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REMARKS ON NONLINEAR NONCOERCIVE PROBLEMS WITH JUMPING NONLINEARITIES

Pavel DRÁBEK

Dedicated to the memory of Svatopluk FUČÍK

Abstract: We are interested in the investigation of the equations of the type

$$(0.1) \qquad J(x) - \mu S(x^+) + \nu S(x^-) + G(x) = f$$

which were intensively studied in the principal Fučík s papers. The purpose of this paper is to give a short survey of the results in this field which have been published during last five years and also to formulate some open problems the solutions of which, in the author s opinion, would lead to the better understanding of the equations in question.

Key words: Boundary value problems for ordinary differential equations, spectral theory of nonlinear operators.

Classification: 34B15, 34B25, 34C10, 47H12

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1. Introduction. In his paper [9], Fučík emphasized the concept of "jumping nonlinearity" and in this framework he studied the solvability of the Dirichlet problem for second order ordinary differential equations

$$(1.1) \qquad \begin{cases} -u''(t) + g(u(t)) = f(t), \ t \in]0,\pi[, \\ u(0) = u(\pi) = 0, \end{cases}$$

with nonlinearity $g: \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$(1.2) \qquad \lim_{s \to -\infty} \frac{g(s)}{s} = \nu \ , \quad \lim_{s \to +\infty} \frac{g(s)}{s} = \mu \ ,$$

where $\mu \neq \nu$ are real numbers. These results were afterwards generalized by Fučík himself and by many other authors in various directions (i.e. μ and ν acquire values $+\infty$ or $-\infty$, or the partial differential operator of elliptic type is considered instead of $-u''$, e.t.c.). An exhaustive list of references up to 1980 is given in the monography [11]. In the last two years many papers have appeared which deal with the multiplicity of the solutions of the problem (1.1). For the most recent results in this direction and also for an other bibliography see [16],[20].

In this paper we shall concentrate on the case of finite limits (1.2) and on existence results. The following parts of the paper are organized as follows. In Section 2 the abstract formulation of the problems in question is given and there is shown the connection between the problems with jumping nonlinearities and the nonlinear Fredholm alternative developed independently by Nečas [17] and Pochožajev [18] (see also [7]). Section 3 contains some applications of abstract results to Dirichlet and periodic boundary value problems for ordinary differential equations of second and fourth order. We mention also some local results for partial differential equations. Finally, in Section 4 we formulate some open problems which are mostly motivated by the known results in some particular cases.

2. Operator equation with jumping nonlinearity. Let us suppose that X, Y, Z are Banach spaces with zero elements 0_X, 0_Y, 0_Z and with norms $\|x\|_X$, $\|y\|_Y$, $\|z\|_Z$, respectively. A subset C of Z is called a cone if it is closed, convex, invariant under multiplication by nonnegative real numbers and if $C\cap(-C) = \{0_Z\}$. We shall suppose that the following assumptions are fulfilled.

(Z1) C induces the semiordering $x \leq y$ (i.e. $(y - x) \in C$) such that $z^+ = \max\{z, 0_Z\} \in C$, $z^- = \max\{-z, 0_Z\} \in C$ exists for every $z \in Z$.

(Z2) The mapping $z \mapsto z^+$ is continuous.

(Z3) $X \subset Z$ and the identity mapping $X \hookrightarrow Z$ is continuous.

Let us suppose that $a > 0$ is a fixed real number and $J: X \longrightarrow Y$ is the mapping which satisfies the following properties:

(J1) J is positively a-homogeneous, i.e. $J(tx) = t^a J(x)$ for all $x \in X$, $t > 0$.

(J2) J is a homeomorphism X onto Y.

(J3) J is odd, i.e. $x \in X \Rightarrow J(-x) = -J(x)$.

Let $S: Z \longrightarrow Y$ be the operator defined on Z and satisfying

(S1) S is positively a-homogeneous.

(S2) S is continuous.

(S3) $x \mapsto S(x^+)$, $x \mapsto S(x^-)$ are completely continuous mappings from X onto Y.

Suppose that $G: X \longrightarrow Y$ is a completely continuous operator. According to the works of Dancer [2, 3] and Fučík [9, 10, 11] we shall denote

$\mathcal{R}(G) = \{f \in Y;\ \exists x_0 \in X:\ J(x_0) - \mu S(x_0^+) + \nu S(x_0^-) + G(x_0) = f\}$,

$\mathcal{R}(0)$ is written in case $G \equiv 0$;

$A_{-1} = \{(\mu, \nu) \in \mathbb{R}^2;\ \exists x_0 \neq 0_X:\ J(x_0) - \mu S(x_0^+) + \nu S(x_0^-) = 0_Y\}$;

$A_0 = \mathbb{R}^2 \setminus A_{-1}$;

$A_1 = \{(\mu, \nu) \in A_0;\ \deg[y - \mu S(J^{-1}(y))^+ + \nu S(J^{-1}(y))^-;\ B_Y(1), 0_Y] \neq 0\}$;

$A_2 = \{(\mu, \nu) \in A_0;\ \mathcal{R}(0) \neq Y\}$,

$A_3 = \{(\mu, \nu) \in \mathbb{R}^2;\ \mathcal{R}(0) = Y\}$.

We refer to [4],[11] where the basic properties of the sets A_i, $i = -1,\dots,3$, are proved. In the sequel we shall show how the nonlinear Fredholm alternative for quasihomogeneous operators (see [8]) may be generalized using the classification of parameters μ and ν in the sense of the sets A_i, $i = -1,\dots,3$.

Definition 2.1. The mapping $T:X \longrightarrow Y$ is said to be regularly surjective from X onto Y if $T(X) = Y$ and for any $R > 0$ there exists $r > 0$ such that $\|x\|_X \leq r$ for all $x \in X$ with $\|T(x)\|_Y \leq R$.

The following is proved in [8, Chapt. II].

Theorem 2.1. The operator $J - \lambda S$ is regularly surjective from X onto Y if and only if λ is not an eigenvalue of $J - \lambda S$, i.e. $J(x) - \lambda S(x) = 0_Y$ implies $x = 0_X$.

