Aleš Pultr

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POINTLESS UNIFORMITIES I. COMPLETE REGULARITY

A. PULTR

Abstract: The equivalence of complete regularity and uniformizability is shown for general locales. Also, a characterisation of complete regularity by means of the behavior of diameters is presented.

Key words: Locale, uniformity and uniformizability, complete regularity, diameter.

Classification: 54E15, 06D10

The definition of uniformity in the form of a system of covers can be extended in an obvious way to locales (see also [5]). In this paper we prove the fact one could expect, namely that, also in locales, uniformisability coincides with complete regularity.

More explicitly: A system of covers $\mathcal{A}$ of a locale L gives naturally rise to a subset $L_{\mathcal{A}} \subset L$ (see Section 2). A locale is shown to be completely regular iff there is a uniformity $\mathcal{A}$ such that $L_{\mathcal{A}} = L$. (By the way, it is regular iff there is a system of covers $\mathcal{A}$ such that $L_{\mathcal{A}} = L$.) Moreover, another characterization of complete regularity by means of separation by diameter functions is presented.

The paper is divided into five sections. The first one contains the necessary definitions and basic facts. Section 2 deals with the sublocales induced by systems of covers. In

Section 3 the notions of uniformity and weak uniformity are introduced, Section 4 deals with diameters. The characterization theorem is proved in the last, fifth section.

1. Preliminaries

1.1. A locale (see, e.g. [6]) is a complete lattice L satisfying the distribution law $x \wedge \bigvee A = \bigvee\{x \wedge a \mid a \in A\}$. The bottom resp. top of L will be denoted by

$$0 \text{ resp. } e,$$

the pseudocomplement of $x \in L$ by

$$\bar{x}.$$

An element x is said to be complemented if $\bar{x} \vee x = e$.

1.2. One writes $x \lhd y$ if there is a z such that

$$x \wedge z = 0 \text{ and } y \vee z = e$$

(or, equivalently, if $\bar{x} \vee y = e$).

Note that $x \lhd x$ iff x is complemented. Consequently, $y \lhd x$ with non-complemented x implies $y \neq x$.

1.3. A locale is said to be regular iff

$$x = \bigvee\{z \mid z \lhd x\} \text{ for each } x \in L$$

(see, e.g.,[1],[3]).

1.4. One writes

$$x \lhd\!\!\lhd y$$

if there is a family x_{ik} of elements of L such that

$i = 0,1,\ldots,\ k = 0,1,\ldots,2^i,\ x_{00} = x,\ x_{01} = y,$

$x_{ik} \lhd x_{i,k+1}$, and finally $x_{ik} = x_{i+1,2k}$.

A locale is said to be completely regular iff

$$x = \bigvee\{z \mid z \lhd\!\!\lhd x\} \text{ for each } x$$

(see e.g. [1], cf. [3],[2]).

1.5. Lemma: Let R be a binary relation on L such that

(1) $xRy \Rightarrow x \lhd y$, and (2) $xRy \Rightarrow \exists z,\ xRzRy$. Then

$$xRy \Rightarrow x \lhd\!\lhd y.$$

Proof: Put $x_{00} = x$, $x_{01} = y$. Let us have x_{ij} defined for all $i < n$ and all $k = 0,1,\dots,2^i$ so that $x_{ik}Rx_{i,k+1}$. Put

$$x_{n,2k} = x_{n-1,k}$$

and choose $x_{n,2k+1}$ such that $x_{n,2k}Rx_{n,2k+1}Rx_{n,2(k+1)}$. □

1.6. A cover of a locale L is a subset $A \subset L$ such that

$$\bigvee A = L.$$

The system of all covers of L will be denoted by

$$\mathscr{C}(L).$$

For $A,B \in \mathscr{C}(L)$ we write

$$A \prec B$$

if for each $a \in A$ there is a $b \in B$ such that $a \leq b$.

For $A,B \in \mathscr{C}(L)$ set

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

(Obviously, $A \wedge B \in \mathscr{C}(L)$.)

Finally, take an $A \in \mathscr{C}(L)$. Put

$$A^{(2)} = \{a \vee b \mid a,b \in A, a \wedge b \neq 0\},$$

$$A^* = \{\bigvee B \mid B \subset A,\ (a,b \in B \Rightarrow a \wedge b \neq 0)\};$$

for $x \in L$ put

$$Ax = \bigvee\{a \mid a \in A, a \wedge x \neq 0\}.$$

1.7. Proposition. 1. $A \prec B \Rightarrow Ax \leq Bx$.

2. $(A \wedge B)x \leq Ax \wedge Bx$.

