t y(s)\, ds$, and

$$y(0) = y(\pi) = 0, \quad -x''' - x' = -y'' - y.$$

Therefore, by Example 1 applied to y, we have, for all $y = \bar{y} + \tilde{y} \in D(L)$ with $\bar{y}(t) = c \sin t$ and $\int_0^\pi \tilde{y}(t) \sin t \, dt = 0$,

$$|\tilde{y}|_{C^0} \le \Lambda |\tilde{y}'' + \tilde{y}|_{\sin} = \Lambda |y'' + y|_{\sin} = \Lambda |x''' + x'|_{\sin}.$$

Consequently, as

$$x'(t) = y(t) = c \sin t + \tilde{y}(t)$$

we have

$$|x'(\cdot) - c \sin(\cdot)|_{C^0} \le \Lambda |x''' + x'|_{\sin}$$

$$|x(\cdot) - c(1 - \cos(\cdot))|_{C^0} = |\int_0^{\cdot} \tilde{y}(s)ds|_{C^0} \leq \pi\Lambda |x''' + x'|_{\sin}$$

for all $x \in D(L_1)$. Thus, if we define the bounded linear operator $P: C^1(I) \longrightarrow C^1(I)$ by

$$(Px)(t) = [(2/\pi) \int_0^{\pi} x'(s) \sin s \, ds] (1 - \cos t)$$

it is easy to check that $\operatorname{Im} P = \ker L_1$ and that $P^2 = P$ so that we can take $\tilde{C}^1(I) = \ker P$. Thus, if we write $x(t) = \bar{x}(t) + \tilde{x}(t)$ with $\bar{x} = Px$ and $\tilde{x} = (I - P)(x)$, then, with the notations above, $\bar{x}(t) = c(1 - \cos t)$, and the above inequalities can be written

$$|\tilde{x}'|_{C^0} \leq \Lambda |x''' + x'|_{\sin}, \quad |\tilde{x}|_{C^0} \leq \pi\Lambda |x''' + x'|_{\sin}$$

i.e.

$$|\tilde{x}|_{C^1} \leq (1 + \pi)\Lambda |x''' + x'|_{\sin}.$$

3. <u>An existence theorem for abstract boundary value problems.</u> Let now $f: I \times R^{k+1} \longrightarrow R$ be such that $f(t,\cdot)$ is continuous on R^{k+1} for a.e. $t \in I$ and $f(\cdot,y)$ is measurable on I for each $y \in R^{k+1}$. Assume moreover that for each $r > 0$, there exists $a_r \in L^1(I)$ such that

$$|f(t,y)| \leq a_r(t)$$

whenever $t \in I$ and $|y| \leq r$. Such an f will be called a Carathéodory function for $L^1(I)$.

Let us introduce the following condition of Ward type (see [9] and [2 - 4, 10] for various extensions).

(f_1) There exists $\gamma \in L^1(I)$ and $\varepsilon = \pm 1$ such that $|\gamma|_{\psi} > 0$ and

$$|f(t,y)| \leq \varepsilon f(t,y) + \gamma(t)$$

for a.e. $t \in I$ and all $y \in R^{k+1}$.

Such a condition expresses the fact that f is either bounded below or bounded above with respect to y.

We shall consider the following abstract boundary value problem

$$(3.1) \qquad (Lx)(t) = f(t, x(t), x'(t), \ldots, x^{(k)}(t)), \quad t \in I$$

where L is a linear operator of the type considered in section 2 and satisfying moreover the following compactness condition.

(L_4) The inverse of the operator $L: D(L) \subset \tilde{C}^k(I) \longrightarrow \tilde{L}^1(I)$ is compact.

The conditions (L_1) to (L_4) and (f_1) are not sufficient to insure the existence of a solution for (3.1), as shown by the trivial example

$$x''(t) + x(t) = \sin t$$
$$x(0) = x(\pi) = 0$$

for which they are satisfied and which has no solution.

We introduce a supplementary sign condition upon f.

