Commentationes Mathematicae Universitatis Carolinae

Luc Vrancken-Mawet

The lattice of R-subalgebras of a bounded distributive lattice

Commentationes Mathematicae Universitatis Carolinae, Vol. 25 (1984), No. 1, 1--17

Persistent URL: `http://dml.cz/dmlcz/106275`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
25,1 (1984)

THE LATTICE OF R-SUBALGEBRAS OF A BOUNDED DISTRIBUTIVE LATTICE

L. VRANCKEN-MAWET

Abstract: Using Priestley's duality, we investigate the lattice $S_R(L)$ of the $\{0,1\}$-sublattices of a given bounded distributive lattice L which are closed under relative complementation. We characterize those bounded distributive lattices L such that $S_R(L)$ is semimodular, modular, distributive or Boolean.

Key words: Distributive lattice - Relative complementation - Priestley's duality - Congruences on partially ordered spaces.

Classification: 06D05.

Introduction. In his study on Boolean lattices R-generated by distributive lattices, Grätzer considers particular $\{0,1\}$-sublattices of a given bounded distributive lattice, namely those which are closed under relative complementation. The purpose of this paper is to study these sublattices, which we call R-subalgebras. It turns out that Priestley's duality is a well-adapted tool to achieve this aim.

In Section 1, we introduce the concept of congruence on a Priestley space, which is dual to that of R-subalgebra; the lattice of all R-subalgebras of a bounded distributive lattice is dually isomorphic to the lattice Con(X) of all congruences on the dual X of L.

Section 2 is devoted to the study of $\mathrm{Con}(X)$. In particular we characterize those Priestley spaces whose congruence lattice is semi-modular, modular or distributive respectively. We translate these results in terms of R-subalgebras in Section 3.

We adopt standard set theoretic notations. Let us however recall some of them. For a set X, we denote by $|X|$ its cardinal and by Eq(X) its equivalence lattice. If $\Theta \in \mathrm{Eq}(x)$, $x \in X$ and $E \subseteq X$, we write x^{Θ} for the Θ-class of x and $E^{\Theta} = \bigcup\{x^{\theta} \mid x \in E\}$; E is Θ-saturated if $E^{\Theta} = E$. If $X = (X, \leq)$ is a poset, $p \prec q$ means that q covers p and $p \parallel q$ means that p and q are not comparable. We say that $E \subseteq X$ is convex if $x \leq z \leq y$ and $x, y \in E$ imply that $z \in E$. An order connected component (o.c.c.) of X is a subset E of X which is minimal with respect to the property of being both increasing and decreasing. Finally, the n-element chain is denoted by $\underline{\underline{n}}$.

1. A Duality for R-subalgebras of a bounded distributive lattice

1.1. Definition. Let $\mathbb{D}$ denote the category of bounded distributive lattices and $\{0,1\}$-homomorphisms. If $L \in \mathbb{D}$ and A is a $\{0,1\}$-sublattice of L, then A is said to be an R-subalgebra of L if it is closed under relative complementation (when the latter is defined). Other ways of defining R-subalgebras are given in [4]. The set of all R-subalgebras of a lattice L in $\mathbb{D}$, ordered by inclusion, is an algebraic lattice, the study of which is the purpose of the present paper. We denote it by $\mathcal{S}_R(L)$.

In [6], H.A. Priestley establishes a duality between $\mathbb{D}$ and the category $\mathbb{P}$ of Priestley (i.e. compact totally order disconnected) spaces and order-preserving continuous maps. The functors

$\mathcal{P}: \mathbb{D} \to \mathbb{P}$ and $\mathcal{O}: \mathbb{P} \to \mathbb{D}$ which realize the duality are described as follows: if $L \in \mathbb{D}$, $\mathcal{P}(L)$ is the ordered set of all prime ideals of L, suitably topologized, whereas, for $X \in \mathbb{P}$, $\mathcal{O}(X)$ is the lattice of all clopen order-ideals of X. If f is a morphism in $\mathbb{D}$ (resp. $\mathbb{P}$), its dual map is defined by $\mathcal{P}(f) = f^{-1}$ (resp. $\mathcal{O}(f) = f^{-1}$). We refer to [7] for the fundamental facts on Priestley's duality but we usually follow the notations of [2].

1.2. <u>Definition</u>. Let $X \in \mathbb{P}$. Suppose (X', τ') is a topological space and π is an onto continuous map $X \to X'$. An order $\leq'$ on X' is said to be <u>compatible</u> with τ' and π if $X' = (X', \tau', \leq')$ is in $\mathbb{P}$ and if π is order-preserving (hence a morphism in $\mathbb{P}$).

1.3. <u>Lemma</u>. <u>Let</u> $L \in \mathbb{D}$ <u>and</u> $A \in \mathcal{S}_R(L)$. <u>Denote by</u> $X = (X, \tau, \leq)$ <u>the dual of</u> A <u>and by</u> π <u>the dual map of the inclusion map</u> $A \to L$. <u>Then</u> $\leq$ <u>is the least order on</u> X <u>which is compatible with</u> τ <u>and</u> π.