3. $A(Ax) \leq A^{(2)}x$.

Proof is straightforward. □

1.8. Proposition: Let there be an $A \in \mathscr{C}(L)$ such that $Ax \leq y$. Then $x \lhd y$.

Proof: Put $z = \bigvee\{a \mid a \in A, a \wedge x = 0\}$. We have $z \wedge x = 0$ and $z \vee y \geq z \vee Ax = \bigvee A = e$. □

2. Systems of covers, induced sublocales and a characterization of regularity

2.1. Let $\mathcal{A}$ be a subset of $\mathscr{C}(L)$. We write

$$x \overset{\mathcal{A}}{\lhd} y$$

iff there is an $A \in \mathcal{A}$ such that $Ax \leq y$.

2.2. By 1.8 we immediately obtain

Proposition: $x \overset{\mathcal{A}}{\lhd} y \Rightarrow x \lhd y$. □

2.3. Also the following statement is obvious:

Proposition: Let $\mathcal{A} \subset \mathcal{B}$. Then

$$x \overset{\mathcal{A}}{\lhd} y \Rightarrow x \overset{\mathcal{B}}{\lhd} y. \quad \square$$

2.4. We set

$$L_{\mathcal{A}} = \{x \in L \mid x = \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x\}\}$$

2.5. From 2.3 we immediately obtain

Proposition: $\mathcal{A} \subset \mathcal{B} \Rightarrow L_{\mathcal{A}} \subset L_{\mathcal{B}}$. □

2.6. Lemma: Let $\mathcal{A} \subset \mathscr{C}(L)$ be such that

$$A,B \in \mathcal{A} \Rightarrow A \wedge B \in \mathcal{A}.$$

Then $u \overset{\mathcal{A}}{\lhd} x \;\&\; v \overset{\mathcal{A}}{\lhd} y \Rightarrow u \wedge v \overset{\mathcal{A}}{\lhd} x \wedge y$.

Proof: We have $A,B \in \mathcal{A}$ such that

$$Au \leq x \text{ and } Bv \leq y.$$

Thus, by 1.7.1, $(A \wedge B)(u \wedge v) \leq (A \wedge B)u \leq Au \leq x$,

$$(A \wedge B)(u \wedge v) \leq (A \wedge B)v \leq Bv \leq y,$$

and hence $(A \wedge B)(u \wedge v) \leq x \wedge y$. □

2.7. Theorem: Let $\mathcal{A} \subset \mathscr{C}(L)$ be non-void and such that

$$A,B \in \mathcal{A} \Rightarrow A \wedge B \in \mathcal{A}$$

Then $L_{\mathcal{A}}$ is a sublocale of L.

Proof: Obviously, $0, e \in L_{\mathcal{A}}$. Now, let $x, y \in L_{\mathcal{A}}$. By the distributivity and by 2.6 we have

$$x\wedge y = \bigvee\{u \mid u \overset{\mathcal{A}}{\lhd} x\} \wedge \bigvee\{v \mid v \overset{\mathcal{A}}{\lhd} y\} = \bigvee\{u\wedge v \mid u \overset{\mathcal{A}}{\lhd} x \,\&\, v \overset{\mathcal{A}}{\lhd} y\} \leq$$
$$\leq \bigvee\{u\wedge v \mid u\wedge v \overset{\mathcal{A}}{\lhd} x\wedge y\} \leq \bigvee\{w \mid w \overset{\mathcal{A}}{\lhd} x\wedge y\} \leq x\wedge y.$$

Let $x_i = \bigvee\{u \mid u \overset{\mathcal{A}}{\lhd} x_i\}$ for $i\in J$. Then we have

$$x_j \leq \bigvee\{u \mid u \overset{\mathcal{A}}{\lhd} \bigvee x_i\}$$

for all j and hence

$$\bigvee x_j \leq \bigvee\{u \mid u \overset{\mathcal{A}}{\lhd} \bigvee x_j\} \leq \bigvee x_j. \quad \square$$