Using the properties of A_i, $i = 1,\dots,3$ (see [4],[11]) it is easy to see that the following generalization of the previous theorem is true.

Theorem 2.2. (i) The operator

(2.1) $\quad x \longmapsto J(x) - \mu S(x^+) + \nu S(x^-)$

is regularly surjective from X onto Y if and only if $(\mu,\nu) \in A_0 \cap A_3$.

(ii) If $(\mu,\nu) \in T_\lambda$, where T_λ is the component of A_0 containing the diagonal point (λ,λ) then the operator (2.1) is regularly surjective.

Investigation of homogeneous equation

(2.2) $\quad J(x) - \mu S(x^+) + \nu S(x^-) = 0_Y$

is also useful for proving existence results for the equations containing operators which are asymptotically close to J and S.

Definition 2.2. The mapping $T:X \longrightarrow Y$ is said to be a

(K,L,a)-homeomorphism of X onto Y if

(1) T is a homeomorphism of X onto Y;

(2) there exist real numbers $K>0$, $L>0$ such that

$$L\,\|x\|_X^a \leq \|T(x)\|_Y \leq K\,\|x\|_X^a,$$

for each $x \in X$.

Definition 2.3. Let $T_0 : X \longrightarrow Y$ be an a-homogeneous operator.

(i) T is said to be a-quasihomogeneous with respect to T_0 if $t_n \searrow 0, x_n \rightharpoonup x_0, t_n^a\, T\left(\frac{x_n}{t_n}\right) \longrightarrow y_0 \in Y$ imply $T_0(x_0) = y_0$.

(ii) T is said to be a-strongly quasihomogeneous with respect to T_0 if

$t_n \searrow 0, x_n \rightharpoonup x_0$ imply $t_n^a\, T\left(\frac{x_n}{t_n}\right) \longrightarrow T_0(x_0)$.

Note that the symbols " $\rightharpoonup$ " and " $\longrightarrow$ " denote as usual the weak and the strong convergence, respectively.

Using the homotopy invariance property of the Leray-Schauder degree it is possible to prove the following assertion.

Theorem 2.3. Let X be a reflexive Banach space and A an odd (K,L,a)-homeomorphism of X onto Y which is a-quasihomogeneous with respect to J. Let F be a completely continuous operator from X into Y which is a-strongly quasihomogeneous with respect to the operator $x \longmapsto \mu S(x^+) - \nu S(x^-)$. Then if $(\mu, \nu) \in T_\lambda \subset A_0$, where T_λ is some component containing the point (λ, λ), the equation

$$A(x) - F(x) = f \tag{2.3}$$

has at least one solution for arbitrary right hand side $f \in Y$.

Proof. We shall prove at first that there exists a sufficiently large ball $B_X(r) \subset X$ such that

$$(2.4) \qquad \mathcal{H}(x,\tau) \neq 0_Y$$

for all $x \in \partial B_X(\rho)$, $\tau \in [0,1]$, where
$\mathcal{H}(x,\tau) = A(x) - (1-\tau)F(x) - \tau\mu S(x^+) + \tau\nu S(x^-) - (1-\tau)f.$

Let us suppose by contradiction that there are $\tau_n \in [0,1]$, $\|x_n\|_X \longrightarrow \infty$ such that

$$(2.5) \qquad \mathcal{H}(x_n,\tau_n) = 0_Y.$$

Then at least for some subsequences, $\tau_n \longrightarrow \tau_0 \in [0,1]$, $x_n/\|x_n\|_X = v_n \longrightarrow v_0 \in X$ and $F(\|x_n\|_X v_n)/\|x_n\|_X^a \longrightarrow$ $\longrightarrow \mu S(v_0^+) - \nu S(v_0^-)$, $S(v_n^+) \longrightarrow S(v_0^+)$, $S(v_n^-) \longrightarrow S(v_0^-)$.

Hence dividing (2.5) by $\|x_n\|_X^a$ we obtain $(A(\|x_n\|_X v_n)/\|x_n\|_X^a) \longrightarrow \mu S(v_0^+) - \nu S(v_0^-)$, i.e. letting $n \longrightarrow \infty$,

$$(2.6) \qquad J(v_0) - \mu S(v_0^+) + \nu S(v_0^-) = 0_Y.$$

Since A is (K,L,a)-homeomorphism, we have

$$\frac{\|A(\|x_n\|_X v_n)\|_Y}{\|x_n\|_X^a} \geq L$$

for all $n \in \mathbb{N}$ and hence $v_0 \neq 0_X$, which together with (2.6) contradicts the assumption $(\mu,\nu) \in A_0$. This proves (2.4).

Let us denote, now, by $\eta(\tau) = (\eta_1(\tau),\eta_2(\tau))$, $\tau \in [1,2]$, the smooth curve which lies in T_λ and such that $\eta(2) = (\lambda,\lambda)$, $\eta(1) = (\mu,\nu)$. Let us consider
$\mathcal{H}(x,\tau) = A(x) - \eta_1(\tau)S(x^+) + \eta_2(\tau)S(x^-) - \tau t$, $\tau \in [1,2]$, $x \in \partial B_X(r)$. By contradiction we shall show that for $r > 0$ large enough it is

$$(2.7) \qquad \mathcal{H}(x,\tau) \neq 0_Y$$

for all $x \in \partial B_X(r)$, $\tau \in [1,2]$. Let us suppose that for the

suitable subsequences $\tau_n \longrightarrow \tau_0 \in [1,2]$, $x_n/\|x_n\|_X = v_n \longrightarrow$
$\longrightarrow v_0 \in X$, $v_0 \neq 0_X$,

$A(\|x_n\|_X v_n)/\|x_n\|_X^a \longrightarrow \eta_1(\tau_0)S(v_0^+) - \eta_2(\tau_0)S(v_0^-)$,

i.e. $J(v_0) - \eta_1(\tau_0)S(v_0^+) + \eta_2(\tau_0)S(v_0^-) = 0_Y$.

