Josef Voldřich

On solvability of the Stokes problem in Sobolev power weight spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 2, 325--336

Persistent URL: `http://dml.cz/dmlcz/106305`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
25,2 (1984)

ON SOLVABILITY OF THE STOKES PROBLEM IN SOBOLEV POWER WEIGHT SPACES

Josef VOLDŘICH

Abstract: This paper deals with the solvability of the Stokes problem in Sobolev power weight spaces.

Key words: Weighted spaces, Stokes problem, generalized Lax-Milgram lemma.

Classification: 76D07

---

1. Introduction. This paper deals with the solvability of the Stokes problem

$$(1)\qquad \begin{aligned} -\nu\,\Delta\vec{u} + \operatorname{grad} p &= \vec{f} \quad \text{in } \Omega,\\ \operatorname{div}\vec{u} &= g \quad \text{in } \Omega,\\ \vec{u} &= \vec{\phi} \quad \text{on } \partial\Omega, \end{aligned}$$

in a bounded domain $\Omega\subset R^N$ with a Lipschitz boundary, where $\nu > 0$ and $\int_\Omega g\,dx = \int_{\partial\Omega}\vec{\phi}.\vec{\nu}\,dS$. In comparison with the classical case we assume that right-hand sides $\vec{f}, g, \vec{\phi}$ of (1) include certain singularities which are described by weighted spaces. Those circumstances make impossible to find a weak solution in (classical) Sobolev spaces. Moreover, from the properties of the right-hand sides of (1) we are able to describe the behaviour of the solution of (1) near the boundary using the methods of weighted spaces.

In order to avoid technical difficulties, we shall consider spaces with weights related to the whole boundary $\partial\Omega$. In the case of weights related to a part M of the boundary $\partial\Omega$ where M is a manifold of the dimension less than or equal to N-1 we can use the same ideas of the proofs.

Fundamental properties of weighted spaces we shall use can be found e.g. in [1],[2].

Section 2. Troughout this paper Ω will be a bounded domain in the Euclidean N-space R^N with a Lipschitz boundary $\partial\Omega$. We shall use the distance d(x) of a point $x \in \Omega$ from $\partial\Omega$ defined by $d(x) = \inf_{y\in\partial\Omega} |x-y|$. The Sobolev power weight space $W^{1,2}(\Omega;d,\varepsilon)$ is defined to be the set of all functions u defined a.e. on Ω whose (distributional) derivatives $D^{\alpha}u$ with $|\alpha| \leq 1$ belong to the weighted Lebesgue space $L_2(\Omega;d,\varepsilon)$ endowed with the norm

$$|\varphi|_{\varepsilon} = \left(\int_{\Omega} |\varphi(x)|^2 \, d^{\varepsilon}(x)\, dx \right)^{1/2} .$$

The space $[W^{1,2}(\Omega;d,\varepsilon)]^N$ with the norm

$$\|\vec{u}\|_{\varepsilon} = \left(\sum_{i,j=1}^{N} \int_{\Omega} \left| \frac{\partial u_j}{\partial x_i} \right|^2 d^{\varepsilon}\, dx + \sum_{j=1}^{N} \int_{\Omega} |u_j|^2 \, d^{\varepsilon}\, dx \right)^{1/2}$$

is a Hilbert space. The set $[C^{\infty}(\bar{\Omega})]^N$ is dense in $[W^{1,2}(\Omega;d,\varepsilon)]^N$ for $\varepsilon \in (-1,1)$ and therefore we can consider traces of functions from this space on the boundary $\partial\Omega$ (see e.g. [1]).

The weighted analogy of the Sobolev space $[W_0^{1,2}(\Omega)]^N$

is defined by the formula $[W_0^{1,2}(\Omega;d,\varepsilon)]^N = \overline{C_0^\infty(\Omega)}$ where the closure is taken with respect to the norm $\|.\|_\varepsilon$.