2.8. Theorem: A locale L is regular iff there is a system of covers $\mathcal{A}$ such that $L = L_{\mathcal{A}}$.

Proof: If $L = L_{\mathcal{A}}$, L is regular by 2.2. On the other hand, let L be regular. Put

$$\mathcal{A} = \{\{\bar{x},y\} \mid x,y\in L,\ x\lhd y\}.$$

We have $\{\bar{x},y\}x\leq y$ so that now

$$x\lhd y \Rightarrow x \overset{\mathcal{A}}{\lhd} y,$$

and hence $L_{\mathcal{A}} = L$. $\square$

3. Uniformities and uniformizability

3.1. A non-void system $\mathcal{U}\subset\mathcal{C}(L)$ is said to be a uniformity on the locale L if

(i) $A\in\mathcal{U} \,\&\, A\prec B \Rightarrow B\in\mathcal{U}$,

(ii) $A\in\mathcal{U} \,\&\, B\in\mathcal{U} \Rightarrow A\wedge B\in\mathcal{U}$,

(iii*) for each $A\in\mathcal{U}$ there is a $B\in\mathcal{U}$ such that

$$B^{*}\prec A \text{ (cf. [5]).}$$

A non-void $\mathcal{U}$ is said to be a weak uniformity on L if there hold (i),(ii) and

(iii2) for each $A\in\mathcal{U}$ there is a $B\in\mathcal{U}$ such that

$$B^{(2)}\prec A.$$

3.2. A non-void system $\mathcal{U}\subset\mathcal{C}(L)$ is said to be a uniformity basis (resp. a weak uniformity basis; briefly, u-basis resp. wu-basis) if it satisfies (iii*) resp. (iii2).

3.3. For $\mathcal{A} \subset \mathcal{C}(L)$ put

$$\widetilde{\mathcal{A}} = \{A \mid \exists A_1, \dots, A_k \in \mathcal{A},\ A_1 \wedge \dots \wedge A_k \prec A\}.$$

3.4. Lemma: We have

$$(A_1 \wedge \dots \wedge A_n)^{(2)} \prec A_1^{(2)} \wedge \dots \wedge A_n^{(2)},$$
$$(A_1 \wedge \dots \wedge A_n)^* \prec A_1^* \wedge \dots \wedge A_n^*.$$

Proof: Obviously, it suffices to prove the statement for $n=2$.

I. Let $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$ be such that $(a_1 \wedge a_2) \wedge \wedge (b_1 \wedge b_2) \neq 0$. Then $a_1 \wedge b_1 \neq 0 \neq a_2 \wedge b_2$ and hence $a_i \vee b_i \in A_i^{(2)}$ $(i = 1,2)$. We have

$$(a_1 \wedge a_2) \vee (b_1 \wedge b_2) \leq (a_1 \vee b_1) \wedge (a_2 \vee b_2).$$

II. Let $C \subset A_1 \wedge A_2$ be such that

$$a_1 \wedge a_2,\ b_1 \wedge b_2 \in C \Rightarrow (a_1 \wedge a_2) \wedge (b_1 \wedge b_2) \neq 0.$$

Define C_i $(i = 1,2)$ as follows:

$$C_1 = \{a_1 \in A_1 \mid \exists a_2 \in A_2,\ a_1 \wedge a_2 \in C\},$$
$$C_2 = \{a_2 \in A_2 \mid \exists a_1 \in A_1,\ a_1 \wedge a_2 \in C\}.$$

Obviously, $a_i, b_i \in C_i \Rightarrow a_i \wedge b_i \neq 0$ so that

$$\bigvee C_1 \wedge \bigvee C_2 \in A_1^* \wedge A_2^*.$$

We have, however, obviously $\bigvee C \leq \bigvee C_i$ and hence $\bigvee C \leq \bigvee C_1 \wedge \wedge \bigvee C_2$. □

3.5. Theorem: If $\mathcal{U}$ is a u-basis, $\widetilde{\mathcal{U}}$ is a uniformity. If $\mathcal{U}$ is a wu-basis, $\widetilde{\mathcal{U}}$ is a weak uniformity.