This contradicts $(\eta_1(\tau), \eta_2(\tau)) \in A_0$, for all $\tau \in [1,2]$, and hence (2.7) is proved. Using (2.4),(2.7), homotopy invariance property of the Leray-Schauder degree and the fact that A is (K,L,a)-homeomorphism we obtain that there is some $R > 0$ and a ball $B_Y(R) \subset Y$ such that

(2.8) $\deg[y - F(A^{-1}(y)); B_Y(R), 0_Y] = \deg[y - \lambda S(A^{-1}(y))^+ + \lambda S(A^{-1}(y))^-; B_Y(R), 0_Y]$.

Borsuk theorem and oddness of A and S imply that

(2.9) $\deg[y - \lambda S(A^{-1}(y))^+ + \lambda S(A^{-1}(y))^-; B_Y(R), 0_Y] \neq 0$.

Then (2.8),(2.9) and the basic property of the Leray-Schauder degree imply that (2.3) has at least one solution for arbitrary $f \in Y$. Q.E.D.

Remark 2.1. The previous Theorem 2.3 may be understood as a completion of the results contained in [8] concerning the solvability of operator equations with quasihomogeneous and strongly quasihomogeneous operators.

3. Some applications. Let us suppose that $p \geq 2$, $q = p/(p-1)$ are real numbers. Let a and b be real functions defined on $[0,\pi]$. Suppose that $a(t) > 0$, for all $t \in [0,\pi]$, $a \in C^1([0,\pi])$, $b(t) > 0$, for all $t \in [0,\pi]$, $b \in C([0,\pi])$. Put

$$X = Z = W_0^{1,p}(0,\pi),\ Y = X^* = W^{-1,q}(0,\pi)$$

and denote

$$(3.1)\begin{cases} \langle J(u),v\rangle = \int_0^{\pi} a(t)|u'(t)|^{p-2}u'(t)\ v'(t)dt, \\ \langle S(u),v\rangle = \int_0^{\pi} b(t)|u(t)|^{p-2}\ u(t)\ v(t)dt, \\ \langle f,v\rangle = \int_0^{\pi} h(t)\ v(t)dt, \end{cases}$$

$h\in L_1(0,\pi)$, for all $v\in X$, where $\langle .,.\rangle$ is used for the duality between X and Y.

Remark 3.1. See [15] for the usual function spaces used in this section.

Remark 3.2. It is possible to verify that the operators J and S defined by (3.1) satisfy the conditions (J1)-(J3),(S1)-(S3) from Section 2 (see [4]) and the equation

$$(3.2)\qquad J(u) - \mu S(u^+) + \nu S(u^-) = f$$

is the operator representation of the boundary value problem

$$(3.3)\begin{cases} -(a(t)|u'(t)|^{p-2}\ u'(t))' - b(t)|u(t)|^{p-2}(\ \mu u^+(t) - \\ -\nu u^-(t)) = h(t),\ t\in [0,\pi]\ , \\ u(0) = u(\pi) = 0. \end{cases}$$

Definition 3.1. The solution of the operator equation (3.2) is called the weak solution of BVP (3.3).

Remark 3.3. It is possible to prove that the weak solution of (3.3) has more regularity than $u\in X$. In fact we have $u\in C^1([0,\pi])$ and if $h\in C([0,\pi])$ then $(a(t)|u'(t)|^{p-2}\ u'(t))\in$ $\in C^1([0,\pi])$ (for the proof see [4, Th. 3.3]).

The following assertion is proved in [8].

Theorem 3.1. The real numbers λ for which there exists a nontrivial solution of $J(u) - \lambda S(u) = 0_Y$ form a sequence $\sigma =$ $= \{\lambda_n\}_{n=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \ldots .,\ \lim_{n\to\infty} \lambda_n = \infty$.

Remark 3.4. Note that $\lambda\in\sigma$ is equivalent to $(\lambda,\lambda)\in A_{-1}$.

Then using the basic properties of the sets A_i, $i = -1,\dots,3$ (see [11]) we can prove

<u>Theorem 3.2.</u> Let $\lambda \notin \sigma$. Then there exists a positive real number $c(\lambda) > 0$ (depending on the distance λ from σ) such that BVP (3.3) has at least one weak solution for arbitrary right hand side $h \in L_1(0,\pi)$ if $|\mu - \lambda| + |\nu - \lambda| < c(\lambda)$.

We shall suppose now that $g: [0,\pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Carathéodory's conditions, i.e. $g(t,s)$ is measurable in t for all $s \in \mathbb{R}$ and continuous in s for a.a. $t \in [0,\pi]$, and let us consider perturbed BVP:

$$(3.4)\quad \begin{cases} -(a(t)|u'(t)|^{p-2}u'(t))' - b(t)|u(t)|^{p-2}(\mu u^+(t) - \\ - \nu u^-(t)) + g(t,u(t)) = h(t),\ t \in]0,\pi[, \\ u(0) = u(\pi) = 0. \end{cases}$$

<u>Theorem 3.3.</u> Let $(\mu,\nu) \in A_1$. Then there exists real positive $c_1(\mu,\nu) > 0$ with the following property: the BVP (3.4) has at least one weak solution for arbitrary right hand side $h \in L_1(0,\pi)$ if there is some function $r(t) \in L_1(0,\pi)$ such that

$$|g(t,z)| \le r(t) + c_1(\mu,\nu)\,|z|^{p-1},$$

for a.a. $t \in [0,\pi]$ and all $z \in \mathbb{R}$.

<u>Remark 3.5.</u> The proof of this assertion follows again from the basic properties of the set A_1. On the other hand if $(\mu,\nu) \in A_2$ (i.e. there exists such $h \in L_1(0,\pi)$ that BVP (3.3) has no weak solution) then there is $c_2(\mu,\nu) > 0$ such that BVP (3.4) has no solution for some right hand sides provided

$$|g(t,z)| \le r(t) + c_2(\mu,\nu)\,|z|^{p-1},$$

for a.a. $t \in [0,\pi]$ and all $z \in \mathbb{R}$.