Further we shall use the shorter notation

$$L_2(\varepsilon) = L_2(\Omega;d,\varepsilon),\quad L_2^0(\varepsilon) = \left\{\varphi \in L_2(\Omega;d,\varepsilon);\ \int_\Omega \varphi\, dx=0\right\},$$

$$V(\varepsilon) = [W^{1,2}(\Omega;d,\varepsilon)]^N,\quad V_0(\varepsilon) = [W_0^{1,2}(\Omega;d,\varepsilon)]^N,$$

where $\varepsilon \in (-1,1)$. Let B_ε be the space $\{\vec{v} \in V_0(\varepsilon);\ \operatorname{div}\vec{v} = 0\}$ with the norm $\|.\|_\varepsilon$ and $B_\varepsilon^\perp$ its orthogonal complement in $V_0(\varepsilon)$.

According to the following consequence of Hardy's inequality

$$(2)\qquad \int_\Omega d^{\varepsilon-2}|u_j|^2 dx \leqq c_1(\Omega)\,\frac{1}{|\varepsilon-1|^2}\int_\Omega d^\varepsilon\,|\nabla u_j|^2 dx,$$

$$j = 1,\dots,N,\ \vec{u}\in V_0(\varepsilon) \text{ with } \varepsilon\in(-1,1),$$

we can consider the norm equivalent to $\|.\|_\varepsilon$,

$$|||\vec{u}|||_\varepsilon = \Big(\sum_{i,j=1}^N \int_\Omega d^\varepsilon \left|\frac{\partial u_j}{\partial x_i}\right|^2 dx\Big)^{\frac{1}{2}}$$

on the space $V_0(\varepsilon)$.

In the proof of Theorem 2 we shall use the following lemma proved for example in [2].

Lemma 1. If the derivatives $D_i p$, $1 \leqq i \leqq N$, of a distribution p belong to $H^{-1}(\Omega)$ $(=[W_0^{1,2}(\Omega)]^*)$, then $p \in L_2(\Omega)$ and

$$\|p\|_{L_2(\Omega)/R} \leqq c_2(\Omega)\ \|\operatorname{grad} p\|_{[H^{-1}(\Omega)]^N}.$$

The following Theorems 2-6 imply some properties of mappings grad and div defined in weighted spaces. Analogous results concerning classical spaces are proved in [4],[5].

<u>Theorem 2</u>. There exists a symmetric interval I, $0 \in \operatorname{int} I$, such that for every $\varepsilon \in I$ the operator grad is an isomorphism of the space $L_2(\varepsilon)_{/R}$ onto its range in $[V_0(-\varepsilon)]^*$.

<u>Proof</u>. The continuity of the operator grad follows from the estimate

$$\|\operatorname{grad} p\|_{[V_0(-\varepsilon)]^*} = \sup_{\substack{\vec{v} \in V_0(-\varepsilon) \\ \|\vec{v}\|_{-\varepsilon} \leq 1}} \langle \operatorname{grad} p, \vec{v} \rangle =$$

$$= \sup_{\substack{\vec{v} \in V_0(-\varepsilon) \\ \|\vec{v}\|_{-\varepsilon} \leq 1}} \left(- \int_\Omega p \operatorname{div} \vec{v}\, dx \right) \leq N\, |p|_\varepsilon \cdot \sup_{\substack{\vec{v} \in V_0(-\varepsilon) \\ \|\vec{v}\|_{-\varepsilon} \leq 1}} \|\vec{v}\|_{-\varepsilon} \leq$$

$$\leq N\, |p|_\varepsilon .$$

Let V be the orthogonal complement of the subspace $\{\mathrm{const}\} + \{d^{-\varepsilon/2}\,\mathrm{const}\}$ in $L_2(\varepsilon)$ and let $p \in V$, i.e. $\int_\Omega d^\varepsilon p\, dx = 0$, $\int_\Omega d^{\varepsilon/2} p\, dx = 0$. As the mapping $\vec{\varphi} \longrightarrow d^{\varepsilon/2}\vec{\varphi}$ is an isomorphism of $[L_2(\varepsilon)]^N$ onto $[L_2(\Omega)]^N$ and of $V_0(0)$ onto $V_0(-\varepsilon)$ (see e.g. [3]) and moreover as $d^{\varepsilon/2}p$ is orthogonal to the subspace $\{\mathrm{const}\}$ in $L_2(\Omega)$, using Lemma 1, Hölder's inequality and the inequality (2) we obtain the estimate

