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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Abstract: Let E be a Banach space, K a nonempty closed convex subset of E. Suppose that we have a continuous operator T which maps K into a compact subset of E and an operator F from $\overline{T[K]}\times K$ into K. Melvin [7] proved that if for each x, $F(x,\cdot)$ is continuous and $F(\cdot,x)$ is contraction, then the equation $F(T(x),x) = x$ has a solution. The purpose of this note is to give some generalizations of the Melvin's result for multivalued mappings.

Key words: Multivalued mappings, fixed point, uniformly convex Banach space.

Classification: 54C60, 47H10

Let E be a Banach space, K a nonempty closed convex subset of E. Suppose that we have a continuous operator T which maps K into a compact subset of E and an operator F from $\overline{T[K]}\times K$ into K. W.R. Melvin [7] proved that if for each x, $F(x,\cdot)$ is continuous and $F(\cdot,x)$ is contraction, then the equation $F(T(x),x) = x$ has a solution. For $F(x,y) = x + G(y)$ we obtain the fixed point theorem of Krasnoselskii [6] which combines both the Banach contraction principle and the Schauder fixed point theorem.

Problems of the said paper are in a tight connection with the results of J. Daneš (especially Proposition 9, Theorem 9 and the consequent Remarks 1 - 6, pp. 34-37) in the work [2]. These results should be made generalized for multivalued mappings. In this note, we consider the relation $x\in F(T(x),x)$ with F taking values in the family of nonempty bounded closed convex subsets of an uniformly convex Banach space. For other generalizations see [3] - [5], [8] and [9].

The set of all nonempty subsets of a set X is denoted by 2^X. Let $\mathfrak{X}(X)$ be the family of all nonempty closed bounded convex subsets of a real normed linear space X. $\mathfrak{X}(X)$ will be regarded as a metric space endowed with the Hausdorff metric d_H, i.e.

$$d_H(A,B) = \max\left(\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\right);$$

here the distance between any point $x\in X$ and subset Z of X is denoted by $d(x,Z)$ $(= \inf\{\|x - z\| : z\in Z\})$. For metric spaces X and Y, $C(X,Y)$ stands for the space of continuous bounded functions from X to Y endowed with the usual supremum metric σ.

We begin with

<u>Definition</u>. Let X and Y be metric spaces and let Φ be a real-valued nonnegative function defined on $C(X,Y)$. A multivalued mapping $F: X\times Y \to 2^Y$ is called a K_Φ-mapping if (1) for each fixed $x\in X$, $F(x,\cdot)$ is closed on Y (i.e. $y_n \to y_o$ and $z_n \to z_o$ with $z_n\in F(x,y_n)$ for $n\geq 1$ implies that $z_o\in F(x,y_o)$), and (2) for every $f\in C(X,Y)$ there exists $h_f\in C(X,Y)$ such that $h_f(x)\in F(x,f(x))$ for $x\in X$ and $\sigma(f,h_f) \leq \Phi(f) - \Phi(h_f)$.

<u>Proposition</u>. Let X be a metric space, Y a complete metric space and let $F: X\times Y \to 2^Y$ be a K_Φ-mapping. Then there is a function $h\in C(X,Y)$ with $h(x)\in F(x,h(x))$ for $x\in X$.

<u>Proof</u>. Let $h_o\in C(X,Y)$. Since F is a K_Φ-mapping we obtain $h_n\in C(X,Y)$ $(n = 1,2,\dots)$ such that $h_n(x)\in F(x,h_{n-1}(x))$ for $x\in X$, and

$$\sigma(h_k,h_m) \leq \sum_{i=k}^{m-1}\sigma(h_i,h_{i+1}) \leq \Phi(h_k) - \Phi(h_m)$$

for $0 \leq k \leq m$. Hence $\Phi(h_0) \geq \Phi(h_1) \geq \Phi(h_2) \geq \ldots$, and consequently (h_n) is a Cauchy sequence in $C(X,Y)$.

Let $h = \lim_{n \to \infty} h_n$ in $C(X,Y)$. For $x \in X$, we have $h_n(x) \longrightarrow h(x)$ as $n \longrightarrow \infty$ and $h_{n+1}(x) \in F(x,h_n(x))$ $(n = 1,2,\ldots)$ and, since $F(x,\cdot)$ is closed, it follows that $h(x) \in F(x,h(x))$.

Now we are able to state the following

<u>Theorem 1</u>. Suppose we are given: E - a Banach space, Y - a nonempty closed convex subset of E, and T - a singlevalued continuous mapping from Y into a compact subset of E. If $F:\overline{T[Y]} \times Y \longrightarrow 2^Y$ is a K_Φ-mapping, then the relation $x \in F(T(x),x)$ has a solution in Y.