Let us suppose that $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function which has finite limits

(3.5) $\mu = \lim_{s \to +\infty} \frac{\varphi(s)}{|s|^{p-2}s}$ and $\nu = \lim_{s \to -\infty} \frac{\varphi(s)}{|s|^{p-2}s}$.

Define the operator $F: X \longrightarrow Y$ by

(3.6) $\langle F(u), v \rangle = \int_0^{\pi} \varphi(u(t))v(t)dt, \quad u, v \in X,$

and the operator $A: X \longrightarrow Y$ by

(3.7) $\langle A(u), v \rangle = \int_0^{\pi} (1 + a(t)|u'(t)|^{p-2})\, u'(t)\, v'(t)dt,$

$u, v \in X.$

Then the solution of the operator equation

(3.8) $$A(u) - F(u) = f$$

is the weak solution of BVP

(3.9) $$\begin{cases} -[(1 + a(t)|u'(t)|^{p-2})\, u'(t)]' = \varphi(u(t)) + h(t), \\ \qquad\qquad t \in [0, \pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

It is not difficult to see that A is odd, $(K,L,p-1)$-homeomorphism X onto Y which is $(p-1)$-quasihomogeneous with respect to J and F is completely continuous operator from X into Y which is $(p-1)$-strongly quasihomogeneous with respect to the operator

$$u \longmapsto \mu S(u^+) - \nu S(u^-).$$

Using the properties of A_1 and applying Theorem 2.3 we obtain the following existence result.

Theorem 3.4. Let $\lambda \notin \sigma$. Then there exists $\gamma(\lambda) > 0$ such that BVP (3.9) has at least one weak solution for arbitrary right hand side $h \in L_1(0, \pi)$ provided $|\mu - \lambda| + |\nu - \lambda| < \gamma(\lambda)$.

Let us suppose that the functions a, b are the same as at the beginning of this section and put $X = W_0^{2,p}(0, \pi)$, $Y = W^{-2,q}(0, \pi)$, $Z = L_p(0, \pi)$. Let us define operator $S: X \longrightarrow Y$ and an element $f \in Y$ by the same way as in (3.1) and an operator

$J: X \longrightarrow Y$ by the relation

$$(3.10) \quad \langle J(u), v \rangle = \int_0^{\pi} a(t)|u''(t)|^{p-2} u''(t)v''(t)dt, u, v \in X.$$

Remark 3.6. It is possible to verify that the operators J and S satisfy again the conditions (J1)-(J3), (S1)-(S3) from Section 2 and the solution of (3.2) is the weak solution of BVP

$$(3.11) \quad \begin{cases} (a(t)|u''(t)|^{p-2}u''(t))'' - b(t)|u(t)|^{p-2}(\mu u^+(t) - \\ - \nu u^-(t)) = h(t), \; t \in [0, \pi], \\ u(0) = u'(0) = u(\pi) = u'(\pi) = 0. \end{cases}$$

Remark 3.7. Also in this case the assertion of Theorem 3.1 is still valid (see [8]). That is why analogous results to that formulated in Theorems 3.2 - 3.4 may be proved also for the weak solvability of BVP (3.11).

Remark 3.8. Let us remark that all the results formulated above have the local character in the sense that we obtain the solvability of BVP (3.3), resp. (3.11), when (μ, ν) is "near" to some diagonal point (λ, λ), $\lambda \notin \sigma$. In order to obtain more global results we need some information about the structure of the set A_{-1} which plays the key role in the classification of real parameters μ and ν.

It is possible to prove such global results for BVP (3.3) under the assumption of constant coefficients, i.e. $a(t) = b(t) = 1$ for all $t \in [0, \pi]$.

Theorem 3.5. BVP

$$(3.12) \quad \begin{cases} -(|u'|^{p-2}u')' - |u|^{p-2}(\mu u^+ - \nu u^-) = 0, \\ u(0) = u(\pi) = 0 \end{cases}$$

has a nontrivial weak solution if and only if one of the following conditions holds:

(i) $\mu = \lambda_1$, ν is arbitrary,

(ii) μ is arbitrary, $\nu = \lambda_1$,

(iii) $\mu > \lambda_1$, $\nu > \lambda_1$,

$$\frac{(\mu)^{\frac{1}{p}}(\nu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} = k,$$

$$\frac{((\mu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\nu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} = k,$$

$$\frac{((\nu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\mu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} = k,$$

$k = 1, 2, 3, \dots$.

Remark 3.9. The previous assertion gives the precise description of the set A_{-1} for BVP (3.12). The proof of this theorem with the sketch of the figure of A_{-1} may be found in [4].

Using the description of A_{-1} we may formulate the global analog of Theorems 3.2 - 3.4.

Let us suppose that continuous function φ satisfies (3.5) and consider BVP

$$(3.13) \quad \begin{cases} -[(1 + |u'|^{p-2})u']' = \varphi(u(t)) + h, \text{ in } [0, \pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

Theorem 3.6. Let us suppose that one of the following conditions is fulfilled:

(i) $\mu < \lambda_1$, $\nu < \lambda_1$,

(ii) $\mu > \lambda_1$, $\nu > \lambda_1$ and

$$\frac{((\mu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\nu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} < 1, \quad \frac{((\nu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\mu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} < 1,$$

or

$$k-1 < \frac{((\mu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\nu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} < k, k-1 <$$

$$< \frac{((\nu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\mu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} < k,$$

with some $k \in \mathbb{N}$, $k \geq 2$. Then the BVP (3.13) has at least one weak solution for arbitrary right hand side $h \in L_1(0,\pi)$.

<u>The proof</u> of this assertion follows immediately from Theorem 2.3 because the above inequalities (i),(ii) are equivalent to $(\mu,\nu) \in T_\lambda$, where T_λ is a component of A_1 containing diagonal point (λ,λ), $\lambda \notin \sigma$.

On the other hand using the shooting method we obtain the following nonexistence result.