Proof. Suppose $\leq'$ is an order on X which is compatible with τ and π. Denote by A' the dual lattice of $(X, \tau, \leq')$, considered as a $\{0,1\}$-sublattice of L. By [7], both A and A' R-generate the Boolean algebra whose dual is (X, τ). Since A is closed under relative complementation, it follows from [3], p. 89, that A contains A'. In other words, $\mathcal{O}(X, \tau, \leq') \supseteq \mathcal{O}((X, \tau, \leq')$, which implies that $\leq$ is contained in $\leq'$.

By 1.3, the dual of an R-subalgebra A of a lattice $L \in \mathbb{D}$ is determined by the canonical epimorphism $\pi : \mathcal{P}(L) \to \mathcal{P}(A)$. Therefore, the general concept of separating set [1] may be advantageously replaced by the simpler one of congruence.

1.4. <u>Definition</u>. Let $X \in \mathbb{P}$ and $\Theta \in Eq(X)$. Then Θ deserves the name of <u>congruence</u> if there exists a topology τ' and an order $\leq'$ on X such that

i) the natural map $\pi : X \to X/\Theta$ is continuous, and

ii) $\leq'$ is compatible with τ' and π.

We denote by Con(X) the set of all congruences on X. Obviously, ω (the identity relation) and ι (the universal relation) are always congruences. Since the intersection of any subfamily of Con(X) is again a congruence, Con(X) is a complete lattice, but it need not be a sublattice of Eq(X). It is also worth to note that, if $\Theta \in Con(X)$, the topology τ' of the definition is necessarily the quotient topology, which we shall denote by τ_Θ. Moreover, among all orders $\leq'$ compatible with τ' and π, there always exists a least one, that we shall denote by $\leq_\Theta$ (it suffices to consider the R-subalgebra of $\mathcal{O}(X)$ generated by $\mathcal{O}((X/\Theta, \tau', \leq'))$ and to apply 1.3). We shall now describe $\leq_\Theta$.

1.5. <u>Notation</u>. Let $X \in \mathbb{P}$ and $\Theta \in Eq(X)$. We denote by $\mathcal{O}(X, \Theta)$ the set of all clopen order-ideals of X which are Θ-saturated and we define on X/Θ a quasi-order $\leq_\Theta$ as follows: $x^\Theta \leq_\Theta y^\Theta$ if, for all $U \in \mathcal{O}(X, \Theta)$, $U \ni y$ implies $U \ni x$.

1.6. <u>Lemma</u>. <u>Let</u> $X \in \mathbb{P}$ <u>and</u> $\Theta \in Eq(X)$. <u>The following assertions are equivalent</u>:

(i) $\Theta \in Con(X)$;

(ii) $\leq_\Theta$ <u>is antisymmetric</u>;

(iii) $\leq_\Theta$ <u>is the least order compatible with</u> τ_Θ <u>and</u> π;

(iv) if $x \, \Theta \, y$ <u>fails, then</u> x <u>and</u> y <u>can be separated by some member of</u> $\mathcal{O}(X, \Theta)$.

Proof. (i) $\Rightarrow$ (ii). Let $\leq'$ be an order on X/Θ which is compatible with τ_Θ and π. If $x^\Theta \not\leq' y^\Theta$, there exists $V \in \mathcal{O}(X/\Theta, \tau_\Theta, \leq'))$ with $V \ni v^\Theta$ and $V \not\ni x^\Theta$. If $U = \pi^{-1}(V)$, then

$U \in \mathcal{O}(X,\Theta)$, $U \ni y$ and $U \not\ni x$, which shows that $x^\Theta \not\leq_\Theta y^\Theta$. Consequently, $\leq_\Theta$ is contained in $\leq'$ and therefore is antisymmetric.

(ii) $\Rightarrow$ (iii). It is clear that $\leq_\Theta$ is an order on X/Θ which is compatible with τ_Θ and π. The proof of (i) $\Rightarrow$ (ii) shows that it is the least one.

Finally, (iii) $\Rightarrow$ (i) and (ii) $\Leftarrow$ (iv) are trivial.

1.7. <u>Theorem</u>. <u>If</u> $L \in \mathbb{D}$ <u>and if</u> X <u>is its dual space, then there exists a canonical dual isomorphism</u> $\mathcal{S}_R(L) \to \mathrm{Con}(X)$. <u>In particular</u>, $\mathrm{Con}(X)$ <u>is dually algebraic</u>.

Proof. We may assume that $L = \mathcal{O}(X)$. Let us define $h:\mathrm{Con}(X) \to \mathcal{S}_R(L)$ by $h(\Theta) = \mathcal{O}(X,\Theta)$. Clearly, h is one-to-one and order-preserving (see 1.6(iv)).