(f_2) There exists $\sigma = \pm 1$ and $\rho > 0$ such that

$$\sigma \int_I f(t, c\varphi(t) + \tilde{v}(t), \ldots, c\varphi^{(k)}(t) + \tilde{v}^{(k)}(t))\psi(t)dt \leq 0$$

whenever $c \leq -\rho$ and $|\tilde{v}|_{C^k} \leq \Lambda |\gamma|_\psi$, and

$$\sigma \int_I f(t, c\varphi(t) + \tilde{v}(t), \ldots, c\varphi^{(k)}(t) + \tilde{v}^{(k)}(t))\psi(t)dt \geq 0$$

whenever $c \geq \rho$ and $|\tilde{v}|_{C^k} \leq \Lambda |\gamma|_\psi$, where Λ is given by Proposition 1 and $\tilde{v} \in D(L)$ with $\int_I \tilde{v}(t)\varphi(t)\, dt = 0$.

We can now prove the following existence theorem.

<u>Theorem 1</u>. Assume that L satisfies the conditions (L_1) to (L_4) and that f satisfies the conditions (f_1) and (f_2). Then problem (3.1) has at least one solution.

<u>Proof</u>. Let $F: C^k(I) \longrightarrow L^1(I)$ be the Nemitsky operator

associated to f and defined by

$$Fx = f(\cdot,x(\cdot),x'(\cdot),\ldots,x^{(k)}(\cdot)),$$

so that (3.1) is equivalent to the abstract equation

$$Lx = Fx$$

in $C^k(I)$, and F is L-completely continuous on $C^k(I)$. Let $x(t) = \bar{x}(t) + \tilde{x}(t)$, with $\bar{x} \in \ker L$ and $\tilde{x} \in \tilde{C}^k(I)$, and define $G: C^k(I) \to L^1(I)$ by

$$G\,x = (1 - |\bar{x}|_{C^k})^{-1}(\delta\eta/2)\bar{x}(\cdot), \quad \eta = |\gamma|_\psi / |\psi|_{L^1},$$

so that G is odd, L-completely continuous and

$$|(Gx)(t)| \leq \eta/2$$

for a.e. $t \in I$. By Theorem IV.3 and Proposition II.18 of [8], (3.1) will have a solution if the set of possible solutions of the family of equations

(3.2) $$Lx = (1 - \lambda)Gx + \lambda Fx, \quad \lambda \in [0,1[,$$

is a priori bounded independently of λ. Let $\lambda \in [0,1[$ and x be a possible solution of (3.2). Then,

(3.3) $$0 = (1 - \lambda)\int_I (Gx)(t)\psi(t)\,dt + \lambda\int_I (Fx)(t)\psi(t)\,dt$$

and

$$\int_I |(Lx)(t)|\psi(t)\,dt \leq (1-\lambda)\int_I |(Gx)(t)|\psi(t)\,dt + \lambda\int_I |(Fx)(t)|\psi(t)\,dt.$$

Using condition (f_1) and (3.3), the last inequality implies that

$$|Lx|_\psi \leq (1 - \lambda)|\gamma|_\psi/2 + \lambda\,\varepsilon\int_I (Fx)(t)\psi(t(dt) +$$
$$+ \lambda\int_I \gamma(t)\,\psi(t)\,dt = (1 - \lambda)|\gamma|_\psi/2 - \varepsilon(1 - \lambda)\int_I (Gx)(t)\psi$$
$$(t)\,dt + \lambda\int_I \gamma(t)\psi(t)dt \leq (1 - \lambda)|\gamma|_\psi + \lambda|\gamma|_\psi = |\gamma|_\psi.$$

Consequently, using Proposition 1, we have

(3.4) $$|\tilde{x}|_{C^k} \leq \Lambda|\gamma|_\psi.$$

If we set $\bar{x}(t) = c\,\varphi(t)$, then by (3.4) and condition (f_2) we get, if $c \leq -\rho$,

$$(1 - \lambda)\eta\int_I (1 + |\bar{x}|_{C^k})^{-1}\bar{x}(t)\psi(t)\,dt + \delta\lambda\int_I f(t,x(t),\ldots$$

$\ldots, x^{(k)}(t))\psi(t)\ dt = (1-\lambda)(1+|\bar{x}|_{C^k})^{-1}\eta c +$

$+ \delta\lambda \int_I f(t, c\varphi(t) + \tilde{x}(t), \ldots, c\varphi^{(k)}(t) + \tilde{x}^{(k)}(t))\psi(t)\ dt \leq$

$\leq (1-\lambda)(1+|\bar{x}|_{C^k})^{-1}\eta c \leq -(1-\lambda)(1+|\bar{x}|_{C^k})^{-1}\eta\rho < 0$

so that (3.3) cannot hold. Similarly if $c \geq \rho$, which implies that we have necessarily

$$|c| < \rho,$$

and hence, by (3.4),

$$|x|_{C^k} \leq |c\varphi|_{C^k} + |\tilde{x}|_{C^k} < \rho|\varphi|_{C^k} + \Lambda|\gamma|_\psi = R$$

and the proof is complete.