<u>Theorem 3.7</u>. Let us suppose that one of the following conditions is fulfilled:

(i) $\mu > \lambda_1$, $\nu < \lambda_1$;

(ii) $\mu < \lambda_1$, $\nu > \lambda_1$;

(iii) $$\frac{((\mu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\nu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} < k,$$

$$\frac{((\nu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\mu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} > k;$$

(iv) $$\frac{((\mu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\nu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} > k, \quad \frac{((\nu)^{\frac{1}{p}} - (\lambda_1)^{\frac{1}{p}})(\mu)^{\frac{1}{p}}}{((\mu)^{\frac{1}{p}} + (\nu)^{\frac{1}{p}})(\lambda_1)^{\frac{1}{p}}} < k;$$

$k = 1,2,3,\ldots$. Then there exists right hand side $h \in L_1(0,\pi)$ such that

$$(3.14) \quad \begin{cases} -(|u'|^{p-2}u')' - |u|^{p-2}(\mu u^+ - \nu u^-) = h \text{ in } [0,\pi], \\ u(0) = u(\pi) = 0 \end{cases}$$

has no weak solution.

<u>For the proof</u> see [4].

<u>Remark 3.10</u>. Note that under the assumptions of Theorem 3.6 the BVP (3.14) has the weak solution for arbitrary right hand side $h \in L_1(0,\pi)$. We have complete description of the set A_{-1} for BVP (3.14) which is given by conditions (i) - (iii) from Theorem 3.5 (the system of curves in the plane (μ,ν)). The set A_{-1} divides the plane (μ,ν) into some open unbounded components. These components are of two different types - some of them have nonempty intersection with the diagonal (λ,λ), $\lambda \in \mathbb{R}$, and some of them have empty intersection with this diagonal. Theorem 3.6 then implies that the components of the first type belong to A_1 (and hence also to A_3) and Theorem 3.7 implies that the components of the second type belong to A_2.

It is possible to prove some more precise results in the case $p = 2$, i.e. for the solvability of BVP

$$(3.15) \quad \begin{cases} -u''(t) - \mu u^+(t) + \nu u^-(t) = h(t), \ t \in [0,\pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

Let us suppose $(\mu,\nu) \in A_{-1}$, i.e. μ and ν satisfy the assumptions of Theorem 3.5 (with $p = 2$), and denote $v_{\mu,\nu} \in W_0^{1,2}(0,\pi)$ the normed nontrivial solution of BVP

$$\begin{cases} -u''(t) - \mu u^+(t) + \nu u^-(t) = 0, \ t \in [0,\pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

The standard regularity argument for ODE's shows that $v_{\mu,\nu} \in$ $\in W^{2,2}(0,\pi)$.

Theorem 3.8. Let $(\mu,\nu) \in A_{-1}$. Then for given $h_1 \in$ $\in [v''_{\mu,\nu}]^{\perp}$ (an orthogonal complement in the space $L_2(0,\pi)$) there exists an $\alpha(h_1) \in \mathbb{R}$ such that (3.15) has at least one weak solution for $h = h_1 + \alpha(h_1)v''_{\mu,\nu}$.

Theorem 3.9. Let us suppose that $(\mu,\nu) \in A_2$ (i.e. μ and ν satisfy the assumptions of Theorem 3.7 with $p = 2$). Then there exists $w_{\mu,\nu} \in L_2(0,\pi)$ such that for any given $h_1 \in$ $\in [w_{\mu,\nu}]^{\perp}$ there exists a constant $T(h_1)$ such that (3.15) has at least two weak solutions for $h = h_1 + tw_{\mu,\nu}$ provided that $t > T(h_1)$.

The proofs of the previous two theorems may be found in [19]. Note that $p = 2$ is essential here.

Some global results (concerning the classification of parameters μ and ν) it is possible to prove also in the case of ODE of the fourth order. Let us consider the equation

$$u^{IV} = \mu u^+ - \nu u^-, \tag{3.16}$$

with periodic boundary conditions. The regularity argument shows that the description of A_{-1} is equivalent to finding a nonconstant 2π-periodic solution $u \in C^4(\mathbb{R})$ solving (3.16). It is useful to put $\mu = a^4$, $\nu = b^4$, $(a,b) \in]0,+\infty[\times]0,+\infty[=]0,+\infty[^2$. Let us denote by $\psi \in](3/4)\pi, \pi[$ the smallest positive root of the equation

$$\tan(x) + \mathrm{th}(x) = 0,$$

and for $z \in]0,\psi[$

$$g(z) = \frac{ch(z)\ \sin(z) - sh(z)\cos(z)}{ch(z)\ \sin(z) + sh(z)\cos(z)} .$$

The following localization of the set A_{-1} is proved in [14].

Theorem 3.10. The set $\widetilde{A}_{-1} = \{(a,b) \in]0,+\infty[^2;\ \mu = a^4,\ \nu = b^4,\ (\mu,\nu) \in A_{-1}\}$ is the system $\{S_k, k \in \mathbb{N}\}$ of C^∞-curves, where S_1 is a curve $(a,b(a))$; $b(a)$ is decreasing C^∞-function defined in $]\frac{\psi}{\pi}, +\infty[$ with $\lim_{a \to \infty} b(a) = \frac{\psi}{\pi}$. The curve S_1 is symmetrical with respect to the straight line $b = a$ and fulfils $S_1 \subset G_1$, where G_1 is the set of all pairs $(a,b) \in]0,+\infty[^2$ such that

$b \geq a,\ (\frac{b}{a})^2 - g(\pi a(1 - \frac{1}{2b})) \geq 0 \geq (\frac{a}{b})^2 - g(\pi b(1 - \frac{1}{2a}))$,

or

$b \leq a,\ (\frac{a}{b})^2 - g(\pi b(1 - \frac{1}{2a})) \geq 0 \geq (\frac{b}{a})^2 - g(\pi a(1 - \frac{1}{2b}))$.