Proof: The conditions (i) and (ii) are obviously satisfied. (iii*) resp. (iii2): Let $A_1 \wedge \dots \wedge A_k \prec A$, $A_i \in \mathcal{U}$. Choose $B_i \in \mathcal{U}$ such that $B_i^* \prec A_i$ resp. $B_i^{(2)} \prec A_i$. Put $B = B_1 \wedge \dots \wedge B_k$. By 3.4 we have $B^* \prec A$ resp. $B^{(2)} \prec A$. □

3.6. A locale L is said to be uniformizable (resp. weakly uniformizable) if there is a uniformity (resp. weak uniformity) on L such that $L_{\mathcal{U}} = L$ (cf. [5]).

3.7. Remark: According to 2.5 and 3.6 we see that for uniformizability (resp. weak uniformizability) it suffices to have a u-basis (resp. a wu-basis) $\mathcal{U}$ on L such that $L_{\mathcal{U}} = L$.

3.8. For a complemented $x \in L$ and an $A \in \mathcal{C}(L)$ put

$$A \circ x = \{a \wedge x \mid a \in A\} \cup \{a \wedge \bar{x} \mid a \in A\}.$$

If $\mathcal{U} \subset \mathcal{C}(L)$, define $\mathcal{U}^{\circ}$ as

$$\{A \circ x \mid A \in \mathcal{U}\ ,\ x \text{ complemented}\}.$$

3.9. Proposition: If $\mathcal{U}$ is a u-basis (resp. wu-basis), $\mathcal{U}^{\circ}$ is a u-basis (resp. wu-basis).

Proof: It suffices to realize that if $B^{*} \prec A$ resp. $B^{(2)} \prec\prec A$, we have also $(B \circ x)^{*} \prec A \circ x$ resp. $(B \circ x)^{(2)} \prec A \circ x$. □

4. Diameter functions and separation

4.1. As usual, $\mathbb{R}_+$ is the set of non-negative real numbers. A mapping

$$d: L \longrightarrow \mathbb{R}_+$$

is said to be a weak diameter on L if it satisfies the following three conditions:

(I) d is non-decreasing and $d(0) = 0$,

(II) for each $\varepsilon > 0$, $\{a \mid d(a) < \varepsilon\}$ is a cover,

(W) if $d(a), d(b) \leq \alpha$ and $a \wedge b \neq 0$ then $d(a \vee b) \leq 2\alpha$.

A mapping $d: L \longrightarrow \mathbb{R}_+$ is said to be a metric diameter if it satisfies (I),(II),

(A) if $a \wedge b \neq 0$ then $d(a \vee b) \leq d(a) + d(b)$, and

(M) for every $a \neq 0$ and each $\varepsilon > 0$ there are x, y such that $d(x), d(y) \leq \varepsilon$, $x \wedge a \neq 0 \neq y \wedge a$ and $d(x \vee y) > d(a) - \varepsilon$.

4.2. Remark: The role of the diameters in general locales is to simulate the distance functions in the spatial ones. In our context the two definitions are, roughly speaking, the weakest and the strongest among the suitable ones (in the spatial case, the metric diameters are already exactly those given by $d(a) = \sup\{\rho(x,y) \mid x,y \in a\}$ with ρ a pseudometric). In the literature one encounters diameter functions defined for other purposes and hence subjected to other kind of conditions (see, e.g., [4]).

4.3. <u>Construction</u>: Let D be a dense subset of the unit interval I, let it contain 0 and 1. Let us have a family $(u_\alpha \mid \alpha \in D)$ of elements of L such that

$$u_0 = v, \quad \beta < 1 \Rightarrow u_\beta \leq u, \quad u_1 = e \quad \text{and}$$

$$\alpha < \beta \Rightarrow u_\alpha \lhd u_\beta .$$

For $x \in L$ put

$$d_+(x) = \inf\{\alpha \mid x \leq u_\alpha\},$$

$$d_-(x) = \sup\{\alpha \mid x \wedge u_\alpha = 0\}.$$

Finally define

$$d(0) = 0, \text{ and}$$

$$d(x) = d_+(x) - d_-(x) \text{ for } x \neq 0.$$

4.4. <u>Lemma</u>: If $x \neq 0$, we have $d_-(x) \leq d_+(d)$.

<u>Proof</u>: Let $d_+(x) < d_-(x)$. Then there is an $\alpha \in D$ such that $d_+(x) < \alpha < d_-(x)$ and such that $x \leq u_\alpha$. Since $\alpha < d_-(x)$, we have a β with $\alpha < \beta < d_-(x)$ such that $x \wedge u_\beta = 0$. Thus, $x = x \wedge u_\alpha \leq x \wedge u_\beta = 0$. □

4.5. <u>Lemma:</u> $d_+(a \vee b) = \max(d_+(a), d_+(b))$,

$d_-(a \vee b) = \min(d_-(a), d_-(b))$.

