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COMPACTNESS AND HOMOGENEITY OF SATURATED STRUCTURES I

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Abstract: We investigate $\aleph_1$-saturated models of power $\aleph_1$. In such a structure we define, using monads, some topologies. To clear up some relations among these topologies we introduce a notion of homogeneity of structure. This notion of homogeneity is connected with the existence of certain types of automorphisms of the structure in question. The mentioned notions give us possibilities to study some typical syntactic problems, e.g., to decide whether a given theory is complete, what functions or predicates are undefinable in it and whether every formula is equivalent to a formula of a certain form.

Key words: Saturated models, totally disconnected relation, homogeneous structure, isomorphism, undefinability.

Classification: 03C50, 03C65

Introduction. We work with $\aleph_1$-saturated models of power $\aleph_1$ for a countable first-order language L. Having such a model A we define for every $k \geq 1$, $\Psi \subseteq Form_L$ and $C \leq |A|$, the relation $\overset{\Psi, A^k}{\underset{C}{=}}$ on $|A|^k$:

$\langle a_1, \dots, a_k \rangle \overset{\Psi, A^k}{\underset{C}{=}} \langle b_1, \dots, b_k \rangle$ iff

$A \models \psi(a_1, \dots, a_k, c_1, \dots) \leftrightarrow \psi(a_1, \dots, a_k, c_1 \dots)$ holds for every $\psi(x_1, \dots, x_k, y_1, \dots) \in \Psi$ and $c_1, \dots \in C$.

On the assumption that C is at most countable, we prove a compactness principle for these relations. To clear it up, the L(C)-definable classes of $|A|^k$ can be seen as clopen sets in a compact topology of $|A|^k$. the neighbourhood generating system of a point $a \in |A|^k$ of which has the form $\{X \subseteq |A|^k : X$ is L(C)-definable in A and $\sim$" $\{a\} \subseteq X\}$. We deduce from this compactness principle and from the equality

$$(*) \qquad \frac{\Psi, A^k}{\overline{\overline{C}}} = \frac{\mathrm{Form}_L, A^k}{\overline{\overline{C}}}$$

that every L(C)-formula $\varphi(x_1, \dots, x_k)$ is equivalent in A to a boolean combination of formulas from $\Psi(C)$.

We give a notion of Ψ-homogeneity of a structure A such that for every structure A having this property, the relations (*) are satisfied for all $k \geq 1$. Note that our notion of homogeneity is connected with the existence of certain automorphisms on A. We present a criterion of homogeneity in § 3.

We can prove, as a consequence of our investigations, that the theory of real (algrebraically resp.) closed fields and Presbourgher arithmetic are complete. These results are presented in the part II of this article. We discuss there some questions of undefinability in models of Presbourgher arithmetic, too. We prove, e.g., that in the "additive part" A^+ of a given model of Peano arithmetic, the predicate "x is prime" is not definable.

These investigations have been inspired by the possibilities which afford the universe of the alternative set theory.

§ 1. Preliminaries. Writing L we mean a countable first-order language (with =). If $A \models L$ and $C \subseteq |A|$ we denote by L(C) the language $L \cup \{\underline{c} : c \in C\}$, where $\underline{c}$ is a new constant symbol for c. We expand A to the structure $\langle A, c \rangle_C$ for L(C). If there is no danger of confusion we shall write A instead of $\langle A, c \rangle_C$. Let Φ be a class of formulas of L. We shall sometimes denote this fact by the symbol $\Phi \subseteq L$. We shall use $\Phi, \Psi, \nabla, \Phi_0, \ldots$ as names for an arbitrary set of formulas. $\Phi(C)$ is the set of all formulas of $L(C)$ of the form $\varphi(\underline{c}_1, \ldots, \underline{c}_n)$, where $\varphi(x_1, \ldots, x_n) \in \Phi$ and $x_1, \ldots, x_n$ are the list of some free variables in φ. We use $\Phi^{(k)}$ to denote the set $\{\varphi \in \Phi : \varphi$ has exactly k free variables$\}$.

We now come to an extremely important convention. By a saturated model for L we mean an $\aleph_1$-saturated model for L of power $\aleph_1$.