For $k \geq 2$ it is $S_k = \{(a,b) \in]0,+\infty[^2;\ (a/k, b/k) \in S_1\}$ and $S_k \subset G_k$, where $G_k = \{(a,b) \in]0,+\infty[^2;\ (a/k,b/k) \in G_1\}$. In particular, $\widetilde{A}_{-1} \subset \bigcup_{k=1}^{\infty} G_k$ and for $(a,b) \in S_k$ the corresponding 2π-periodic solution has exactly 2k-"semi-waves" in an interval of length 2π. This solution is unique if translations and positive multiples are not considered.

Remark 3.11. See [14] for the picture of the system $\{G_k\}_{k=1}^{\infty}$.

Let us consider now the equation (3.16) with boundary conditions

(3.17) $\quad u(0) = u''(0) = u(\pi) = u''(\pi) = 0.$

Then the following information about the set $\widetilde{A}_{-1}$ (for BVP (3.16), (3.17)) may be got.

Theorem 3.11. The set $\widetilde{A}_{-1}$ is a system of continuous curves $\{S_i^+, S_i^-; i \in \mathbb{N}\}$ such that

(i) for $(a,b) \in S_i^+$, resp. S_i^-, the solution u satisfies $u'(0) > 0$, resp. $u'(0) < 0$. This solution is uniquely determined by the choice of $u'(0)$ and it has exactly $i + 1$ zeros in $[0,\pi]$;

(ii) S_i^+ is symmetrical to S_i^- with respect to the straight line $a = b$. If i is even then $S_i^+ = S_i^-$;

(iii) for each $i \in \mathbb{N}$ we have $(S_i^+ \cup S_i^-) \cap (S_{i+1}^+ \cup S_{i+1}^-) = \emptyset$.

For the proof of this assertion see [14].

Remark 3.12. Using the assertion of Theorem 3.10 (i.e. the localization of A_{-1}) and the abstract Theorem 2.3 we may formulate the global existence results (analogous to that from Theorem 3.6) for the periodic BVP for the equation

$$u^{IV} = \varphi(u(t)) + h(t).$$

The situation concerning the description of the set A_{-1} in the case of PDE's seems to be much more complicated. This fact implies that investigation of the solvability of the corresponding BVP with jumping nonlinearity is very difficult. The most recent results in this direction may be found in [12],[13]. The authors study the following problem

$$(3.18) \qquad u \in D(J),\ J(u) = \mu u^+ - \nu u^- + \varphi(.,u) + h,$$

under the assumptions: $\Omega \subset \mathbb{R}^N$ is an open set, $h \in L_2(\Omega)$, J is a linear selfadjoint operator with compact resolvent, the domain of J is $D(J) \subset L_2(\Omega)$ and J maps $D(J)$ into $L_2(\Omega)$, $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function,

$$\lim_{|s| \to \infty} \frac{\varphi(.,s)}{s} = 0, \quad \sup_{s \in \mathbb{R}} \left| \frac{\varphi(.,s)}{s} \right| \in L_\infty(\Omega).$$

There is proved in [12] that if $\mu \neq \nu$ and interval $[\mu,\nu]$ (resp. $[\nu,\mu]$ if $\mu > \nu$) does not contain any eigenvalue of J then (3.18) has at least one solution for every $h \in L_2(\Omega)$.

Also in the case $\mu = \nu = \lambda$ and λ is not an eigenvalue of J the problem (3.18) has at least one solution for every $h \in L_2(\Omega)$. It is also proved there that the set A_{-1} associated with (3.18) ($\varphi \equiv 0, h \equiv 0$) in the neighbourhood of the simple eigenvalue λ has the character of a continuous curve (or two continuous curves) passing through the point (λ, λ).

In the second paper [13] there is studied the case when the interval $[\mu, \nu]$ contains one simple eigenvalue λ of the operator J and $(\mu, \nu) \in A_{-1}$, (μ, ν) lies "near" to (λ, λ). The authors have obtained sufficient conditions of Landesman-Lazer type for the solvability of (3.18).

At the end of this section let us mention two results concerning the solvability of BVP's for ODE's containing nonlinearities introduced by Fučík [9] (see (1.2)).

Let us suppose that $\varphi(t,s): [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function, there is some constant $c > 0$ and a function $m \in L_d(0,\pi)$ $(d > 1)$ such that

$$(3.19) \qquad |\varphi(t,s)| \le m(t) + c|s|^{p-1}$$

for all $s \in \mathbb{R}$ and a.a. $t \in [0,\pi]$. We shall suppose that there exist functions $\chi^{+\infty}, \chi^{-\infty}, \chi_{+\infty}, \chi_{-\infty} \in L_\infty(0,\pi)$ such that

$$(3.20) \quad \begin{cases} \limsup\limits_{s \to \pm\infty} \dfrac{\varphi(t,s)}{|s|^{p-2}s} = \chi^{\pm\infty}(t), \\[2ex] \liminf\limits_{s \to \pm\infty} \dfrac{\varphi(t,s)}{|s|^{p-2}s} = \chi_{\pm\infty}(t), \end{cases}$$

for a.a. $t \in [0,\pi]$. Then using the description of the set A_1 for the BVP (3.12) (see Theorem 3.5) we obtain the following existence result for BVP

$$(3.21)\quad \begin{cases} -(|u'(t)|^{p-2}\, u'(t))' = \varphi(t,u(t)) + h(t), & t \in [0,\pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

<u>Theorem 3.12</u>. Let us suppose that either

(i) there exists some $\delta > 0$ such that

$$\chi_{\pm\infty}(t),\quad \chi^{\pm\infty}(t) \le \lambda_1 - \delta,$$

for a.a. $t \in]0,\pi[$, or

(ii) there are two couples (μ_1, ν_1) and (μ_2, ν_2) lying in the same component of A_1 and

$$\mu_1 \le \chi_{+\infty}(t) \le \chi^{+\infty}(t) \le \mu_2,$$

$$\nu_1 \le \chi_{-\infty}(t) \le \chi^{-\infty}(t) \le \nu_2,$$

holds for a.a. $t \in]0,\pi[$.