It has already been noticed that the sign condition (f_2) contains as special case Landesman-Lazer conditions of the following type.

(f_2') $k = 0$, there exist functions $\delta_+ \in L^1(I)$ and $\delta_- \in L^1(I$ such that

$$f(t,y) \geq \delta_+(t) \text{ if } y \geq 0$$

and

$$f(t,y) \leq \delta_-(t) \text{ if } y \leq 0,$$

and the measurable functions μ_1 and μ_2 defined by

$$\mu_1(t) = \limsup_{y \to -\infty} f(t,y), \quad \mu_2(t) = \liminf_{y \to +\infty} f(t,y),$$

are such that

$$\int_I \mu_1(t)\psi(t)dt < 0 < \int_I \mu_2(t)\psi(t)dt.$$

We give a proof for completeness.

<u>Proposition 3</u>. Condition (f_2') implies condition (f_2) with $k = 0$ and $\delta = 1$.

<u>Proof</u>. If it is not the case, there will exist sequences (c_n) and $(\tilde{x}_n)$ with $|\tilde{x}_n|_{C^0} \leq \Lambda|\gamma|_\psi$ such that either $c_n \to -\infty$

and

$$\int_I f(t, c_n \varphi(t) + \tilde{v}_n(t))\,\psi(t)\,dt > 0$$

or such that $c_n \to +\infty$ and

$$\int_I f(t, c_n \varphi(t) + \tilde{v}_n(t))\,\psi(t)\,dt < 0.$$

Considering, say, the first case, we have, for a.e. $t \in I$,

$$c_n \varphi(t) + \tilde{v}_n(t) \to -\infty$$

if $n \to \infty$, and hence, by Fatou's lemma.

$$0 \le \limsup_{n \to \infty} \int_I f(t, c_n \varphi(t) + \tilde{v}_n(t))\,\psi(t)\,dt \le$$

$$\le \int_I \limsup_{n \to \infty} f(t, c_n \varphi(t) + \tilde{v}_n(t))\,\psi(t)\,dt \le$$

$$\le \int_I \limsup_{x \to -\infty} f(t,x)\,\psi(t)\,dt = \int_I \mu_1(t)\,\psi(t)\,dt < 0,$$

a contradiction.

One can show similarly that the following condition (f_2'') implies condition (f_2) with $k = 0$ and $\delta = -1$.

(f_2'') $k = 0$, there exist functions $\delta_+ \in L^1(I)$ and $\delta_- \in L^1(I)$ such that

$$f(t,y) \le \delta_+(t) \text{ if } y \ge 0$$
$$f(t,y) \ge \delta_-(t) \text{ if } y \le 0$$

and the measurable functions μ_1 and μ_2 defined by

$$\mu_2(t) = \liminf_{x \to -\infty} f(t,x), \quad \mu_1(t) = \limsup_{x \to +\infty} f(t,x)$$

are such that

$$\int_I \mu_1(t)\,\psi(t)\,dt < 0 < \int_I \mu_2(t)\,\psi(t)\,dt.$$

Let us mention the following obvious Corollary of Theorem 1.

Corollary 1. Assume that L satisfies conditions (L_1), (L_2), (L_3'), (L_4) and that f satisfies condition (f_1) and (f_2') or (f_2''). Then the problem

$$(Lx)(t) = f(t.x(t))$$

has at least one solution.

In particular, if $f(t,x) = h(t) - g(x)$, with $h \in L^1(I)$, $g: R \to R$ is continuous, $g(x) = 0$ for $x \geq 0$, $\limsup_{x \to -\infty} g(x) = -\infty$, then condition (f_1) holds with $\varepsilon = 1$ and

$\liminf_{x \to -\infty} f(t,x) = h(t) - \limsup_{x \to -\infty} g(x) = +\infty$,

$\limsup_{x \to +\infty} f(t,x) = h(t) - \liminf_{x \to +\infty} g(x) = h(t)$,

so that condition (f_2'') becomes here

$$\int_I h(t)\psi(t)\ dt < 0.$$

In the special case where $Lx = x'' + x$ with the Dirichlet boundary conditions on $[0,\pi]$, $\psi(t) = \sin t$, all conditions (L_1), (L_2), (L_3'), (L_4) are satisfied (see Example 1 in Section 2) and we obtain the generalization of the results of Aguinaldo-Schmitt and Castro announced in the Introduction.