$$|p|_\varepsilon = |d^{\varepsilon/2}p|_0 = \|d^{\varepsilon/2}p\|_{L_2(\Omega)_{/R}} \leq c_2\, \|\operatorname{grad} d^{\varepsilon/2}p\|_{[H^{-1}(\Omega)]^N} \leq$$

$$\leq c_2 \sup_{\substack{\vec{v} \in [W_o^{1,2}(\Omega)]^N \\ \|\vec{v}\|_o \leq 1}} \langle \operatorname{grad} d^{\varepsilon/2} p, \vec{v} \rangle =$$

$$= c_2 \sup_{\substack{\vec{v} \in [W_o^{1,2}(\Omega)]^N \\ \|\vec{v}\|_o \leq 1}} [\langle \operatorname{grad} p, d^{\varepsilon/2} \vec{v} \rangle - \langle p, \vec{v} . \operatorname{grad} d^{\varepsilon/2} \rangle] \leq$$

$$\leq c_3 [\sup_{\substack{\vec{v} \in V_o(-\varepsilon) \\ \|\vec{v}\|_{-\varepsilon} \leq 1}} \langle \operatorname{grad} p, \vec{v} \rangle +$$

$$+ \sup_{\substack{\vec{v} \in [W_o^{1,2}(\Omega)]^N \\ \|\vec{v}\|_o \leq 1}} |\varepsilon| \left| \int_\Omega d^{\varepsilon/2-1} p \; \vec{v} . \operatorname{grad} d \, dx \right|] \leq$$

$$\leq c_3 \|\operatorname{grad} p\|_{[V_o(-\varepsilon)]^*} + c_4 \cdot |\varepsilon| \cdot |p|_\varepsilon$$

(we use that $|\nabla d| \leq 1$ a.e. in Ω). Hence there exists a symmetric interval I, $0 \in \operatorname{int} I$, such that for every $\varepsilon \in I$ we have

$$|p|_\varepsilon \leq c_5(\Omega) \|\operatorname{grad} p\|_{[V_o(-\varepsilon)]^*} \text{ whenever } p \in V.$$

Therefore, the set grad$[V]$ is a closed subspace of $[V_o(-\varepsilon)]^*$. Since grad$[\{d^{-\varepsilon/2}\text{const}\}]$ is also a closed subspace of the same space, the subspace grad$[L_2(\varepsilon)]$ = grad$[V]$ + + grad $[\{d^{-\varepsilon/2}\text{const}\}]$ is closed as well. Now, the null-space of the operator grad is the space of constants and the assertion of Theorem 2 is a consequence of the open mapping theorem.

<u>Theorem 3</u>. Let $\varepsilon \in I$. Then the operator div acts from $V_o(-\varepsilon)$ onto $L_2^o(-\varepsilon)$.

Proof. The subspace $\operatorname{grad}[L_2(\varepsilon)_{/R}]$ is closed in $[V_0(-\varepsilon)]^*$ and hence the adjoint operator div maps the space $V_0(-\varepsilon)$ onto the anihilator of the subspace Ker[grad] = $=\{\text{const}\}$ which has the form $\{u \in L_2(-\varepsilon);\ \int_\Omega u\,dx = 0\}$.