Then BVP (3.21) has at least one weak solution for arbitrary right hand side $h \in L_1(0,\pi)$.

<u>Remark 3.13</u>. The proof of this assertion may be found in [1], the sketch of the proof of this assertion is given also in [5]. Note that the method of the proof is topological in nature (it is based on the homotopy invariance property of the Leray-Schauder degree) and therefore it is possible to consider more general differential operator of second order than that considered in (3.21) (in the sense of Definition 2.2 and Definition 2.3(i)), i.e. the assertion of Theorem 3.12 remains also valid in the case of BVP:

$$\begin{cases} -[(1 + |u'(t)|^{p-2})\, u'(t)]' = \varphi(t,u(t)) + h(t), t \in]0,\pi[, \\ u(0) = u(\pi) = 0. \end{cases}$$

Let us consider now the periodic BVP for forced Duffing equation

$$(3.22)\quad \begin{cases} u'' + \tilde{c}\, u' + \varphi(t,u) = h(t) \text{ in }]0,\pi[\,, \\ u(\pi) - u(0) = u'(\pi) - u'(0) = 0, \end{cases}$$

$\tilde{c} \in \mathbb{R}$, $h \in L_1(0,\pi)$, φ is again the Carathéodory's function satisfying (3.19) with $m \in L_1(0,\pi)$, $p = 2$ and (3.20) (also with $p = 2$).

Theorem 3.13. Let us suppose that either

(i) there exists some $\delta > 0$ such that

$$\chi_{\pm\infty}(t) - \frac{\tilde{c}^2}{4}\ , \quad \chi^{\pm\infty}(t) - \frac{\tilde{c}^2}{4} \leq -\delta,$$

for a.a. $t \in]0,\pi[$, or

(ii) there are two couples (μ_1, ν_1) and (μ_2, ν_2) lying in the same component of $\tilde{A}_1$ and

$$\mu_1 \leq \chi_{+\infty}(t) - \frac{\tilde{c}^2}{4} \leq \chi^{+\infty}(t) - \frac{\tilde{c}^2}{4} \leq \mu_2,$$

$$\nu_1 \leq \chi_{-\infty}(t) - \frac{\tilde{c}^2}{4} \leq \chi^{-\infty}(t) - \frac{\tilde{c}^2}{4} \leq \nu_2,$$

$$(0,0) \notin [\mu_1, \mu_2] \times [\nu_1, \nu_2]\ ,$$

for a.a. $t \in]0,\pi[$. Then periodic BVP (3.22) has at least one solution for arbitrary $h \in L_1(0,\pi)$.

Remark 3.14. $\tilde{A}_1 = \{(\mu,\nu) \in \mathbb{R}^2;\ \mu \cdot \nu > 0, (\mu,\nu) \notin \bigcup_{k=1}^{\infty} C_k\}$, where

$$C_k = \{(\mu,\nu) \in \mathbb{R}^2;\ \ \mu > 0,\ \nu > 0,\ \frac{\sqrt{\mu} \cdot \sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} = k\},$$

$k = 1,2,3,\ldots$. See [6] for the picture of $\tilde{A}_1$ and $\bigcup_{k=1}^{\infty} C_k$.

Remark 3.15. The method of the proof of Theorem 3.23 is based on the homotopy invariance property of the Leray-Schauder degree and the shooting argument. In the first step, proceeding via contradiction, we obtain the limit equation

$$v'' + \tilde{c}\, v' + \gamma_+(t)v^+ - \gamma_-(t)v^- = 0 \text{ a.e. on } [0,\pi]\,,$$

with $v(0) = v(\pi) = 0$, $v'(0) = v'(\pi) = 1$ and with γ_+, γ_- verifying $\mu_1 \le \gamma_+(t) \le \mu_2$, $\nu_1 \le \gamma_-(t) \le \nu_2$, for a.a. $t \in [0, \pi]$. By a substitution $z(t) = \exp((c/2)t)v(t)$ we transform this BVP to

$$\begin{cases} z'' + (\gamma_+(t) - \frac{\gamma^2}{4}) z^+ - (\gamma_-(t) - \frac{\gamma^2}{4}) z^- = 0, \\ z(0) = z(\pi) = 0, \end{cases}$$

which verifies also $\operatorname{sign} z'(\pi) = \operatorname{sign} z'(0) \neq 0$. Here we get a contradiction using the description of A_{-1} for BVP (3.15) using the shooting argument (for complete proof of Theorem 3.13 see [6]).

4. <u>Open problems</u>. In this last section we shall formulate some open problems. Note that some open problems concerning solvability of general operator equation (0.1) are formulated in [5].

Let us consider (3.14) and the sets A_i, $i = -1, \dots, 3$, associated with this BVP. Then the following open problems concerning the solvability of (3.14) may be formulated.

<u>Problem 4.1</u>. (i) $A_{-1} \subset \mathbb{R}^2 \setminus A_3$? (ii) To find sufficient conditions upon $h \in L_1(0, \pi)$ in order (3.14) to be solvable if the answer to (i) is positive and $(\mu, \nu) \in A_{-1}$.

<u>Problem 4.2</u>. Let us suppose that $(\mu, \nu) \in A_2$. Find sufficient conditions (or necessary and sufficient conditions) upon h in order (3.14) to be solvable.

<u>Problem 4.3</u>. Let us suppose that $(\mu, \nu) \in A_{-1}$ and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and bounded function with finite limits $\lim_{s \to \pm\infty} \varphi(s) = \varphi(\pm\infty)$. The problem is to find by means of $\varphi(\pm\infty)$ sufficient conditions upon $h \in L_1(0, \pi)$ in order

$$\begin{cases} -(|u'|^{p-2}u')' - |u|^{p-2}(\mu u^+ - \nu u^-) + \varphi(u(.)) = h \text{ in } [0,\pi], \\ u(0) = u(\pi) = 0, \end{cases}$$

to be solvable.

Remark 4.1. The last problem was motivated by the result [13] where Landesman-Lazer-type sufficient conditions are derived for solvability of semilinear problem (3.18).