<u>Proof</u>: Obviously $\alpha = \max(d_+(a), d_+(b)) \leq d_+(a \vee b)$. Now let $\beta \in D$ be such that $\beta > \alpha$. Then $a \leq u_\beta$, $b \leq u_\beta$ and hence $a \vee b \leq u_\beta$. Hence, $d_+(a \vee b) \leq \beta$.

Obviously $\alpha = \min(d_-(a), d_-(b)) \geq d_-(a \vee b)$. Let $\beta \in D$ be such that $\beta < \alpha$. Then $a \wedge u_\beta = 0$ and $b \wedge u_\beta = 0$ and hence $(a \vee b) \wedge u_\beta = 0$ so that $d_-(a \vee b) \geq \beta$. $\square$

4.6. **Theorem:** The function d is a metric diameter.

Proof: (I) is obvious.

(II): Let $\varepsilon > 0$ be given. Choose $\alpha_i \in D$ so that $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = 1$ and $\alpha_{i+2} - \alpha_i < \varepsilon$ for all i. Since $u_{\alpha_i} \lhd u_{\alpha_{i+1}}$, we have $z_i \in L$ such that

$$u_{\alpha_i} \wedge z_i = 0 \text{ and } u_{\alpha_{i+1}} \vee z_i = e.$$

Put $b_0 = v = u_0$ and, for $i > 0$, $b_i = u_{\alpha_{i+1}} \wedge z_{i-1}$. We have

$$\bigvee_{i=0}^{k-1} b_i = e.$$

(Indeed, let us prove by induction that $\bigvee_{i=0}^{j-1} b_i = u_{\alpha_j}$: This is obvious for j=1. Now, $\bigvee_{i=0}^{j} b_i = \bigvee_{i=0}^{j-1} b_i \vee b_j = u_{\alpha_j} \vee (u_{\alpha_{j+1}} \wedge z_{j-1}) =$
$= (u_{\alpha_{j+1}} \wedge u_{\alpha_j}) \vee (u_{\alpha_{j+1}} \wedge z_{j-1}) = u_{\alpha_{j+1}} \wedge (u_{\alpha_j} \vee z_{j-1}) = u_{\alpha_{j+1}}$.)

Obviously, $d_+(v) = 0$ and hence $d(b_0) = d(v) = 0$. Further,

$b_i \wedge u_{\alpha_{i-1}} \leq z_{i-1} \wedge u_{\alpha_{i-1}} = 0$ so that $d_-(b_i) \geq \alpha_{i-1}$.

$b_i \leq u_{\alpha_{i+1}}$ so that $d_+(b_i) \leq \alpha_{i+1}$

and hence $d(b_i) \leq \alpha_{i+1} - \alpha_{i-1} < \varepsilon$.

(A): If $a \wedge b \neq 0$ we have, by 4.4,

(1) $d_-(a) \leq d_-(a \wedge b) \leq d_+(a \wedge b) \leq d_+(b)$ and similarly $d_-(b) \leq d_+(a)$.

Using 4.5 we obtain

$$d(a) + d(b) = d_+(a)-d_-(a)+d_+(b)-d_-(b) = \max(d_+(a),d_+(b))-$$
$$- \min(d_-(a),d_-(b)) + \min(d_+(a),d_+(b)) - \max(d_-(a),d_-(b)) =$$
$$= d(a \vee b) + (d_+(x)-d_-(y))$$

and the second summand is non-negative by (1).

(M) : We can assume $d_+(a) > 0$ and $d_-(a) < 1$ since otherwise we could put $x=y=a$.

Choose $\alpha_1, \alpha_2, \alpha \in D$ so that

$$d_+(a) - \frac{1}{2}\varepsilon < \alpha_1 < \alpha_2 < d_+(a) \leq \alpha < d_+(a) + \frac{1}{2}\varepsilon$$

and $a \leq u_\alpha$.

Since $u_{\alpha_1} \lhd u_{\alpha_2}$ there is a z such that $u_{\alpha_1} \wedge z = 0$ and $u_{\alpha_2} \vee z = e$.

Put $x = u_\alpha \wedge z$.