Theorem 4. There exists a symmetric interval I', $0 \in \operatorname{int} I'$, and a constant c_6 such that for every $\varepsilon \in I'$ the inverse of the operator $\operatorname{div}: B_\varepsilon^\perp \longrightarrow L_2^0(\varepsilon)$ satisfies the estimate

$$\|\operatorname{div}^{-1}\|_{\mathcal{L}(L_2^0(\varepsilon);\ V_0(\varepsilon))} \leq c_6 .$$

Proof. Since $\|\operatorname{div}\|_{\mathcal{L}(V_0(0);\ L_2^0(0))} = L > 0$, there exists an element $\vec{y} \in V_0(0)$ with $\|\vec{y}\|_0 \leq 2$ satisfying $|\operatorname{div}\vec{y}|_0 = L$. If P denotes the projection of $V_0(\varepsilon)$ onto $B_\varepsilon^\perp$, we have

$$\|P(d^{-\varepsilon/2}\vec{y})\|_\varepsilon \leq \|d^{-\varepsilon/2}\vec{y}\|_\varepsilon \leq c_7\|\vec{y}\|_0 \leq 2c_7,$$

for every $\varepsilon \in I$ and therefore

$$\|\operatorname{div}\|_{\mathcal{L}(B_\varepsilon^\perp;\ L_2(\varepsilon))} \geq \frac{1}{2c_7}\ |\operatorname{div}(d^{-\varepsilon/2}\vec{y})|_\varepsilon =$$

$$= \frac{1}{2c_7}\ |d^{-\varepsilon/2}\operatorname{div}\vec{y} + \vec{y}.\operatorname{grad} d^{-\varepsilon/2}|_\varepsilon \geq$$

$$\geq \frac{1}{2c_7}\ [\,|d^{-\varepsilon/2}\operatorname{div}\vec{y}|_\varepsilon - |\vec{y}.\operatorname{grad} d^{-\varepsilon/2}|_\varepsilon] \geq$$

$$\geq \frac{1}{2c_7}\ [L - |\tfrac{\varepsilon}{2}|\ (\int_\Omega d^{-2}|\vec{y}.\operatorname{grad} d|^2\,dx)^{\frac{1}{2}}] \geq \frac{1}{2c_7}\ [L - |\varepsilon|\sqrt{c_1}\,].$$

We can choose now a symmetric interval I', $0 \in \operatorname{int} I'$ in such a way that $\|\operatorname{div}\|_{\mathscr{L}(B_\varepsilon^\perp\,;\;L_2^0(\varepsilon))} \geq \frac{L}{4c_7}$ for all $\varepsilon \in I'$.

This completes the proof.

Theorem 5. Let $\varepsilon \in I$, $g \in L_2(\varepsilon)$ and $\vec{\varphi} \in V(\varepsilon)$ satisfy the condition

$$\int_\Omega g\, dx = \int_{\partial\Omega} \vec{\varphi}.\vec{\nu}\; dS .$$

Then there exists $\vec{u} \in V(\varepsilon)$ such that $\operatorname{div}\vec{u} = g$ in Ω, $\vec{u} = \vec{\varphi}$ on $\partial\Omega$.

Proof. Since the set $[C^\infty(\bar{\Omega})]^N$ is dense in $V(\varepsilon)$ (see e.g. [1]) the trace of the vector function $\vec{\varphi}$ on $\partial\Omega$ makes sence and it holds $\int_{\partial\Omega}\vec{\varphi}.\vec{\nu}\; dS = \int_\Omega \operatorname{div}\vec{\varphi}\; dx$. Therefore we have $g - \operatorname{div}\vec{\varphi} \in L_2^0(\varepsilon)$ and with respect to Theorem 3 there exists $\vec{w} \in V_0(\varepsilon)$ such that $g - \operatorname{div}\vec{\varphi} = \operatorname{div}\vec{w}$. It is sufficient to put $\vec{u} = \vec{w} + \vec{\varphi}$.

Theorem 6. Let $\varepsilon \in I$, $\vec{f} \in [V_0(-\varepsilon)]^*$. Then the following conditions are equivalent

1/ $\langle \vec{f}, \vec{v} \rangle = 0$ for every $\vec{v} \in B_{-\varepsilon}$,

2/ $\vec{f} = \operatorname{grad} p$ for some $p \in L_2(\varepsilon)$.