Let $A \models L$ and suppose ψ is an L-formula with exactly $k \geq 1$ variables.

$$\psi[A^k] = \{\langle a_1, \ldots, a_k \rangle \in |A|^k : A \models \psi(a_1, \ldots, a_k)\}.$$

Suppose $\Psi \subseteq Form_L$. Then we put

$\text{bool}(\Psi) = \{\psi : \psi$ is a boolean combination of formulas from $\Psi\}$

$\neg(\Psi) = \Psi \cup \{\neg\psi : \psi \in \Psi\}$

and $\vee(\Psi) = \{\psi : \psi$ is a finite disjunction of formulas from $\Psi\}$.

Let $\mathfrak{D}^k_A$ designate the set $\{\psi[A^k] : \psi \in Form^{(k)}_{L(A)}\}$, $k \geq 1$. Suppose $C \subseteq |A|$, $P_i \subseteq |A|^{n_i}$, $i \in \omega$. The expansion of A on the predicates P_i, $i \in \omega$ and the constants $c \in C$ is designated by $\langle A, P_i \rangle_C$.

A C-expansion of A is a structure $\langle A, P_i \rangle_K$, where $K \subseteq C$ and, for every i, P_i is an n_i-place relation such that $P_i \subseteq C^{n_i}$.

Suppose that $\sim$ is an equivalence on a set X, $a \in X$. We put $[a]_\sim = \{x \in X : x \sim a\}$. We use $i, j, k, l, m, n, i_0, \ldots$ as names for elements of ω. Having $A \models L$ we shall write sometimes A^n instead of $|A|^n$.

§ 2. Compactness theorem for saturated structures. A relation $\sim$ on A^n is totally disconnected iff there exists a sequence $\{X_i\}_{i\in\omega}$ such that

(i) $X_i \in \mathfrak{D}_A^{2n}$, $i \in \omega$,

(ii) X_i is an equivalence on A^n and the set $\{[a]_{X_i} : a \in A^n\}$ of factor-classes modulo X_i is finite, $i \in \omega$,

(iii) $\sim = \bigcap\{X_i, i \in \omega\}$.

Suppose that the relation $\sim$ is an equivalence on A^n and $a \in A^n$. A set $U \in \mathfrak{D}_A^n$ is said to be a $\sim$ -neighbourhood of a iff $\sim'' \{a\} \subseteq U$. Let $\mathcal{U} \subseteq \mathcal{P}(\mathfrak{D}_A^n)$ and $X \subseteq A^n$. $\mathcal{U}$ is a $\sim$-covering of X iff $\bigcup\mathcal{U} = X$ and $(\forall x \in X)(\exists U \in \mathcal{U})$ (U is a $\sim$-neighbourhood of a). We say that a class $Y \subseteq A^n$ is dense in X w.r.t. $\sim$ iff $Y \subseteq X$ and $(\forall x \in X)(\forall U)$ (U is a $\sim$-neighbourhood of $x \rightarrow U \cap Y \neq 0$).

Theorem. (Compactness theorem for saturated structures.) Let A be a saturated model for L and let $\sim$ be a totally disconnected equivalence on A^n. Suppose $X \in \mathfrak{D}_A^n$ and let $\mathcal{U}$ be a $\sim$-covering of X.

Then there exists a finite part $\mathcal{U}' \subseteq \mathcal{U}$ such that $\bigcup\mathcal{U}' = X$.

Proof: follows from the statements (a),(b):

(a) There is $\overline{\mathcal{U}} \subseteq \mathcal{U}$ such that $\overline{\mathcal{U}}$ is at most countable and $\bigcup\overline{\mathcal{U}} = X$.

(b) Suppose $\overline{\mathcal{U}}$ is as above. Then there exists a finite class $\mathcal{U}' \subseteq \overline{\mathcal{U}}$ such that $\bigcup\mathcal{U}' = X$.

To prove (b) we denote by $\mathcal{V}$ the set $\{X-U : U \in \overline{\mathcal{U}}\}$ and suppose: if $\mathcal{U}' \subseteq \overline{\mathcal{U}}$ is finite then $\bigcup\mathcal{U}' \neq X$. We deduce from the relation $X - \bigcup\mathcal{U}' = \bigcap\{X-U : U \in \mathcal{U}'\}$ that $\mathcal{V}$ is a centred set. By using the saturativity of A we obtain $\bigcap\mathcal{V} \neq 0$, which is a contradiction.