If we had $a \wedge z = 0$ we would have $a = a \wedge (u_{\alpha_2} \vee z) = a \wedge u_{\alpha_2}$, i.e. $a \leq u_{\alpha_2}$ contradicting the choice of α_2. Thus, $a \wedge z \neq 0$ and hence

$$a \wedge x = (a \wedge u_\alpha) \wedge z = a \wedge z \neq 0.$$

Since $x \wedge u_{\alpha_1} = 0$ and $x \leq u_\alpha$, we have

$$\alpha_1 \leq d_-(x) \leq d_+(x) \leq \alpha$$

and hence

$$d(x) \leq \alpha - \alpha_1 < \varepsilon .$$

Now choose a $\beta \in D$ such that

$$d_-(a) < \beta < d_-(a) + \frac{1}{2}\varepsilon .$$

If $d_-(a) > 0$, choose, moreover, $\beta_1, \beta_2 \in D$ such that

$$d_-(a) - \frac{1}{2}\varepsilon < \beta_1 < \beta_2 < d_-(a)$$

and a $w \in L$ such that $w \wedge u_{\beta_1} = 0$ and $w \vee u_{\beta_2} = e$. Put

$$y = \begin{cases} u_\beta & \text{if } d_-(a) = 0, \\ u_\beta \wedge w & \text{otherwise.} \end{cases}$$

In the first case we have obviously $y \wedge a \neq 0$. In the second one, $w \wedge a = (w \wedge a) \vee (u_\beta \wedge a) = (w \vee u_\beta) \wedge a = a$ so that also here

$$y \wedge a = u_\beta \wedge w \wedge a = u_\beta \wedge a \neq 0.$$

In the first case obviously

$$d_-(y) = 0, \; d_+(y) \leq \beta ,$$

in the second one we have $y \wedge u_{\beta_1} = 0$ and $y \leq u_\beta$ so that

$$\beta_1 \leq d_-(y) \leq d_+(y) \leq \beta$$

and hence

$$d(y) \leq \beta - \beta_1 < \varepsilon .$$

Finally, by 4.5,

$$d(x \vee y) = \max(d_+(x), d_+(y)) - \min(d_-(x), d_-(y)) \geq \alpha_1 - \beta > d_+(a) - \frac{1}{2}\varepsilon - (d_-(a) + \frac{1}{2}\varepsilon) = d_+(a) - \varepsilon . \quad \square$$

4.7. We say that a function $d: L \longrightarrow \mathbb{R}_+$ separates v from u if

(a) $d(v) = 0$ and

(b) whenever $x \wedge v \neq 0$ and $d(x) < 1$ then $x \leq u$.

4.8. Proposition: If $v \lhd\!\lhd u$ in L there exists a metric diameter separating v from u.

Proof: Consider a system x_{ij} from the definition of $\lhd\!\lhd$ and put

$$u_{j,2^{-i}} = x_{ij} \text{ with the exception of } u_1 = e.$$

The function d from 4.3 separates v from u. $\square$

4.9. Proposition: Let $\mathcal{U}$ be a wu-basis. If $x \overset{\mathcal{U}}{\lhd} y$, there is a z such that $x \overset{\mathcal{U}}{\lhd} z \overset{\mathcal{U}}{\lhd} y$.

Proof: Take an $A \in \mathcal{U}$ such that $Ax \leq y$ and choose a $B \in \mathcal{U}$ such that $B^{(2)} \prec A$. Put $z = Bx$. Thus, $x \overset{\mathcal{U}}{\lhd} z$. Now $Bz = B(Bx) \leq B^{(2)}x \leq Ax$ by 1.7 so that also $z \overset{\mathcal{U}}{\lhd} y$. $\square$

4.10. Propositions 4.9, 2.2 and 1.5 immediately yield

Corollary: If $\mathcal{U}$ is a wu-basis then

$$x \overset{\mathcal{U}}{\lhd} y \Rightarrow x \lhd\!\lhd y.$$

4.11. Theorem: The following statements are equivalent:

(i) v is separated from u by a metric diameter,

(ii) v is separated from u by a weak diameter,

(iii) $v \overset{\mathcal{U}}{\lhd} u$ for some wu-basis $\mathcal{U}$,

(iv) $v \lhd\!\lhd u$.