Let now $A \in \mathcal{S}_R(L)$. Define Θ to be the kernel of $\mathcal{P}(\mathrm{id})$, where id is the inclusion map $A \to L$. In other words, $x \,\Theta\, y$ if and only if $U \ni x \Leftarrow U \ni y$ for all $U \in A$. We wish to show that $h(\Theta) = A$. It is clear that $A \subseteq h(\Theta)$. Let $U \in h(\Theta)$. For each $x \in U$ and $y \notin U$, there exists by 1.6(iv) either $U_{xy} \in A$ such that $U_{xy} \ni x$ and $U_{xy} \not\ni y$, or $V_{xy} \in A$ such that $V_{xy} \ni y$ and $V_{xy} \not\ni x$. If y is fixed, the sets U_{xy} and $-V_{xy} (x \in U)$ form an open covering of U which, by compactness, has a finite subcover. This gives rise to elements $U_y \in A$, $V_y \in A$ such that $U \subseteq U_y \cup -V_y \subseteq -\{y\}$. Hence, $-U = \cup\{V_y \cap - U_y | y \notin U\}$. Again by compactness, it follows that U is the intersection of finitely many $U_y \cup -V_y$. Therefore, U is in the Boolean algebra R-generated by A. Since $U \in \mathcal{O}(X)$, $U \in A$ by [3].

The map $A \mapsto \Theta$ is obviously order-preserving and the proof is over.

2. The congruence lattice of a Priestley space

2.1. <u>Notation</u>. Let $X \in \mathbb{P}$. If $E \subseteq X$, we denote by $\Theta(E)$ the equivalence on X generated by $E \times E$ and by $\Phi(E)$ the equivalence $\Theta(E) \cup \Theta(-E)$. If $E = \{p,q\}$ we write $\Theta(p,q)$ instead of $\Theta(\{p,q\})$.

2.2. <u>Lemma. If</u> $X \in \mathbb{P}$ <u>and</u> $E \subseteq X$, <u>then</u> $\Theta(E) \in \mathrm{Con}(X)$ <u>if and only if</u> E <u>is closed and convex</u>.

Proof. Any congruence class is closed and convex. Hence the condition is necessary. Suppose now that $(x,y) \notin \Theta(E)$ where E is closed and convex. To find $U \in \mathcal{O}(X,\theta)$ separating x and y, it suffices to distinguish the possible positions of x and y relatively to E.

2.3. <u>Theorem. If</u> $X \in \mathbb{P}$, <u>then</u> Con(X) <u>is atomistic. Its atoms are the equivalences</u> $\Theta(p,q)$ <u>where</u> $p \parallel q$ <u>or</u> $p \prec q$.

Proof. Let us first show that any closed and convex subset E of X which is not reduced to a singleton contains a pair $\{p,q\}$ where $p \parallel q$ or $p \prec q$. This is clear if E is not a chain. If E is a chain, it is a Boolean chain, in which jumps $p \prec q$ exist in abondance ([5]).

To prove atomisticity, note that one has $\theta = \bigvee\{\Theta(E) \mid E$ is a θ-class$\}$ for each $\theta \in \mathrm{Con}(X)$. Hence it remains to prove that, if E is closed and convex, $\Theta(E) = \bigvee\{\Theta(p,q) \in \mathrm{Con}(X) \mid p,q \in E\}$. Let $\Phi \in \mathrm{Con}(X)$ be such that $\Phi \geq \Theta(p,q)$ for all $p,q \in E$ with $\Theta(p,q) \in \mathrm{Con}(X)$. If $\not\geq \Theta(E)$, there exist x, y in E for which $x \Phi y$ fails. Consequently, x and y can be separated by some $U \in \mathcal{U}(X,\Phi)$. If p is maximal in $U \cap E$ and q minimal in $-U \cap E$, then $p,q \in E$ and $\Theta(p,q) \in \mathrm{Con}(X)$ whence $\Theta(p,q) \leq \Phi$, a contradiction.

2.4. Theorem. If $X \in \mathbb{P}$, then $\mathrm{Con}(X)$ is dually atomistic. Its dual atoms are the equivalences $\Phi(U)$, where $U \in \mathcal{O}(X) - \{\emptyset, X\}$.

Proof. It is clear that $\Phi(U) \in \mathrm{Con}(X)$ if and only if $U \in \mathcal{O}(X)$ (use 1.6(iv). It suffices now to show that, if $\Theta \in \mathrm{Con}(X)$, then $\Theta = \bigwedge\{\Phi(U) \mid U \in \mathcal{O}(X,\Theta)\}$. If $U \in \mathcal{O}(X,\Theta)$, then $\Theta \leq \Phi(U)$. Conversely, if $\Phi \leq \Phi(U)$ for each $U \in \mathcal{O}(X,\Theta)$, and if $x \Phi y$, then $x \Theta y$ by 1.6(iv). Hence $\Phi \leq \Theta$, which completes the proof.

The following result shows that the semimodularity of $\mathrm{Con}(X)$ depends only on the order on X and not on its topology. (A lattice L is called semimodular if and only if it satisfies the following condition for all $a,b \in L$: $a \wedge b \prec a \Rightarrow b \prec a \vee b$.)

