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ON UNIFORM CONNECTION PROPERTIES

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Abstract: We show that (i) uniform local connectedness and property S are both closed under uniform quotients; (ii) the uniform product has property S (is uniformly locally connected) iff each co-ordinate space has property S (is uniformly locally connected) and all but finitely many of the co-ordinate spaces are connected; (iii) a uniform space has property S iff its coreflection (in the subcategory of uniformly locally connected spaces) has property S.

Key words: Uniform local connectedness, uniform quotient.

Classification: 54D05, 54E15

Introduction: The concept of uniform local connectedness and property S were introduced into the theory of uniform spaces by A.M. Gleason ([2]) and P.J. Collins ([1]) respectively. These concepts, both of which imply local connectedness in general and which are equivalent to local connectedness for compact Hausdorff spaces ([2]), are well known in the theory of metric spaces (e.g. see [4]). A.M. Gleason ([2]) has shown that the subcategory of uniformly locally connected spaces and uniformly continuous maps is coreflective in the category Unif of uniform spaces and uniformly continuous maps. Although not explicitly stated it is evident from Gleason's construction that the uniformly locally connected coreflection of a uniform space $(X,\mathcal{U})$ has the same topology as that generated by $\mathcal{U}$ iff $(X,\mathcal{U})$ is locally connected.

In this paper we give a direct proof of the fact that uniform local connectedness is closed under uniform quotients. We also show that a uniform product has property S (is uniformly locally connected) iff each co-ordinate space has property S (is uniformly locally connected) and all but finitely many of the co-ordinate spaces are connected. Finally we prove that a uniform space has property S iff its coreflection (in the subcategory of uniformly locally connected spaces and uniformly continuous maps) has property S.

Section 1: Throughout this paper we shall use $(X,\mathcal{U})$ to denote a uniform space, with $\mathcal{U}$ the family of entourages of X. If $f:X \to Y$ is a function let

$$\underline{f}: X \times X \to Y \times Y$$

be given by

$$\underline{f}(x,y) = (f(x),f(y)).$$

Recall that $f:(X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous iff $\underline{f}^{-1}(V) \in \mathcal{U} \quad \forall V \in \mathcal{V}$. Unif will denote the category of uniform spaces and uniformly continuous maps. The following two concepts both of which imply local connectedness were introduced by P.J. Collins [1] and A.M. Gleason [2] respectively.

Definition 1.1. $(X,\mathcal{U})$ has property S iff for each $U \in \mathcal{U}$, there exists a finite family $\{A_i\}_{i=1}^{n}$ of connected U-small subsets of X which cover X.

Definition 1.2. $(X,\mathcal{U})$ is said to be uniformly locally connected iff for each $U \in \mathcal{U}$, $\exists V \in \mathcal{U} \ni V \subset U$ and $V[x]$ is connected for each $x \in X$.

By Props we shall mean that subcategory (of Unif) of spaces which satisfy property S while Ulc will denote the subcategory (of Unif) of spaces which are uniformly locally connected.

Since Ulc is coreflective in Unif it follows that Ulc is closed under quotients. Nevertheless we give a direct proof of this fact without using any categorial methods.

Definition 1.3 (see e.g. [3]). Let $\mathcal{U}$ be a uniformity for X and let $f: X \to Y$ be an onto function. Then the largest uniformity $\mathcal{V}$ for Y making f uniformly continuous is called the quotient uniformity for Y relative to f, and f is called a uniform quotient map.

$\mathcal{V}$ is defined as follows:

Let $\mathcal{V}_1 = \{V \subset Y \times Y \mid V$ contains the diagonal of $Y \times Y$ and $\underline{f}^{-1}(V) \in \mathcal{U}\}$. Then $\mathcal{V} = \{V_0 \subset Y \times Y \mid$ there exists a sequence $\{V_n\}_{n=1}^{\infty} \subset \mathcal{V}_1$ and $V_n \circ V_n \subset V_{n-1}$ for all $n \geq 1\}$.

We have the following result which is analogous to the well known result for topological spaces that local connectedness is preserved by quotient maps.

Theorem 1.4. Let $f: (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a uniform quotient map. If $(X, \mathcal{U})$ is uniformly locally connected then so is $(Y, \mathcal{V})$.

Proof: Let $V_0 \in \mathcal{V}$ be given. Then there exists a sequence $\{V_n\}_{n=1}^{\infty}$ in $\mathcal{V}_1$ such that $V_n \circ V_n \subset V_{n-1}$ for all $n = 1, 2, \ldots$. We may assume without loss of generality that V_n is symmetric for all $n = 1, 2, \ldots$. Since $\underline{f}^{-1}(V_1) \in \mathcal{U}$ there are surroundings U and U' such that U' is symmetric, $U \subset U' \subset U' \circ U' \subset \underline{f}^{-1}(V_1)$ and $U[x]$ is connected for each $x \in X$. For each $y \in Y$ let C_y^1 be the connected component of y in $V_1[y]$ and $W_1 = \bigcup \{C_y^1 \times C_y^1 \mid y \in Y$. Then