Proof. Since the range of the operator grad acting from $L_2(\varepsilon)$ is a closed subspace of $[V_0(-\varepsilon)]^*$, it follows from the theory of linear operators that $\vec{f}$ is an element of this range if and only if $\vec{f}$ belongs to the anihilator of the null-space of the adjoint operator, i.e. $\vec{f}$ belongs to the anihilator of $\operatorname{Ker}[\operatorname{div}] = B_{-\varepsilon}$.

Section 3.

Definition. A couple $(\vec{u},p)\in V(\varepsilon)\times L_2^0(\varepsilon)$ is said to be the weak solution of the Stokes problem (1) with $\vec{f}\in[V_0(-\varepsilon)]^*$, $g\in L_2(\varepsilon)$ and $\vec{\varphi}\in V(\varepsilon)$, if

$$\nu\sum_{i,j=1}^{N}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial z_j}{\partial x_i}\,dx-\int_\Omega p\,\operatorname{div}\vec{z}\,dx=\langle\vec{f},\vec{z}\rangle$$

$$\text{for all}\quad \vec{z}\in[C_0^\infty(\Omega)]^N,$$

$$\operatorname{div}\vec{u}=g\quad\text{in }\Omega,$$

$$\vec{u}=\vec{\varphi}\quad\text{on }\partial\Omega.$$

In consequence of the density of $[C_0^\infty(\Omega)]^N$ in $V_0(-\varepsilon)$ we can consider the first equality for all $\vec{z}\in V_0(-\varepsilon)$.

Since $\int_\Omega g\,dx=\int_{\partial\Omega}\vec{\varphi}.\vec{\nu}\,dS$ there exists $\vec{w}\in V_0(\varepsilon)$ satisfying conditions $\operatorname{div}\vec{w}=g$ in Ω, $\vec{w}=\vec{\varphi}$ on $\partial\Omega$ and

$$\|\vec{w}\|_\varepsilon\le c_8(\Omega,\varepsilon)\left[|g|_\varepsilon+|\operatorname{div}\vec{\varphi}|_\varepsilon\right].$$

Putting $\vec{v}=\vec{u}-\vec{w}$ we transform the problem (1) to the homogenous one

$$(3)\qquad \begin{aligned}-\nu\Delta\vec{v}+\operatorname{grad}p&=\vec{h}\quad\text{in }\Omega,\\ \operatorname{div}\vec{v}&=0\quad\text{in }\Omega,\\ \vec{v}&=0\quad\text{on }\partial\Omega,\end{aligned}$$

where $\vec{h}=\vec{f}+\nu\Delta\vec{w}$.

Further, we shall study the solvability of (3). Let us define a bilinear form $a\colon V_0(\varepsilon)\times V_0(-\varepsilon)\to R$ by the relation

$$a(\vec{v},\vec{z}) = \nu \sum_{i,j=1}^{N} \int_{\Omega} \frac{\partial v_j}{\partial x_i} \frac{\partial z_j}{\partial x_i}\, dx \ , \ \vec{v} \in V_0(\varepsilon),\ \vec{z} \in V_0(-\varepsilon).$$

From (3) we obtain the equation

(4) $\quad a(\vec{v},\vec{z}) = \langle \vec{h},\vec{z} \rangle$, for all $\vec{z} \in B_{-\varepsilon}$.

This equation has unique solution $\vec{v} \in B_\varepsilon$ for every $\vec{h} \in [V_0(-\varepsilon)]^*$ with $\|\vec{v}\|_\varepsilon \leq c_9 \|\vec{h}\|_{[V_0(-\varepsilon)]^*}$ if the form $a(.,.)$ is elliptic on $B_\varepsilon \times B_{-\varepsilon}$ in both its components, i.e.