2.5. Theorem. If $X \in \mathbb{P}$, then $\mathrm{Con}(X)$ is semimodular if and only if either

(i) X is order-isomorphic to an ordinal sum $A \oplus C \oplus A'$, where A and A' are (possibly empty) antichains and C is a bounded chain, or

ii) $X = \mathrm{Min}X \cup \mathrm{Max}X$ and either $|X - \mathrm{Min}X| \leq 1$ or $|X - \mathrm{Max}X| \leq 1$.

Proof. Suppose first that $\mathrm{Con}(X)$ is semimodular. We proceed in four steps.

a) There cannot exist in X elements x,y,z,t with $x<y$, $z<t$, $x \parallel t$ and $y \parallel z$ (otherwise $\Theta(x,t) > \Theta(y,z) \wedge \Theta(x,t) = \omega$ and $\Theta(y,z) \vee \Theta(x,t) > \Theta([x,y] \cup \{z\}) > \Theta(y,z)$). In particular, there exists at most one o.c.c. which is not reduced to a singleton. Let us denote it by X_0.

b) If $p,q \in X_0$ and $p \parallel q$, then for each $x \in X_0$, $x>p$ (resp. $x<p$) implies $x>q$ (resp. $x<q$). Suppose on the contrary that,

for some $r \in X_0$, one has $r > p$ and $r \not> q$ (which implies $r \parallel q$). We distinguish three possibilities.

If $[p) \wedge [q) \neq \emptyset$, it contains some minimal element t. Necessarily, either $r \parallel t$ or $r < t$. In the first case, we have $\Theta([p,t]) \vee \Theta(q,r) > \Theta([p,t] \cup [q,t]) > \Theta([p,t])$. In the second case, we have $\Theta([q,t]) \vee \Theta(p,q) > \Theta([q,t] \cup [r,t]) > \Theta([q,t])$. Both inequalities contradict the fact that Con(X) is semimodular.

If $[p) \cap [q) = \emptyset$ and $(r] \cap (q] \neq \emptyset$, choose some maximal element t in $(r] \cap (q]$. If $t < p$, then $\Theta([t,q]) \vee \Theta(q,r) > \Theta([t,q] \cup [t,p]) >$ $> \Theta([t,q])$. If $t \parallel p$, then $\Theta([t,r]) \vee \Theta(p,q) > \Theta([t,r] \cup [p,r]) >$ $> \Theta([t,r])$. Here again this is not possible because of the semimodularity of Con(X).

It remains to consider the case where $[p) \cap [q) = \emptyset$ and $(r] \cap (q] = \emptyset$. Since $q \in X_0$, there exists in $\mathrm{Min}X \cup \mathrm{Max}X$ some ele-- ment $t \neq q$ which is comparable with q, say $q < t$. The existence of the elements p,r,q,t contradicts a).

c) Suppose now $X \neq \mathrm{Min}X \cup \mathrm{Max}X$. The only o.c.c. of X are $\emptyset$ and X itself. Otherwise choose $x < y < z$ and some t not belonging to the same o.c.c. as x. Then $\Theta(x,t) \vee \Theta(z,t) > \Theta([x,y)] \cup \{t\}) >$ $> \Theta(x,t)$, which is not possible.

Moreover, $C = X-(\mathrm{Min}X \cup \mathrm{Max}X)$ is a chain. Indeed if p, q are non comparable elements of C, let t (resp. u) be minimal (resp. maximal) in $[p) \cap [q)$ (resp. $(p] \cap (q]$) (these sets are not empty by b)). Then $\Theta(p,t) \vee \Theta(p,u) > \Theta(\{p,q,t\}) > \Theta(p,t)$ and this again is not possible.

As a consequence, X is of the type i) as required.

d) If $X = \mathrm{Min}X \cup \mathrm{Max}X$, we have to prove that $|X-\mathrm{Min}X| \leq 1$ or $|X-\mathrm{Max}X| \leq 1$. Suppose on the contrary that there exist distinct elements x,y in $\mathrm{Min}X-\mathrm{Max}X$ and z,t in $\mathrm{Max}X-\mathrm{Min}X$. By b), we may assume that $x < z$, $x < t$, $y < z$ and $y < t$. Then $\Theta(x,z) \vee \Theta(y,t) >$

$> \Theta\{(x,yz)\} > \Theta(x,z)$ which is absurd.