(i) $W_1 \subset \bigcup \{V_1[y] \times V_1[y] \mid y \in Y\} = V_1 \circ V_1 \subset V_0$;

(ii) $W_1[y] = \bigcup \{C_z^1 \mid y \in C_z^1\}$ which is connected since each C_z^1 is connected; and

(iii) $W_1 \in \mathcal{V}_1$: for

$$\bigcup\{U[x]\times U[x] \mid x\in X\} \subset \bigcup\{U'[x]\times U'[x] \mid x\in X\} = $$
$$= U'\circ U' \subset \underline{f}^{-1}(V_1).$$

Therefore

$$\underline{f}\,(\bigcup\{U[x]\times U[x] \mid x\in X\}) = \bigcup\{f(U[x])\times f(U[x]) \mid x\in X\} \subset V_1.$$

Since C_y^1 is the connected component of y in $V_1[y]$ we must have that $C_y^1 \supset f(U[x])$ $\forall x$ such that $y = f(x)$. Therefore $\underline{f}(U) \subset f(U[x] \times U[x] \mid x\in X\}) \subset \bigcup\{C_y^1 \mid y\in Y\} = W_1$. Hence $\underline{f}^{-1}(W_1)\in\mathcal{U}$ and therefore $W_1 \in \mathcal{V}_1$. Consider now $\underline{f}^{-1}(V_2)$. As above let C_y^2 be the connected component of y in $V_2[y]$ and let $W_2 = \bigcup\{C_y^2\times C_y^2 \mid y\in Y\} \subset \bigcup\{V_2[y]\times V_2[y] \mid y\in Y\} = V_2\circ V_2 \subset V_1$. Now $W_2[y]$ is connected for all $y\in Y$; hence $W_2\circ W_2 = \bigcup\{W_2[y]\times W_2[y] \mid y\in Y\} \subset \bigcup\{C_y^1\times C_y^1 \mid y\in Y\} = W_1$ and (as above) $W_2 \in \mathcal{V}_1$. Continuing in this manner we obtain a sequence $\{W_n\}_{n=1}$ such that $W_n\circ W_n \subset W_{n-1}$ for all $n = 1,2,3,\ldots$ and $f^{-1}(W_n) \in \mathcal{U}$ $\forall n>1$. We have thus found $W_1 \subset V_0$ such that $W_1 \in \mathcal{V}$ and $W_1[y]$ is connected for each $y\in Y$. This completes the proof.

The subcategory Props is not epireflective in Unif since it is not closed under subobjects; consider, $X = [0,1]$ and $A = [0,1]\cap\mathbb{Q}$, both having the usual metric. In fact Props is not closed under products as well. The same is true for Ulc.

Now let $(X_\alpha, \mathcal{U}_\alpha)$ be non-empty uniform spaces for each $\alpha \in A$, and let $X = \prod_{\alpha\in A} X_\alpha$ be the uniform product of these spaces with uniformity $\mathcal{U}$. Let $P_\alpha : X\times X \longrightarrow X_\alpha\times X_\alpha$ be given by $P_\alpha(x,y) = (\pi_\alpha(x), \pi_\alpha(y))$ where $\pi_\alpha : \prod X_\alpha \longrightarrow X_\alpha$ is the projection map. Then we have:

Theorem 1.5. $(\prod X_\alpha, \mathcal{U})$ has property S iff

(i) each $(X_\alpha, \mathcal{U}_\alpha)$ has property S, and

(ii) all but finitely many X_α are connected.

Proof: Suppose that $(\prod X_\alpha, \mathcal{U})$ has property S. Then since $\pi_\alpha : \prod X_\alpha \to X_\alpha$ is uniformly continuous and onto, each $(X_\alpha, \mathcal{U}_\alpha)$ has property S. Also $\prod X_\alpha$ is locally connected ([1]), and thus all but finitely many X_α are connected.

Conversely, let $U \in \mathcal{U}$ be given. Find $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$ (say) in $\mathcal{U}_{\alpha_i}$ $(i = 1,2,\ldots,n)$ such that $P^{-1}_{\alpha_1}(U_{\alpha_1}) \cap \cap P^{-1}_{\alpha_2}(U_{\alpha_2}) \cap \ldots \cap P^{-1}_{\alpha_n}(U_{\alpha_n}) \subset U$. We may assume that the se $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ has been expanded to include all α for which X_α is not connected. Since $(X_{\alpha_i}, \mathcal{U}_{\alpha_i})$ has property S for each $= 1,2,\ldots,n$, we have that $X_{\alpha_i} = \bigcup_{j=1}^{n(i)} A_{ij}$ (say), where each A_{ij} is connected in X_{α_i} and $A_{ij} \times A_{ij} \subset U_{\alpha_i}$ for $j = 1,2,\ldots \ldots, n(i)$. For each i, $1 \leq i \leq n$, let j_i be a variable such that $1 \leq j_i \leq n(i)$, and consider sets of the form

$$Y_{j_1, j_2, \ldots, j_n} = \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_\alpha \times A_{1j_1} \times A_{2j_2} \times \ldots \times A_{nj_n}.$$

Clearly each sum $Y_{j_1, j_2, \ldots, j_n}$ is connected since each factor in the product is connected. Furthermore $X = \bigcup\{Y_{j_1, j_2, \ldots, j_n} \mid 1 \leq j_i \leq n(i)$ for each $i = 1,2,\ldots,n\}$ is clearly a finite union. As can be easily verified each such $Y_{j_1, j_2, \ldots, j_n}$ is U-small. This completes the proof.