Proof: Trivially, (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Let d be a weak diameter function separating v from u. Put $\mathcal{U} = \{\{a \mid d(a) < \varepsilon\} \mid \varepsilon > 0$. It is a wu-basis since $\{a \mid d(a) < \frac{1}{2}\varepsilon\}^{(2)} \subset \{a \mid d(a) < \varepsilon\}$. Consider the $A = \{a \mid d(a) < 1\}$. For $a \in A$ and $a \wedge v \neq 0$ we have $a < u$ so that $Av \leq u$ and hence $v \overset{\mathcal{U}}{\lhd} u$.

(iii) $\Rightarrow$ (iv) is contained in 4.10 and (iv) $\Rightarrow$ (i) in 4.8.□

5. Characterizations of complete regularity

5.1. **Lemma:** A metric diameter function d has the following property:

If $S \subset L$ is such that $a, b \in S \Rightarrow a \wedge b \neq 0$, then

$$d(\bigvee S) \leq 2 \sup \{d(a) \mid a \in S\}.$$

Proof: Let $d(\bigvee S) > 2 \sup d(a) + 3\varepsilon$. Consider some x, y such that $x \wedge \bigvee S \neq 0 \neq y \wedge \bigvee S$, $d(x), d(y) < \varepsilon$ and

$$d(x \vee y) > d(\bigvee S) - \varepsilon .$$

Choose $a, b \in S$ so that $a \wedge x \neq 0 \neq b \wedge y$. Thus,

(2) $d(x \vee y) > d(a) + d(b) + 2\varepsilon$.

On the other hand,

(3) $d(x \vee y) \leq d(a \vee b \vee x \vee y) \leq d(a \vee b \vee x) + \varepsilon \leq d(a \vee b) + 2\varepsilon$.

From (2) and (3) we obtain

$$d(a \vee b) > d(a) + d(b)$$

and hence $a \wedge b = 0$. □

5.2. **Proposition:** Put

$\mathcal{U} = \{\{a \mid d(a) < \varepsilon\} \mid \varepsilon > 0$, d a metric diameter on L$\}$.

Then $\mathcal{U}$ is a u-basis and

$$v \lhd\!\lhd u \text{ iff } v \overset{\mathcal{U}}{\lhd} u.$$

Proof: Take an $A = \{a \mid d(a) < \varepsilon\}$ and put $B = \{a \mid d(a) < \frac{1}{2}\varepsilon\}$. By 5.1, $B^* \subset A$. Hence $\mathcal{U}$ is a u-basis and, by 4.10,

$$v \overset{\mathcal{U}}{\lhd} u \Rightarrow v \lhd\!\lhd u.$$

Now, let $v \lhd\!\lhd u$. By 4.8 there is a metric diameter d separating v from u. Take $A = \{a \mid d(a) < 1\}$. If $a \wedge v \neq 0$ and $d(a) < 1$ we have $a \leq u$ so that $Av \leq u$. Thus, $v \overset{\mathcal{U}}{\lhd} u$. □

5.3. Theorem: Let L be a locale. Then the following statements are equivalent:

(i) L is completely regular,

(ii) each $x \in L$ is covered by the elements $y \leq x$ separated from x by weak diameters,

(iii) each $x \in L$ is covered by the elements $y \leq x$ separated from x by metric diameters,

(iv) there is a u-basis $\mathcal{U}$ such that $L_{\mathcal{U}} = L$,

(v) L is uniformizable,

(vi) there is a wu-basis $\mathcal{U}$ such that $L_{\mathcal{U}} = L$,

(vii) L is weakly uniformizable.

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) by 4.11.

(iii) $\Rightarrow$ (iv): Take the u-basis $\mathcal{U}$ from 5.2.

(iv) $\Leftrightarrow$ (v) by 3.5 and 2.5.

(iv) $\Rightarrow$ (vi) trivially.

(vi) $\Leftrightarrow$ (vii) by 3.5 and 2.5.

(vii) $\Rightarrow$ (i): Let $u \in L$. We have a uniformity $\mathcal{U}$ such that $L_{\mathcal{U}} = L$. Thus, for an arbitrary $u \in L$, by 4.10,

$$u = \bigvee\{v \mid v \overset{\mathcal{U}}{\lhd} u\} \leq \bigvee\{v \mid v \lhd\!\lhd u\} \leq u. \quad \square$$

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Matematicko-fyzikální fakulta, Univerzita Karlova, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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