Assume now that X satisfies either i) or ii). We have to prove that if $\Phi, \Theta \in \mathrm{Con}(X)$, then $\Phi\wedge\Theta \prec \Theta$ implies $\Phi\wedge\Theta \prec \Theta$. It is not difficult to show that the third isomorphism theorem holds in $\mathbb{P}$ and we may assume $\Phi\wedge\Theta = \omega$. We shall prove the following stronger result: if $\Phi \in \mathrm{Con}(X)$ and if Θ is an atom in $\mathrm{Con}(X)$, then the supremum $\Phi\wedge_q\Theta$ of Φ and Θ in $\mathrm{Eq}(X)$ is a congruence. To achieve this result, let us suppose $\Theta = \Theta(p,q)$ where $p \parallel q$ or $p \prec q$. We first show that it is not possible to have $(*)$ $p^{\Phi} < y^{\Phi} < q^{\Phi}$ for some $y \in X$ (here, $<$ is written instead of $<_{\Phi}$). The proof is carried on ab absurdo.

a) Suppose first that X satisfies i). If $p \parallel q$, then $\{p,q\} \subseteq \mathrm{Min}X$ or $\{p,q\} \subseteq \mathrm{Max}X$, say $\{p,q\} \subseteq \mathrm{Min}X$. It results from $(*)$ that $p^{\Phi} \cup y^{\Phi} \subseteq \mathrm{Min}X$. Let t be the least element of X-MinX. Then $y^{\Phi} < t^{\Phi}$, and there exists $V \in \mathcal{O}(X,\Phi)$ such that $V \ni y$ and $V \not\ni t$. Moreover, since $p^{\Phi} < y^{\Phi}$, there exists $W \in \mathcal{O}(X,\Phi)$ such that $W \ni p$ and $W \not\ni y$. If $U = V-W$, then $U \in \mathcal{O}(X,\Phi)$, $U \ni y$ and $U \not\ni p$, which contradicts $p^{\Phi} < y^{\Phi}$.

If $p \prec q$ and $p \in \mathrm{Min}X$, then $q = t$ and we have seen that $p^{\Phi} < y^{\Phi} < t^{\Phi}$ is not possible. Hence we may assume that $p \in X - (\mathrm{Min}X \cup \mathrm{Max}X)$. In the same way, we may assume that $q \in X - (\mathrm{Min}X \cup \mathrm{Max}X)$. Since $X-(\mathrm{Min}X \cup \mathrm{Max}X)$ is a chain, y is comparable with p and q and $(*)$ implies $p<y<q$, which contradicts $p \prec q$.

b) Suppose now that X satisfies (ii). Obviously, $(*)$ prevents X from being an antichain. By (ii), we may assume that $X-\mathrm{Max}X = \{m\}$ for some m. Let us show that $x^{\Phi} < y^{\Phi}$ implies x m (and this contradicts $(*)$). If not, then either $x^{\Phi} < m^{\Phi}$ or $x \nleq m$. The first possibility cannot occur because, if $U \in \mathcal{O}(X)$ and $U \not\ni m$, then $-U \in \mathcal{O}(X)$. Hence there exists $V \in \mathcal{O}(X,\Phi)$ such that $V \ni m$ and $V \not\ni x$. Since $x^{\Phi} \prec y^{\Phi}$, there also exists $W \in \mathcal{O}(X,\Phi)$ such that $W \ni x$ and $W \not\ni y$. If $U = V \cup -W$, then $U \in \mathcal{O}(X,\Phi)$, $U \ni y$ and $U \not\ni x$, which contradicts $x^{\Phi} \prec y^{\Phi}$.

We are now in a position to prove that $\Phi \vee_{eq} \Theta(p,q) \in \in \mathrm{Con}(X)$. Let $\alpha = \Phi \vee_{eq} \Theta(p,q)$ and suppose that $x \alpha y$ fails. To separate x and y by some member of $\mathcal{O}(X,\alpha)$ we have to consider the various positions of x and y relative to p and q. As an example, let us assume $x^{\Phi} = p^{\Phi}$, $y^{\Phi} \neq p^{\Phi}$ and $y^{\Phi} \neq q^{\Phi}$.

If $p^{\Phi} \not\leq y^{\Phi}$ and $q^{\Phi} \not\leq y$, there exists $U \in \mathcal{O}(X,\Phi)$ such that $U \ni y$, $U \not\ni x$ and $\{p,q\} \subseteq -U$, which implies $U \in \mathcal{O}(X,\alpha)$. If $p^{\Phi} < y^{\Phi}$ (same argument if $q^{\Phi} < y^{\Phi}$), then $y^{\Phi} \not\leq p^{\Phi}$ and $y \not\leq q^{\Phi}$ (otherwise $p^{\Phi} < y^{\Phi} < q^{\Phi}$) and we may argue as above.

Theorem 2.5 enables us to characterize those $X \in \mathbb{P}$ for which Con(X) is geometric (i.e. Con(X) is semimodular, complete, atomistic and all atoms of Con(X) are compact).

2.6. <u>Theorem</u>. <u>Let</u> $X \in \mathbb{P}$. <u>Then</u> Con(X) <u>is geometric if and only if it has one of the forms</u> (i) <u>or</u> (ii) <u>of 2.5 and moreover,</u> $\mathrm{Min}X \cup \mathrm{Max}X$ <u>is finite</u>.