Theorem 1.6. $(\prod X_\alpha, \mathcal{U})$ is uniformly locally connected iff

(i) Each $(X_\alpha, \mathcal{U}_\alpha)$ is uniformly locally connected, and

(ii) all but finitely many X_α are connected.

Proof: Assume $(\prod X_\alpha, \mathcal{U})$ is uniformly locally connected. Now since $\pi_\alpha : \prod X_\alpha \to X_\alpha$ is uniformly open, uniformly continuous and onto (and hence a uniform quotient map), the fact that each $(X_\alpha, \mathcal{U}_\alpha)$ is uniformly locally connected follows from Theorem 1.4 above. Moreover as $\prod X_\alpha$ is locally connected (see [1]), all but finitely many X_α are connected.

Conversely let $U \in \mathcal{U}$ be given. Find $U_{\alpha_i} \in \mathcal{U}_{\alpha_i}$ ($i = 1,2,\dots,n$) such that $P_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \dots \cap P_{\alpha_n}^{-1}(U_{\alpha_n}) \subset U$. We may clearly assume that the set $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ has been expanded to include all α for which X_α is not connected. Since for each $i = 1,2,\dots,n$, $(X_{\alpha_i}, \mathcal{U}_{\alpha_i})$ is uniformly locally connected, we find for each i, $V_{\alpha_i} \in \mathcal{U}_{\alpha_i}$ such that $V_{\alpha_i} \subset U_{\alpha_i}$ and $V_{\alpha_i}[s]$ is connected for each $s \in X_{\alpha_i}$. Then

$$\bigcap_{i=1}^{m} P_{\alpha_i}^{-1}(V_{\alpha_i}) \subset \bigcap_{i=1}^{m} P_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U$$

and for each $x = (x_\alpha) \in X$ we have

$$(P_{\alpha_1}^{-1}(V_{\alpha_1}) \cap \dots \cap P_{\alpha_n}^{-1}(V_{\alpha_n}))[x] = \prod_{\alpha \neq \alpha_1,\dots,\alpha_m} X_\alpha \times V_{\alpha_1}[x_{\alpha_1}] \times \dots \times V_{\alpha_n}[x_{\alpha_n}]$$

which is connected, as each factor in the product is connected. This completes the proof.

<u>Section 2</u>. A.M. Gleason ([2]) has proved that <u>Ulc</u> is coreflective in Unif. Denote the corresponding functor by UL. His construction of this coreflection is as follows:
Let $(X, \mathcal{U})$ be a uniform space and let $\mathcal{J}$ be the topology of $\mathcal{U}$. Let $(X, \mathcal{J}^*)$ be the locally connected coreflection of $(X, \mathcal{J})$. For each $U \in \mathcal{U}$, let

$V_U = \{(x,y) \in X \times X \mid$ there exists a J^*-connected subset K of X containing both x and y such that $K \times K \subset U\}$.

Then $\{V_U | U \in \mathcal{U}\}$ is a basis for a uniformity $\mathcal{V}$ on X, and $(X, \mathcal{V})$ is the uniformly locally connected coreflection of $(X, \mathcal{U})$ with the associated topology $\mathcal{T}^*$.

Although not stated by Gleason it is easy to prove that ULX and X have the same topology generated by $\mathcal{U}$ if and only if $(X, \mathcal{U})$ is locally connected.

We end this paper by showing how the concept of property S relates to the uniformly locally connected coreflection of a locally connected uniform space.
It is clear that if $\mathcal{U} \subset \mathcal{U}^1$, where $\mathcal{U}$ and $\mathcal{U}^1$ are compatible uniformities on X, and if $(X, \mathcal{U}^1)$ has property S then so does $(X, \mathcal{U})$. However if $(X, \mathcal{U})$ has property S $(X, \mathcal{U}^1)$ need not of course have property S. The significance of the next result is that even though $\mathcal{V} \supset \mathcal{U}$, property S is retained if $(X, \mathcal{U})$ has property S.

Theorem 2.1. $X \in$ Props $\Longleftrightarrow$ (UL)X $\in$ Props.

Proof: It suffices to show necessity only. Let $V \in \mathcal{V}$ and find $U \in \mathcal{U}$ such that $V_U \subset V$. Since $(X, \mathcal{U})$ has property S, X can be written as a finite union of connected sets A_i, $i = 1,2,\dots$ $\dots,n$ (say) such that $A_i \times A_i \subset U$ for each i. Since A_i is connected this means that $A_i \times A_i \subset V_U$ $\forall$ i.
This completes the proof.

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