We shall prove (a). Let $\sim = \bigcap_{i\in\omega} S_i$ where $\{S_i\} \subseteq \mathfrak{D}_A^{2n}$ and each set S_i is an equivalence such that the factorization A^n/S_i is finite. We can assume without loss of generality that $S_{i+1} \subseteq \subseteq S_i$ holds for every $i \in \omega$.

Lemma 1. Let $\bar{x} \in A^n$ and suppose U is a $\sim$-neighbourhood of $\bar{x}$. Then there exists i such that $S_i^n \{\bar{x}\} \subseteq U$.

Proof. Suppose the statement is false. Thus, we have, for all i, $A \models (\exists \bar{y})(\bar{y} \in S_i^n \{\bar{x}\} - U)$. We deduce from the saturativity of A that there exists $\bar{a} \in A^n$ such that $A \models \bar{a} \in S_i^n \{x\} - U$ holds for every i. Thus we have $\bar{a} \sim \bar{x}$, and, consequently, $\bar{a} \in U$, which is a contradiction.

Lemma 2. X contains a dense subclass Y which is at most countable.

Really, let $\{u_i\}_\omega$ be a sequence of finite subsets u_i of X with the following property:

$(\forall i \in \omega)(\forall x \in X)(|u_i \cap S_i^n \{x\}| = 1)$.

The class $Y = \bigcup \{x_i, i \in \omega\}$ has the required properties. Let us finish our proof. Put

$\mathcal{U}_Y = \{S(y,m_y) : y \in Y \,\&\, m_y = \min \{m : (\exists U \in \mathcal{U})(S(y,m) \subseteq U)\}\}$, where S(y,m) denotes the class $S_m^n \{y\}$. The class $\mathcal{U}_Y$ is at most countable. Suppose $x \in X$. Choose $U \in \mathcal{U}$ and m such that $S(x,m) \subseteq \subseteq U$. Let $y \in S(x,m) \cap Y$. We have $x \in S(y,m)(= S(x,m))$ and, consequently, $m_y \le m$ holds. Thus $x \in S(y,m_y) \in \mathcal{U}_Y$ is true and we deduce from this that $\bigcup \mathcal{U}_Y = X$. A class in question is, for example, a class $\overline{\mathcal{U}}$ with the property:

$(\forall \tilde{U} \in \mathcal{U}_y)(\exists! U)(U \in \overline{\mathcal{U}} \,\&\, \tilde{U} \subseteq U)$.

Let $A \models L$, $\Phi \subseteq \mathrm{Form}_L$, $C \subseteq |A|$. We define, for all $k \ge 1$, the relation $\overset{\Phi,A^k}{\underset{C}{=\!=}}$ on A^k as follows:

$\langle a_1,\dots,a_k\rangle \overset{\Phi,A^k}{\underset{C}{=\!=}} \langle b_1,\dots,b_k\rangle$ iff $(\forall\varphi\in\Phi^{(k)}(C))$

$$(A\models\varphi(a_1,\dots,a_k)\leftrightarrow\varphi(b_1,\dots,b_k)).$$

Writing $\overset{A^k}{\underset{C}{=\!=}}$ we mean the relation $\overset{Form_L,A^k}{\underset{C}{=\!=}}$.

Let us present some trivial consequences of our definition.

(1) $\overset{\Phi,A^k}{\underset{C}{=\!=}} = \overset{bool(\Phi),A^k}{\underset{C}{=\!=}}$, (2) $\overset{A^k}{\underset{C}{=\!=}} \subseteq \overset{\Phi,A^k}{\underset{C}{=\!=}}$,

(3) Suppose C is at most countable. Then $\overset{\Phi,A^k}{\underset{C}{=\!=}}$ is a totally disconnected relation on A^k.

<u>Theorem</u>. Let A be a saturated model for L, $\Phi\subseteq Form_L$, $k\geq 1$, $C\subseteq|A|$ at most countable. Suppose that

$$\overset{A^k}{\underset{C}{=\!=}} = \overset{\Phi,A^k}{\underset{C}{=\!=}}$$

holds.

Then $(\forall\psi\in L^{(k)}(C))(\exists\varphi\in bool(\Phi^{(k)}(C)))(A\models\varphi\leftrightarrow\psi)$.