If $f(t,x) = h(t) - \alpha \exp x$, with $h \in L^1(I)$ and $\alpha > 0$, then condition (f_1) holds with $\varepsilon = -1$ and

$$\liminf_{x \to -\infty} f(t,x) = h(t), \quad \limsup_{x \to +\infty} f(t,x) = -\infty ,$$

so that condition (f_2'') becomes

$$0 < \int_I h(t)\psi(t)\ dt.$$

In the special case where $Lx = x'' + x$ with the Dirichlet boundary conditions on $[0,\pi]$, we again find the condition

$$0 < \int_I h(t) \sin t\ dt$$

announced in the Introduction. Notice that when $\alpha < 0$, our result can also be applied and furnishes the existence condition

$$\int_I h(t) \sin t\ dt < 0,$$

but, in contrast to the case where $\alpha > 0$, the situation with $\alpha < 0$ can also be treated by the method of upper and lower solutions (see e.g. [7], Chapter 32). Notice that those conditions are also necessary for the existence of a solution.

As a last example, let us consider the following boundary value problem

$$(3.5)\qquad x'''(t) + x'(t) + \alpha \exp [x(t) + \sin x'(t)] = h(t)$$
$$x(0) = x'(0) = x'(\pi) = 0$$

where $h \in L^1(I)$, $I = [0, \pi]$ and $\alpha \neq 0$. It follows easily from Example 2 that a necessary condition for the existence of a solution of (3.5) is that

$$(3.6)\qquad \alpha \int_0^{\pi} h(t) \sin t \, dt > 0.$$

Combining the results of Example 2 with Theorem 1, it is easy to show that this condition is also sufficient.

<u>Remark 1.</u> In the case of nonlinear perturbations of linear operators whose kernel is made of constant functions, Ward's growth conditions on the nonlinear term f are of the form

$$|f(t,y)| \leq \varepsilon f(t,y) + \beta |y| + \gamma(t)$$

with β sufficiently small. Our approach in the setting of a kernel spanned by a positive function does not seem to extend easily to such a growth condition with $\beta > 0$ and it is an open problem to know if the results of this paper are true or not in this more general setting.

<u>Remark 2.</u> The same method can obviously be applied to boundary value problems for functional-differential equations, as well as to boundary value problems for systems of equations, with generalized Ward conditions in the line of [2, 3].

R e f e r e n c e s

[1] L. AGUINALDO and K. SCHMITT: On the boundary value problem $u'' + u = \alpha u^- + p(t)$, $u(0) = 0 = u(\pi)$, Proc. Amer. Math. Soc. 68(1978), 64-68.

[2] A. CANADA and P. MARTINEZ-AMORES: Periodic solutions of nonlinear vector ordinary differential equations of higher order at resonance, Nonlinear Anal. 7 (1983), 747-761.

[3] A. CANADA and P. MARTINEZ-AMORES: Solvability of some operator equations and periodic solutions of nonlinear functional differential equations, J. Differential Equations 48(1983), 415-429.

[4] A. CANADA and P. MARTINEZ-AMORES: Periodic solutions of neutral functional differential equations, in "Equadiff 82", Knobloch and Schmitt ed., Lecture Notes in Math. n° 1017, Springer, Berlin, 1983, 115-121.

[5] A. CASTRO: A two point boundary value problem with jumping nonlinearities, Proc. Amer. Math. Soc. 79(1980), 207-211.

[6] S. FUČÍK: Boundary value problems with jumping nonlinearities, Časopis pěstov. mat. 101(1976), 69-87.

[7] S. FUČÍK: Solvability of Nonlinear Equations and Boundary Value Problems, Reidel, Dordrecht, 1980.

[8] J. MAWHIN: Topological Degree Methods in Nonlinear Boundary Value Problems, Regional Confer. Series in Math. n° 40, Amer. Math. Soc., Providence, 1979.

[9] J.R. WARD: Asymptotic conditions for periodic solutions of ordinary differential equations, Proc. Amer. Math. Soc. 81(1981), 415-420.

[10] J.R. WARD: Existence for a class of semilinear problems at resonance, J. Differential Equations 45(1982), 156-167.

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