Let us come back to the BVP (3.3). It would be interesting to extend the local result from Theorem 3.2 at least in the following sense.

Problem 4.4. Let λ_i, $\lambda_{i+1} \in \sigma$ be two successive eigenvalues for some $i = 1,2,3,\dots$ (see Theorem 3.1). Prove or disprove:

BVP (3.3) has at least one weak solution for arbitrary right hand side $h \in L_1(0,\pi)$ provided $\lambda_i < \mu < \lambda_{i+1}$, $\lambda_i < \nu < \lambda_{i+1}$.

Remark 4.2. Note that the answer is positive in the case $p = 2$ (see [12] for PDE case).

Remark 4.3. It would be interesting to solve the Problem 4.4 also for BVP (3.11).

Let us suppose that $\lambda_n \in \sigma$ is a simple eigenvalue of $J - \lambda S$; J, S are defined by the relations (3.1). Then any information about the structure of the set A_{-1} in the neighbourhood of (λ_n, λ_n) would be useful, namely we are interested in solving

Problem 4.5. Prove or disprove: the set A_{-1} in the neighbourhood of (λ_n, λ_n) is a continuous curve (or two continuous curves) passing through the point (λ_n, λ_n).

Also global properties of the set A_{-1} associated with J and S defined by (3.1) are not known very well.

Problem 4.6. Has the set A_{-1} an empty interior (with respect to the topology of $\mathbb{R}^2$) ?

Problem 4.7. Is there some connected subset $M \subset A_{-1}$ such that $(\lambda_1, \lambda_1) \in M$ and $(\lambda_2, \lambda_2) \in M$, where $\lambda_1, \lambda_2 \in \sigma$, $\lambda_1 \neq \lambda_2$?

Let us consider the BVP

$$(4.1) \begin{cases} -(a(t)|u'(t)|^{p-2} u'(t))' = \varphi(t,u(t)) + h(t), \ t \in [0,\pi], \\ u(0) = u(\pi) = 0, \end{cases}$$

φ satisfies (3.19),(3.20), $a \in C^1([0,\pi])$, $a(t) > 0$, $t \in [0,\pi]$ and λ_i, $\lambda_{i+1} \in \sigma$, for some arbitrary but fixed $i \in \mathbb{N}$. It would be interesting to prove the following assertion.

Problem 4.8. Let us suppose that either

(i) there exists some $\delta > 0$ such that

$$\chi_{\pm\infty}(t), \quad \chi^{\pm\infty}(t) \leq \lambda_1 - \delta,$$

for a.a. $t \in]0,\pi[$, or

(ii) there exists some $\delta > 0$ such that

$$\lambda_i + \delta \leq \chi_{+\infty}(t) \leq \chi^{+\infty}(t) \leq \lambda_{i+1} - \delta,$$

$$\lambda_i + \delta \leq \chi_{-\infty}(t) \leq \chi^{-\infty}(t) \leq \lambda_{i+1} - \delta,$$

for a.a. $t \in]0,\pi[$.

Then BVP (4.1) has at least one weak solution for arbitrary right hand side $h \in L_1(0,\pi)$.

Let us suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with lipschitzian boundary $\partial\Omega$, $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory's function satisfying the condition

$$|\varphi(x,s)| \leq m(x) + c|s|^{p-1},$$

for all $s \in \mathbb{R}$ and a.a. $x \in \Omega$ with some positive constant c and $m \in L_q(\Omega)$, $q = p/(p-1)(p \geq 2)$. It is possible to define the weak solution of the BVP

$$
(4.2)\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = \varphi(.,u(.)) + h \text{ in } \Omega , \\ u = 0 \text{ on } \partial\Omega \end{cases}
$$

in an analogous way as in Definition 3.1. We shall call by $\tilde{\sigma}$ the set of all real numbers λ for which there exists a nontrivial weak solution of BVP

$$
\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = \lambda |u|^{p-2} u \text{ in } \Omega , \\ u = 0 \text{ on } \partial\Omega . \end{cases}
$$

Using the variational approach it is not difficult to see that $\inf \tilde{\sigma} > 0$. Any other information concerning the set $\tilde{\sigma}$ should be very useful.

<u>Problem 4.9</u>. Is $\tilde{\sigma}$ a countable set, say $\tilde{\sigma} = \{\mu_m\}_{m=1}^{\infty}$, which is isolated and which has the property $\lim_{m\to\infty} \mu_m = \infty$?

<u>Problem 4.10</u>. Is it true that every $\mu_m \in \tilde{\sigma}$ allows the Ljusternik-Schnirelman characterization ?

<u>Remark 4.4</u>. The reason why it is important to have some information about the solution of Problems 4.9 and 4.10 is the following. If the answer to the preceding two questions is positive then the following assertion may be proved using variational method.

<u>Problem 4.11</u>. Let us suppose that $\mu_1, \mu_2 \in \tilde{\sigma}$ and $]\mu_1, \mu_2[\cap \tilde{\sigma} = \emptyset$. Let there be some $\delta > 0$ such that

$$\mu_1 + \delta \le \liminf_{s \to +\infty} \frac{\varphi(x,s)}{|s|^{p-2}s} \le \limsup_{s \to +\infty} \frac{\varphi(x,s)}{|s|^{p-2}s} \le \mu_2 - \delta,$$

(4.3)

$$\mu_1 + \delta \le \liminf_{s \to -\infty} \frac{\varphi(x,s)}{|s|^{p-2}s} \le \limsup_{s \to -\infty} \frac{\varphi(x,s)}{|s|^{p-2}s} \le \mu_2 - \delta,$$

for a.a. $x \in \Omega$. Then BVP (4.2) has at least one weak solution for arbitrary $h \in W^{-1,q}(\Omega)$.

Remark 4.5. Some local sufficient conditions instead of (4.3) are considered in [1] in order to prove solvability of (4.2) for an arbitrary right hand side $h \in W^{-1,q}(\Omega)$.

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