Proof. Suppose Con(X) is geometric. By 2.5, it remains to prove that $\mathrm{Min}X \cup \mathrm{Max}X$ is finite. Assume on the contrary that MinX is infinite. If MinX is not closed, let $p \in \mathrm{Min}X$ and let q be the least element of X-MinX. Then $\Theta(p,q)$ is not compact since $\Theta(p,q) \leq \bigvee T$, where $T = \{\Theta(x,y) \mid x,y \in \mathrm{Min}X\}$ whereas $\Theta(p,q) \not\leq \not\leq T'$ for any finite subset T' of T.

If MinX is closed and thus compact, there exists $p \in \mathrm{Min}X$ such that $\{p\}$ is not open. Let q be an element of $\mathrm{Min}X - \{p\}$. If $T = \{\Theta(x,y) \mid x,y \in \mathrm{Min}X - \{p\}\}$, we conclude as above.

Conversely, if X satisfies i) (case ii) is trivial) of 2.5 and $\mathrm{Min}\,X \cup \mathrm{Max}\,X$ is finite, then each atom is compact. To show this, let T be a set of atoms in Con(X) such that $\theta(p,q) \not\leq$ T. We consider two possibilities.

a) If $\{p,q\} \subseteq (t]$ where t is the least element of X-MinX, then $\theta(p,q) \leq \vee\{\theta(x,y) \in T \mid \{x,y\} \subseteq (t]\}$.

b) If $\{p,q\} \subseteq C$ where C is the Boolean chain described in 2.5 i), then necessarily $\theta(p,q) \in T$ because $(p] \in \mathcal{O}(X,\theta)$ for any $\theta \in T - \{\theta(p,q)\}$.

We now study the modularity of Con(X). We first need to observe that $\theta \in$ Con(X) is dually compact if and only if X/θ is finite (use Priestley's duality).

2.7. <u>Theorem</u>. <u>Let</u> $X \in \mathbf{P}$. <u>If</u> X <u>is not the ordinal sum of two 2-element antichains, then the following assertions are equivalent</u>:

(i) Con(X) <u>is modular</u>;

(ii) Con(X) <u>is dually semimodular</u>;

(iii) Con(X) <u>is dually geometric</u>;

(iv) <u>if</u> Φ <u>and</u> ψ <u>are dual atoms of</u> Con(X), then $\Phi \wedge \psi \prec \Phi$ (<u>and</u> $\Phi \wedge \psi \prec \psi$);

(v) <u>either</u> $|X| \leq 3$ <u>or</u> X <u>is isomorphic to a subspace of</u> $A \oplus C \oplus A'$ <u>where</u> A <u>and</u> A' <u>are two-element antichains and</u> C <u>is a bounded chain</u>.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial. Let us prove (iv) $\Rightarrow$ (v). We proceed in three steps, assuming that (iv) holds.

a) As an ordered set, C = X-(MinX $\cup$ MaxX) is a chain. If not, let $x,y \in C$ be such that $x \parallel y$. Choose $x_0 \in$ MinX $\cap (x]$ and $y_0 \in$ MaxX $\cap$ $\cap [y)$. There exists $U,V \in \mathcal{O}(X)$ such that $V \supset \{x_0,y\}$, $-V \supseteq \{x,y_0\}$, $U \supset \{x_0,y\}$, $-V \supseteq \{x,x_0\}$, $U \supseteq \{x,x_0\}$ and $-U \supseteq \{y,y_0\}$. Hence

$\Phi(V) \wedge \Phi(U) < (\Phi(V) \wedge \Phi(U)) \vee \Theta(V) < \Phi(V)$, which is absurd by (iv).

b) If $|X| > 3$, then $|\mathrm{Min}X| \leq 2$ (and in the same way, $|\mathrm{Max}X| \leq 2$. Otherwise, let $x,y,z \in \mathrm{Min}X$ and $t \in X-\{x,y,z\}$. There exists $U,V \in \mathcal{O}(X)$ such that $U \supseteq \{x,y\}$ and $-U \supseteq \{z,t\}$, $V \supseteq \{x,z\}$ and $-V \supseteq \{y,t\}$. A contradiction arises as in a).

c) If $|X| > 3$ and $p \parallel q$, then for each $x \in X$, $x > p$ (resp. $x < p$) implies $x > q$ (resp. $x < q$). Taking into account that any element of X dominates a minimal element and is dominated by a maximal one, we may assume by a) that $\{p,q\} \subseteq \mathrm{Min}X$ or $\{p,q\} \subseteq \mathrm{Max}X$, say $\{p,q\} \subseteq \mathrm{Min}X$. Suppose that $x > p$ and $x \not> q$ (which implies $x \parallel q$). Choose $y \in X-\{p,q,x\}$. If $y \not\leq x$, there exist $U,V \in \mathcal{O}(X)$ such that $U \supseteq \{p,x\}$, $-U \supseteq \{q,y\}$, $V \supseteq \{p,q\}$ and $-V \supseteq \{x,y\}$ and we conclude as in a), The same argument holds if $x \not\leq y$ (interchanging x and y).