(5) $$\sup_{\substack{\vec{z} \in B_{-\varepsilon} \\ \|\vec{z}\|_{-\varepsilon} \leq 1}} a(\vec{y},\vec{z}) \geq \alpha_1 \|\vec{y}\|_\varepsilon \ , \text{ for all } \vec{y} \in B_\varepsilon \ ,$$

(6) $$\sup_{\substack{\vec{y} \in B_\varepsilon \\ \|\vec{y}\|_\varepsilon \leq 1}} a(\vec{y},\vec{z}) \geq \alpha_2 \|\vec{z}\|_{-\varepsilon} \ , \text{ for all } \vec{z} \in B_{-\varepsilon} \ ,$$

where constants $\alpha_1(\varepsilon), \alpha_2(\varepsilon) > 0$. (The proof of this "generalized Lax-Milgram" lemma can be found in [2], [6].) We shall prove the inequalities (5),(6) for the bilinear form $a(.,.)$ defined above. Since for $\vec{y} \in B_\varepsilon$ we have $d^\varepsilon \vec{y} \in V_0(-\varepsilon)$ and since the operator div: $B_{-\varepsilon}^{\perp} \longrightarrow L_2^0(-\varepsilon)$, $\varepsilon \in I'$, is an isomorphism then there exists an element $\vec{s} = \mathrm{div}^{-1}[\mathrm{div}\ d^\varepsilon \vec{y}] \in B_{-\varepsilon}^{\perp}$.
According to $\mathrm{div}\ \vec{y} = 0$ and to the inequality (2) we obtain
$\|\vec{s}\|_{-\varepsilon} \leq c_6 \,|\mathrm{div}\ d^\varepsilon \vec{y}|_{-\varepsilon} = c_6 \,|d^\varepsilon \,\mathrm{div}\ \vec{y} + \varepsilon\, d^{\varepsilon-1}\, \vec{y}.\mathrm{div}\ d|_{-\varepsilon} \leq$
$\leq c_{10} \cdot |\varepsilon| \cdot \|\vec{y}\|_\varepsilon$. As $d^\varepsilon \vec{y} - \vec{s} \in B_{-\varepsilon}$, $\|d^\varepsilon \vec{y} - \vec{s}\|_{-\varepsilon} \leq \|\vec{y}\|_\varepsilon\, c_{11}(1+|\varepsilon|)$
we can write

$$a(\vec{y}, \frac{d^\varepsilon \vec{y} - \vec{s}}{\|d^\varepsilon \vec{y} - \vec{s}\|_{-\varepsilon}}) \geqq \frac{a(\vec{y}, d^\varepsilon \vec{y}) - c_{12}\|\vec{y}\|_\varepsilon^2 |\varepsilon|}{\|\vec{y}\|_\varepsilon \; c_{11}(1+|\varepsilon|)} \geqq$$

$$\geqq \frac{1}{\|\vec{y}\|_\varepsilon \; c_{11}(1+|\varepsilon|)} \Big[\upsilon \sum_{i,j=1}^{N} \int_\Omega d^\varepsilon \Big|\frac{\partial y_j}{\partial x_i}\Big|^2 dx -$$

$$- |\varepsilon| \, \upsilon \sum_{i,j=1}^{N} \int_\Omega d^{\varepsilon-1} \Big|\frac{\partial d}{\partial x_i}\Big| \Big|\frac{\partial y_j}{\partial x_i}\Big| \, |y_j| \, dx - c_{12} |\varepsilon| \|\vec{y}\|_\varepsilon^2 \Big] \geqq$$

$$\geqq \frac{1}{\|\vec{y}\|_\varepsilon \; c_{11}(1+|\varepsilon|)} \Big[\upsilon |||\vec{y}|||_\varepsilon^2 - \upsilon |\varepsilon| \Big(\sum_{i,j=1}^{N} \int_\Omega d^\varepsilon \Big|\frac{\partial y_j}{\partial x_i}\Big|^2 dx\Big)^{\frac{1}{2}} .$$

$$. \Big(\sum_{i=1}^{N} \int_\Omega d^{\varepsilon-2} |y_i|^2 dx\Big)^{\frac{1}{2}} - c_{12} |\varepsilon| \|\vec{y}\|_\varepsilon^2 \Big] \geqq$$

$$\geqq \|\vec{y}\|_\varepsilon \; \frac{\upsilon c_{13} - |\varepsilon| \, (\upsilon \sqrt{c_1}/|\varepsilon - 1| + c_{12})}{c_{11}(1+|\varepsilon|)} .$$

Hence the inequality (5) is fulfiled for every ε from a suitable interval $J \subset I \cap I'$, $0 \in \operatorname{int} J$. Analogously, the inequality (6) holds for $\varepsilon \in (-J)$.

