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Jiří Souček

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MORSE-SARD THEOREM FOR CLOSED GEODESICS

J. SOUČEK

Abstract: The existence of finite or infinite number of closed geodesics on a compact Riemannian manifold can be proved under suitable assumptions. The paper brings another type of information. It is proved here that the set Γ of lengths of all closed geodesics on a real-analytic compact Riemann manifold is always a discrete set. The proof is based on a version of Morse-Sard theorem for real-analytic maps.

Key words: Closed geodesics, Morse-Sard theorem.

Classification: 58E10

The classical Morse theorem says that the set Γ of critical levels of a function $f \in C^k: R^n \longrightarrow R$

$$\Gamma = \{f(x) \mid \nabla f(x) = 0\}$$

has Lebesgue measure zero; $\mathcal{H}_1(\Gamma) = 0$, if $k \geq n$ [1]. The more refined version [2] asserts that $\mathcal{H}_{n/k}(\Gamma) = 0$, where $\mathcal{H}_{n/k}$ is a Hausdorff measure of dimension $n/k \leq 1$. If $f \in C$, we obtain $\mathcal{H}_\varepsilon(\Gamma) = 0$, $\forall \varepsilon > 0$. If the function f is analytic, we can obtain a better estimate of Γ. Namely, [3], Γ is locally finite; i.e. for every compact set $K \subset R^n$, the set

$$\Gamma_K = \{f(x) \mid \nabla f(x) = 0, x \in K\}$$

is finite. Clearly, then Γ is denumerable.

Analogous theorems hold for a functional defined on a Banach space if its second derivative is a Fredholm map (and some other , rather technical hypotheses are satisfied, see [4 - 7]).

In this paper we prove an analogous theorem for closed geodesics, the functional being the length of a curve.

Theorem 1. Let M be an n-dimensional compact real-analytic Riemannian manifold. Then the set Γ of lengths of all closed geodesics on M is a discrete set.

Remark. The question of existence of closed geodesics was extensively studied (see e.g. [9]), and the existence of infinite number of them can be proved for some manifolds. The theorem 1 brings on the other hand an upper bound on the number of lengths of closed geodesics.

The real-analyticity means that there is an atlas of charts covering M, such that transition maps are real-analytic and the coefficients of a metric written in these charts are real-analytic, too. In the same way one may prove that when M is only a C^{∞} manifold, the set Γ has $\mathcal{H}_{\varepsilon}(\Gamma) = 0, \ \forall \varepsilon > 0$.

The set of closed curves in M does not form a linear space, and therefore the theorems of [7] cannot be applied directly. But this set can be endowed with a structure of a Hilbertian manifold [8, 9].

Proof. We shall divide it into three steps. In the first step we recall the definition and some properties of the Hilbertian manifold of curves. In the second step we prove the theorem using Lemma 1. In the third step we give a rather technical proof of this lemma.

Step 1. Let $S = [0,1] / \{0,1\}$ be the unit circle and let us define the set of closed H^1-curves

$$\Lambda M = \{c : S \to M \mid \int_0^1 \langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)}\, dt < \infty\}$$

where $\langle \cdot , \cdot \rangle_x$ denotes a Riemannian metric on T_xM. Let $c_o : S \to M$ be a fixed C^∞-curve. Let us consider the set of all H^1-vector fields on c_o (parametrized by $t \in S$)

$$H^1(c_o^* TM) = \{\xi : S \to TM \mid \xi(t) \in T_{c_o(t)}M,\ \langle \xi, \xi \rangle_1 < \infty\}$$

where

$$\langle \xi , \eta \rangle_o = \int_0^1 \langle \xi(t), \eta(t) \rangle_{c_o(t)}\, dt,$$

$$\langle \xi , \eta \rangle_1 = \langle \xi , \eta \rangle_o + \langle \nabla \xi, \nabla \eta \rangle_o ;$$

the covariant derivative being taken along the curve

$$(\nabla \xi)(t) = (\nabla_{\dot{c}_o(t)} \xi)(t).$$

There exists an $\varepsilon > 0$ such that [8, 9] the exponential map

$$(\tau, \exp) : TM \to M \times M\ (x, v_x) \mapsto (x, \exp_x v_x),\ v_x \in T_xM$$

is a diffeomorphism, when restricted to the set

$$T^\varepsilon M = \{(x, v_x) \in TM \mid \| v_x \| < \varepsilon\}.$$

Here $\exp_x : T_xM \to M$ is the standard exponential map. Let us define

$$\exp_{c_o} : H^{1,\varepsilon}(c_o^* TM) \to \Lambda M$$

$$\xi = (\xi(t))_{t\in S} \mapsto c = (c(t))_{t\in S}$$

$$c(t) = \exp_{c_o(t)} \xi(t),$$

where

$$H^{1,\varepsilon}(c_o^* TM) = \{\xi \in H^1(c_o^* TM) \mid \| \xi(t) \| < \varepsilon,\ \forall t \in S\}.$$

Then the atlas of charts [8, 9]

$$\{\exp_{c_o}^{-1} \mid c_o \in C^\infty(S, M)\}$$

gives to ΛM the structure of a Hilbertian manifold modelled on Hilbert spaces $H^1(c_o^* TM)$. Moreover, $H^1(c_o^* TM)$ is the tangent space $T_{c_o}\Lambda M$.