<u>Proof</u>. Put $X = \overline{T[Y]}$. By Proposition there exists $h \in C(X,Y)$ with $h(x) \in F(x,h(x))$ for each $x \in X$. Now, we consider the continuous mapping $x \longmapsto h(T(x))$ of Y into itself. It can be easily seen that this operator has values in a compact subset of E. Applying Schauder's theorem we infer that there is a point x_0 in Y such that $h(T(x_0)) = x_0$. Consequently $x_0 \in F(T(x_0),h(T(x_0))) = F(T(x_0),x_0)$, which completes the proof.

The Lemma below is due to Banks and Jacobs [1] and is basic in the proof of the next result.

<u>Lemma</u>. Let E be a uniformly convex Banach space and X a metric space, If $G:X \longrightarrow \mathfrak{X}(E)$ is continuous, then there is a unique continuous function $g:X \rightarrow E$ such that $g(x) \in G(x)$ and $\|g(x)\| = \inf \{\|y\| : y \in G(x)\}$ for each x in X.

<u>Theorem 2</u>. Let E be a uniformly convex Banach space, K a nonempty closed and convex subset of E, T a singlevalued continuous mapping from K into a compact subset of E. Suppose that F is a mapping from $\overline{T[K]} \times K$ to $\mathfrak{X}(K)$ satisfying the fol-

lowing conditions: (1) $F(\cdot,x)$ is continuous on $\overline{T[K]}$ for every x in K, and (2) $d_H(F(x,x_1), F(x,x_2)) \leq k \|x_1 - x_2\|$ for all x in $\overline{T[K]}$ and x_1, x_2 in K and with a constant $k<1$.

Under our assumptions there exists a point x_0 in K such that $x_0 \in F(T(x_0),x_0)$.

Proof. Let $X = \overline{T[K]}$. First of all we note that if $f \in C(X,K)$, then $x \longmapsto F(x,f(x))$ is continuous on X, and, by the Lemma, there is a unique function $h_f \in C(X,K)$ with $h_f(x) \in F(x,f(x))$ and $\|f(x - h_f(x)\| = d(f(x),F(x,f(x)))$ on X.

Assume that $f_0 \in C(X,K)$ is a given function. By the facts above, there exists a uniquely determined sequence (f_n) of functions $f_n \in C(X,K)$ $(n = 1,2,\ldots)$ such that

$f_n(x) \in F(x,f_{n-1}(x))$ and

$$\| f_n(x) - f_{n-1}(x) \| = d(f_{n-1}(x),F(x,f_{n-1}(x)))$$

for all $x \in X$. Hence we obtain

$$\|f_n(x) - f_{n-1}(x)\| \leq d_H(F(x,f_{n-2}(x)), F(x,f_{n-1}(x))) \leq$$

$$\leq k \|f_{n-2}(x) - f_{n-1}(x)\| \leq \ldots \leq k^{n-1} \|f_0(x) - f_1(x))\|$$

and

$$\sum_{n=1}^{\infty} \sigma(f_n,f_{n-1}) \leq \sigma(f_0,f_1) \cdot \sum_{n=1}^{\infty} k^{n-1} < \infty .$$

Consequently, F is a K_Φ-mapping on X with the function $\Phi : C(X,K) \longrightarrow [0,\infty)$ defined by setting

$$\Phi(f_0) = \sum_{n=1}^{\infty} \sigma(f_n,f_{n-1}).$$

So, our result follows from the Theorem 1.

References

[1] H.T. BANKS and M.Q. JACOBS: Differential calculus for multifunctions, J. Math. Anal. Appl. 29(1970), 246-272.

[2] J. DANEŠ: On densifying and related mappings and their application in nonlinear functional analysis, Theory of nonlinear operators, Akademie-Verlag, Berlin 1974, 15-46.

[3] O. HADZIĆ: Some theorems on the fixed points for multivalued mappings in locally convex spaces, Bull. Acad. Polon. Sci., Sér. Sci. Math. 27(1979), 277-285.

[4] O. HADZIĆ: A fixed point theorem for the sum of two mappings, Proc. Amer. Soc. 85(1082), 37-41.

[5] M. KISIELEWICZ and T. JANIK: Existence theorem for functional differential contingent equations, Ann. Polon. Math. 35(1977), 15[illegible]-166.

[6] M.A. KRASNOSELSKII: Two remarks on the method of successive approximations, Uspechi Mat. Nauk 10(1955), 123-127. (Russian).

[7] W.R. MELVIN: Some extensions of the Krasnoselskii fixed point theorem, J. Diff. Equations 11(1972), 335-348.

[8] B. RZEPECKI: An extension of Krasnoselskii's fixed point theorem, Bull. Acad. Polon. Sci., Sér. Sci. Math. 27(1979), 481-488.

[9] B. RZEPECKI: A note on some fixed point theorems, Glasnik Mat. 17(1982), 304-310.

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