Let us now prove that (v) $\Rightarrow$ (i). If $|X| \leq 3$, then clearly Con(X) is modular and we may assume that $X = A \oplus C \oplus A'$ where A and A' are two-element antichains and C is a bounded chain (with least element c and greatest element d). The congruence lattice of C, is dually isomorphic to the lattice of all $\{0,1\}$-sublattices of $\mathcal{O}(C)$, hence it is Boolean. The congruence lattice of $(c]$ (and similarly that of $[d)$) is isomorphic with M_5, the five-elements modular non distributive lattice. It remains to observe that the map $\theta \mapsto (\theta|_{(c]}, \theta|_C, \theta|_{[d)})$ is an isomorphism from Con(X) onto $\mathrm{Con}((c] \times \mathrm{Con}(C) \times \mathrm{Con}([d))$. This is a routine exercise.

<u>Remark</u>. If X is the ordinal sum of two 2-element antichains, it is easy to see that (ii),(iii) and (iv) hold but Con(X) is not modular. Indeed, let x_0, y_0 (resp. x_1, y_1) be the minimal (resp. maximal) elements of X. We have

$$\omega = \theta(x_0,y_1) \wedge \theta(y_0,y_1) \prec \theta(x_0,x_1)$$

while

$$\Theta(y_1,y_0) < \Theta(\{x_0,y_0,y_1\}) < \Theta(x_0,y_1) \vee \Theta(y_0,y_1) = 1.$$

2.8. Theorem. Let $X \in \mathbb{P}$. The following assertions are equivalent:

i) Con(X) is Boolean;

ii) Con(X) is distributive;

iii) Con(X) is uniquely complemented;

iv) either $|X| \leq 2$ or X is a Boolean chain.

Proof. It is clear that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) and it has been said in the proof of 2.7 that (iv) $\Rightarrow$ (i).

Let us prove (ii) $\Rightarrow$ (iv). By 2.7, either $|X| \leq 3$ or X is isomorphic to a subspace of $A \oplus C \oplus A'$ where A, A' are two-element antichains and C is a bounded chain. It is not difficult to check that, if $|X| = 3$, then X must be a three-element chain. We may therefore suppose that $|X| > 3$. In this case, X has a least element (and for a similar reason a greatest one).
Indeed, suppose that $p,q \in \mathrm{Min}X$. Let $r \in X-\{p,q\}$. There exist $U,V \in \mathcal{O}(X)$ such that $U \ni q$, $-U \supseteq \{p,r\}$, $V \ni p$ and $-V \supseteq \{q,r\}$. Then $(\Phi(V) \wedge \Theta(p,q)) \vee (\Phi(U) \wedge \Theta(p,q)) = \omega$ and $(\Phi(V) \vee \Phi(U)) \wedge \Theta(p,q) = \Theta(p,q)$, which is impossible since Con(X) is distributive.

We now prove that (iii) $\Rightarrow$ (iv). First observe that, if $U \in \mathcal{O}(X) - \{\emptyset\}$ and $a \in \mathrm{Max}X-U$, then $\Theta(-U \cup \{a\})$ is a complement of $\Theta(U)$. By (iii), any $U \in \mathcal{O}(X) - \{\emptyset, X\}$ has a greatest element and, for dual reasons, $-U$ has a least element. Now let x, y be non-comparable elements of X. There exist $U,V \in \mathcal{O}(X)$ such that $x \in U-V$ and $y \in U-V$. We claim that $\{U,V\}$ is a partition of X. If not, then for instance $U \cup V \neq X$ and $U \cup V$ has a greatest element, which implies $U \subseteq V$ or $V \subseteq U$ and this is impossible. Let p be the least element of U and q the least element of V. To end the proof, we shall show that $X = \{p,q\}$. If not, let $r \in X - \{p,q\}$ and suppose for instance that $p \nleq r$. There exist $U', V' \in \mathcal{O}(X)$ such that $U' \supseteq \{p,q\}$, $U' \not\ni r$, $V' \supseteq \{q,r\}$ and $V' \ni p$. Then $-(U' \cap V')$ has a least element and this implies $U' \subseteq V'$ or $V' \subseteq U'$, a contradiction.

3. <u>The lattice of R-subalgebras of a bounded distributive lattice</u>

In this section, we dualize the results of the previous section to obtain results on $\mathcal{S}_R(L)$, for $L \in \mathbb{D}$. We omit the proofs which are straightforward.

3.1. <u>Theorem</u>. <u>If</u> $L \in \mathbb{D}$, $\mathcal{S}_R(L)$ <u>is algebraic, atomistic and dually atomistic</u>.

3.2. <u>Theorem</u>. <u>If</u> $L \in \mathbb{D}$, <u>then</u> $\mathcal{S}_R(L)$ <u>is dually semimodular if and only if either</u>

(i) L <u>is isomorphic to an ordinal sum</u> $L' \oplus C \oplus L$, <u>where</u> L' <u>and</u> L <u>are (possibly empty) relatively complemented distributive lattices and</u> C <u>is a chain or</u>

(ii) <u>all prime ideals of</u> L <u>are maximal, except possibly one</u>, <u>or</u>

(ii′) <u>all prime ideals of</u> L <u>are minimal, except possibly one</u>.