The energy of a closed curve $c \in \Lambda M$ is defined by

$$E(c) = \frac{1}{2}\int_0^1 \langle \dot{c}(t), \dot{c}(t)\rangle_{c(t)} dt.$$

Then [8, 9] E is C^∞, its Gateâux differential is given by

$$dE(c;\eta) = \int_0^1 \langle \dot{c}, \nabla\eta\rangle dt$$

and $dE(c;\cdot) = 0$ iff c is a constant map or c is a closed geodesic. Grad $E(c) \in T_c\Lambda M$ is defined by

$$\langle \text{grad } E(c), \eta\rangle_1 = dE(c;\eta), \quad \forall \eta \in T_c\Lambda M.$$

The following basic properties of the energy functional are known [8, 9]:

(i) (so-called Palais-Smale condition.)

If $c_m \in \Lambda M$, $E(c_m) \leq K$, $K > 0$ and $\|\text{grad } E(c_m)\|_1 \to 0$, then there is a subsequence of $\{c_m\}$ converging in ΛM to a closed geodesic c_o,

(ii) Let c_o be a closed geodesic. Then the Hessian A_{c_o} of E at c_o, defined by

$$A_{c_o} : T_{c_o}\Lambda M \to T_{c_o}\Lambda M,$$

$$\langle A_{c_o}\xi, \eta\rangle_1 = d^2E(c_o;\xi,\eta), \quad \forall \xi, \eta \in T_{c_o}\Lambda M,$$

has the form A_{c_o} = identity + compact map.

<u>Step 2</u>. Let us suppose that there are closed geodesics c_m such that $E(c_m) \to K < \infty$, $E(c_m) \neq E(c_k)$, $\forall k \neq m$. Using (i) we can suppose that $c_m \to c_o$ in ΛM, c_o is a geodesic, and that $\{c_m\}$ is in the range of the chart $\exp_{c_o}$. Let us transport the energy functional on the tangent space

$$\bar{E}(\xi) = E(\exp_{c_o}(\xi)), \quad \forall \xi \in T_{c_o}^{\varepsilon}\Lambda M = H^{1,\varepsilon}(c_o^* TM).$$

Then $\xi_m = \exp_{c_o}^{-1}(c_m)$ are critical points of $\overline{E}$ and $\|\xi_m\|_1 \longrightarrow 0$. We want now to apply the infinite dimensional version of the Morse-Sard theorem [7, Theor. 5.1] to $\overline{E}$. But to prove the analyticity of $\overline{E}$, the regularity properties must be used. We follow the method of [7, p. 256]. Let us define spaces

$$X_1 = H^2(c_o^* TM) = \{\xi \in H^1(c_o^* TM) \mid \langle \xi, \xi \rangle_2 < \infty\},$$

$$\langle \xi, \eta \rangle_2 = \langle \xi, \eta \rangle_1 + \langle \nabla\nabla \xi, \nabla\nabla \eta \rangle_o,$$

$$X_2 = H^o(c_o^* TM) = \{\xi : S \longrightarrow TM \mid \xi(t) \in T_{c_o(t)}M, \langle \xi, \xi \rangle_o < \infty\}$$

and consider grad $\overline{E}$ as a map $F: X_1 \longrightarrow X_2$ defined by

$$d\overline{E}(\xi; \eta) = \langle \eta, F(\xi) \rangle_o, \quad \forall \eta \in X_1.$$

By the standard regularity argument (similar to that used in the proof of the regularity of geodesics) we obtain that

$$\xi_m \in X_1, \quad \|\xi_m\|_2 \longrightarrow 0.$$

<u>Lemma 1</u>. <u>The functional</u> $\overline{E}: X_1 \longrightarrow R$ <u>and the operator</u> $F: X_1 \longrightarrow$ $\longrightarrow X_2$ <u>are real-analytic</u>.

Once we have this lemma, all hypotheses of the theorem 5.1 from [7] are fulfilled and it follows that for a sufficiently large m we have

$$E(c_m) = \overline{E}(\xi_m) = \overline{E}(0) = E(c_o).$$

Hence the critical levels of E are isolated. Now, it suffices to show that the energy and the length of a geodesic are interrelated. If c is a critical point of E, then using variations generated by the reparametrizations of c, we can find that $\|\dot{c}(t)\| = \omega$ = constant. Then $E(c) = \frac{1}{2}\omega^2$, while

$$L(c) = \int_0^1 \langle \dot{c}, \dot{c} \rangle^{\frac{1}{2}} dt = \omega,$$

so that $L(c) = (2E(c))^{\frac{1}{2}}$.