Consequently, the equation (4) has a solution $\vec{v} \in B_\varepsilon$, for every $\vec{h} \in [V_0(-\varepsilon)]^*$, with $\varepsilon \in J \cap (-J)$ and $\|\vec{v}\|_\varepsilon \leqq c_{14} \|\vec{h}\|_{[V_0(-\varepsilon)]^*}$. Let $\varepsilon \in J \cap (-J)$. Since $\langle \vec{h} + \upsilon \Delta \vec{v}, \vec{z} \rangle = \langle \vec{h}, \vec{z} \rangle - a(\vec{v}, \vec{z}) = 0$ for all $\vec{z} \in B_{-\varepsilon}$,

by Theorem 6 there exists $p \in L_2^0(\varepsilon)$ such that
grad $p = \vec{h} + \nu \Delta\vec{v}$, i.e. the couple $(\vec{v},p)$ is the weak solution of (3), and according to Theorem 2 we obtain the estimate

$$|p|_\varepsilon \leq c_{14}\left[\|\vec{h}\|_{[V_0(-\varepsilon)]^*} + \|\vec{v}\|_\varepsilon\right].$$

Therefore, the couple $(\vec{u},p) \in V(\varepsilon) \times L_2^0(\varepsilon)$, where $\vec{u} = \vec{v} + \vec{w}$, is the weak solution of the problem (1) and it holds

$$(7) \qquad \|\vec{u}\|_\varepsilon + |p|_\varepsilon \leq c_{15}\left[\|\vec{f}\|_{[V_0(-\varepsilon)]^*} + |g|_\varepsilon + |\operatorname{div}\vec{\varphi}|_\varepsilon\right].$$

Remark. In the last inequality it is possible to write the norm of the trace of $\vec{\varphi}$ on $\partial\Omega$ instead of the norm of div $\vec{\varphi}$.

Let us summarize the results of this Section in

Theorem 7. There exists an interval J, $0 \in \operatorname{int} J$, such that for every $\varepsilon \in J$ the Stokes problem (1) has the unique weak solution $(\vec{u},p) \in [W^{1,2}(\Omega;d,\varepsilon)]^N \times L_2^0(\Omega;d,\varepsilon)$, whenever $\vec{f} \in ([W_0^{1,2}(\Omega;d,-\varepsilon)]^N)$, $g \in L_2(\Omega;d,\varepsilon)$, $\vec{\varphi} \in [W^{1,2}(\Omega;d,\varepsilon)]^N$ (with $\int_\Omega g\,dx = \int_{\partial\Omega} \vec{\varphi}.\vec{\nu}\,dS$).

Moreover, the solution $(\vec{u},p)$ satisfies the estimate (7).

References

[1] Kufner A.: Weighted Sobolev spaces, Teubner, Leipzig, 1980.

[2] Nečas J.: Les méthodes directes en théorie des équa-

tions elliptiques, Academia, Prague, 1967.

[3] Voldřich J.: Solvability of the Dirichlet boundary value problem for nonlinear elliptic partial differential equations in Sobolev power weight spaces, to appear in Časopis Pěst. Mat.

[4] Temam R.: Navier-Stokes equations, North-Holland Publishing Company, Amsterdam, 1979.

[5] Girauld V., Raviart P.-A.: Finite element approximation of the Navier-Stokes equations, Springer Verlag, 1981.

[6] Sallinen P.: A representation theorem for bounded bilinear forms on Banach spaces, Mathematics University of Oulu, reprint, 1979.

Katedra matematiky VŠSE, Nejedlého sady 14, 30614 Plzeň, Czechoslovakia

(Oblatum 6.1. 1984)