Let L_7 be the 7-element lattice of figure 1.

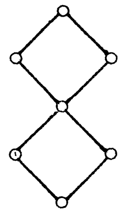

Figure 1

Since the dual of L_7 is the ordinal sum of two 2-element antichains, Theorem 2 7 dualizes as follows.

3.3. *Theorem*. *If* $L \in \mathbb{D}$, *then* $\mathcal{S}_R(L)$ *is dually geometric if and only if* L *is isomorphic to* $(B \oplus 1) \times B'$ *or to* $B \oplus C \oplus B'$, *where* B *and* B' *are finite Boolean algebras and* C *is a not empty chain*.

3.4. *Theorem*. *Let* $L \in \mathbb{D}$.

a) *If* L *is not isomorphic to* L_7, *then the following assertions are equivalent*:

(i) $\mathcal{S}_R(L)$ *is modular*:

(ii) $\mathcal{S}_R(L)$ *is semimodular*;

(iii) $\mathcal{S}_R(L)$ *is geometric*;

(iv) *in* $\mathcal{S}_R(L)$, *the supremum of two atoms covers each of these atoms*;

(v) L *is isomorphic to a sublattice of* $\underline{\underline{2}}^2 \oplus C \oplus \underline{\underline{2}}^2$ (*for some chain* C), *or to* $\underline{\underline{2}}^3$ *or to* $\underline{\underline{2}} \times \underline{\underline{3}}$.

b) *If* L *is isomorphic to* L_7, *then (ii),(iii) and (iv) hold but* $\mathcal{S}_R(L)$ *is not modular*.

3.5. *Theorem*. *Let* $L \in \mathbb{D}$. *Then the following are equivalent*:

(i) $\mathcal{S}_R(L)$ *is Boolean*;

(ii) $\mathcal{S}_R(L)$ is distributive;

(iii) $\mathcal{S}_R(L)$ is uniquely complemented;

(iv) L is a chain or a four-element Boolean algebra.

We conclude by two corollaries of the above results which shed some light on the problem of the characterization of $\mathcal{S}_R(L)$. We are concerned here with the abstract characterization, but there is no difficulty to adapt our results to have information on the concrete characterization problem.

3.6. Theorem. Let $S \in \mathbb{D}$. Then S is isomorphic to $\mathcal{S}_R(L)$ for some $L \in \mathbb{D}$ if and only if S is a complete atomic Boolean lattice.

Proof. If $\mathcal{S}_R(L)$ is distributive, then it is Boolean, complete and atomic by 3.1 and 3.5.

Conversely, let C be a set such that S is isomorphic to the power set of C. Consider any linear ordering on C and define L to be C with supplementary bounds 0 and 1. Then $\mathcal{S}_R(L)$ is isomorphic to S.

3.7. Theorem. Let S be a modular lattice. Then S is isomorphic to $\mathcal{S}_R(L)$ for some L if and only if S is of one of the forms B, $B \times M_5$, or $B \times M_5 \times M_5$, where B is a complete atomic Boolean lattice.

Proof. Theorem 3.4 (and an easy computation) shows that the condition is necessary.

To prove that it is sufficient, let C be a bounded chain, given by 3.6, such that $\mathcal{S}_R(C)$ is isomorphic to B. Disregarding the case where B is trivial, we choose L to be C (resp. $C \oplus \underline{2}^2$, $\underline{2}^2 \oplus C \oplus \underline{2}^2$) and it follows that $\mathcal{S}_R(L)$ is isomorphic to B (resp. $B \times M_5$, $B \times M_5 \times M_5$).

R e f e r e n c e s

[1] ADAMS M.E.: The Frattini sublattice of a distributive lattice, Alg. Univ. vol. 3/2(1973), 216-228.

[2] DAVEY B.A.: Subdirectly irreducible distributive double p-algebras, Alg. Univ. vol. 8/1, 73-88.

[3] GRÄTZER G.: General lattice theory,Birk. Verlag Basel und Stuttgart (1978).

[4] HANSOUL G. and VARLET J.: Essential, strong and perfect extensions of lattices, Bull. Roy. Sci. Liège,52(1983), 22-24.

[5] MAYER R.D. and PIERCE R.S.: Boolean algebras with ordered bases, Pacific J. of Math. 10(1960),925-942.

[6] PRIESTLEY H.A.: Ordered topological spaces and the representation of distributive lattices, Proc. London Math. Soc. 24(1972), 507-530.

[7] PRIESTLEY H.A.: Ordered sets and duality for distributive lattices, to appear in the Proc. of the conference on ordered sets and their applications, Lyon, 1982.

Institut de Mathématique, Université de Liège, Avenue des Tilleuls, 15, B-4000 Liège (Belgium)

(Oblatum 8.11. 1983)