Step 3. We shall prove Lemma 1 using the method of [7, p. 256]. We must show that for each $\xi_0 \in X_1$, $\|\xi_0\|_2 < \varepsilon$ ($\varepsilon > 0$ sufficiently small), there is a $\delta > 0$, such that for every $\chi \in X_1$, $\|\chi\|_2 < \delta$ the Taylor series

$$\overline{E}(\xi_0 + \chi) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n\overline{E}(\xi_0 ; \chi, \ldots, \chi)$$

converges. It suffices to show that $\overline{E}$ is a restriction (to X_1) of a map $\tilde{E}: X_1 + iX_1 \longrightarrow X_2 + iX_2$, which is locally bounded and Gateâux differentiable [7, Theor. 3.7]. Let $I \subset S$ be an interval such that $c_0(I) \subset U$, U is a co-ordinate neighborhood in M (S may be written as a finite union of such intervals). Let us define

$$\overline{E}_I(\xi) = \frac{1}{2} \int_I \langle \dot{c}, \dot{c} \rangle \, dt, \quad c(t) = \exp_{c_0(t)}(\xi(t)).$$

If $(x_1, \ldots, x_n)$ is a co-ordinate system on U we denote by $\overline{x}(x) = (x_1(x), \ldots, x_n(x))$ the co-ordinates of the point $x \in U$. Generally, the bar will denote n-tuples, so that

$$x_i(c_0(t)) = c_{0i}(t), \quad \overline{c}_0(t) = (c_{01}(t), \ldots, c_{0n}(t)).$$

For $\xi \in T_xM$, $x \in U$ we shall write

$$\xi = \xi_i \frac{\partial}{\partial x_i}\Big|_x, \quad \overline{\xi} = (\xi_1, \ldots, \xi_n),$$

$$x_i(\exp_x \xi) = e_i(\overline{x}(x); \overline{\xi}).$$

Using the summation convention we can write

$$\frac{d}{dt} \exp_{c_0} \xi = h_i(t, \overline{\xi}, \dot{\overline{\xi}}) \frac{\partial}{\partial x_i}\Big|_{\exp_{c_0} \xi},$$

$$h_i(t, \overline{\xi}, \dot{\overline{\xi}}) = \frac{d}{dt} e_i(\overline{c}_0; \overline{\xi}) =$$

$$= \frac{\partial e_i}{\partial c_{0k}} (\overline{c}_0; \overline{\xi}) \, \dot{c}_{0k} + \frac{\partial e_i}{\partial \xi_k} (\overline{c}_0; \overline{\xi}) \, \dot{\xi}_k,$$

where the dependence of $c_0, \xi, \overline{c}_0, \overline{\xi}, \dot{\overline{\xi}}$ on t was supressed. Using the co-ordinate form g_{ij} of the metric $\langle \cdot, \cdot \rangle$ we have

$$\overline{E}_I(\xi) = \frac{1}{2} \int_I f(t, \overline{\xi}, \dot{\xi})\, dt,$$

where

$$f(t, \overline{\xi}, \dot{\xi}) = g_{ij}(\overline{e}(\overline{c}_0; \overline{\xi}))h_i(t, \overline{\xi}, \dot{\xi})h_j(t, \overline{\xi}, \dot{\xi}).$$

The functions e_i are real-analytic [10] and g_{ij} are real-analytic, too. Thus, they are restrictions (to real values of their arguments) of holomorphic functions $\tilde{e}_i$, $\tilde{g}_{ij}$ resp. [11; 7, Theor. 3.11]. Then functions h_i and f are also restrictions of $\tilde{h}_i$, $\tilde{f}$ defined for $\overline{\xi}$, $\dot{\xi}$ complex vectors from the neighborhood of allowed real values of $\overline{\xi}$, $\dot{\xi}$, i.e. $|\overline{\xi}|$, $|\dot{\xi}| < \varepsilon$ (the variable t remains always real). At this moment we need the regularity, because we must know that $|\xi|_{C^1} < \varepsilon$; this follows from the embedding $X_1 \subset C^1$. Clearly, h_i and f are C^∞ in t and we can apply Lemma 3.1 from [7, App. VI] which gives us the analyticity of $\overline{E}_I$. Equivalently, we can verify that

$$\tilde{E}_I(\overline{\xi}) = \frac{1}{2} \int_I \tilde{f}(t, \overline{\xi}, \dot{\xi})\, dt, \quad \overline{\xi}, \dot{\xi} \in C^n$$

is locally bounded and Gateâux differentiable. The proof of the analyticity of F_I is similar to the proof of Lemma 3.1 [7, App. VI]. Making a finite sum of E_I's (resp. F_I's) we obtain the analyticity of $\overline{E}$ and F.

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Academy of Sciences, Math. Institute, Žitná 25, 115 67 Prague, Czechoslovakia

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