Proof. Suppose $\psi\in Form^{(k)}_{L(C)}$ and let us denote $\sim = \overset{\Phi,A^k}{\underset{C}{=\!=}}$. At first, we shall prove that $\{\varphi[A^k];\ \varphi\in bool(\Phi^{(k)}(C))\}$ is a neighbourhood generating system of the equivalence $\sim$ i.e. The following statement holds:

$(\forall U)(\forall a\in A^k)(U$ is a $\sim$-neighbourhood of $a\rightarrow$

$\rightarrow(\exists\varphi\in bool(\Phi^{(k)}(C))$ $(\varphi[A^k]$ is a $\sim$-neighbourhood of a & & $\varphi[A^k]\subseteq U))$.

Let $\{\varphi_i\}_\omega$ be a numbering of $\Phi^{(k)}(C)$. Let

$$R_i = \{\langle a,b\rangle\in A^k\times A^k;\ A\models\varphi_i(a)\leftrightarrow\varphi_i(b)\}.$$

We have $\sim = \bigcap\{R_i;i\in\omega\}$ and, moreover, the A/S_i, where $S_i = \bigcap\{R_j;j\leq i\}$, is the class of all atoms of the boolean algebra which is generated by $\{\varphi_j[A^k]; j \leq i\}$. Thus, we can see that the following statement is true:

$(\forall a \in A^k)(\exists \{\chi_i\}_\omega \subseteq \mathrm{bool}(\Phi^{(k)}(C)))(\sim''\{a\} = \bigcap \chi_i[A^k])$.

Suppose now that $X = \psi[A^k]$ for some $\psi \in L^{(k)}(C)$ and let $a \in X$. If $b \sim a$ then $b \in X$ and, consequently, X is a $\sim$-neighbourhood of a. Assume $U = \chi[A^k]$ is a $\sim$-neighbourhood of a, $\chi \in L^{(k)}(C)$. Let $\{\chi_i\}_\omega \subseteq \mathrm{bool}(\Phi^{(k)}(C))$ be such a sequence that $\sim''\{a\} = \bigcap_\omega \chi_i[A^k]$. Suppose that

$$A \models (\exists \bar{z})(\bigwedge_{i \leq j} \chi_j(\bar{z}) \mathbin{\&} \neg \chi(z))$$

holds for each $j \in \omega$. We deduce from the saturativity of A that there exists $b \in A^k$ such that

$$b \in \bigcap_\omega \chi_j[A^k] \cap \chi[A^k],$$

which is a contradiction. Thus, there is $j \in \omega$ for which

$$\bigcap_{i \leq j} \chi_i[A^k] \subseteq \chi[A^k] \; (= U).$$

We deduce from these facts that there exists, to a given formula $\varphi \in L^{(k)}(C)$, a $\sim$-covering $\{\chi[A^k]; \chi \in \Gamma\}$ of the class $\varphi[A^k]$, where Γ is a part of $\mathrm{bool}(\Phi^{(k)}(C))$. Now, the statement in our theorem follows from the compactness theorem for saturated structures.

§ 3. Φ -homogeneity. In this section we want to present a notion of Φ-homogeneity of a structure A, which is connected with the relation $\overset{A^k}{\underset{C}{=}} = \overset{\Phi, A^k}{\underset{C}{=}}$ and which is, under assumption that A is saturated and C is at most countable, equivalent with it.

Before we give the definition of homogeneity, we define the notion of a sort over a class $\mathcal{M}$ of models. Let $\mathcal{M}$ be a class of models for L. A sort over $\mathcal{M}$ is a class $\mathcal{S} \subseteq \bigcup\{\{A\} \times P(|A|); A \in \mathcal{M}\}$

with the following properties: if $A \in \mathfrak{M}$ then

(1) $\mathcal{S}(A,0)$,

(2) $a \in |A|,\ X \subseteq |A| \Rightarrow (\mathcal{S}(A,X) \Rightarrow \mathcal{S}(A, X \cup \{a\}))$,

(3) $\mathcal{S}(A,X_n) \& X_n \subseteq X_{n+1}$ holds for each $n \Rightarrow \mathcal{S}(A, \bigcup_{\omega} X_n)$.

Writing $X \in \mathcal{S}^A$ we mean $\mathcal{S}(A,X)$ and we say that X is of the sort $\mathcal{S}$ in A.

For example, the class

$\{\langle A,X\rangle;\ X$ is an at most countable part of $|A|\ \&\ A \in \mathfrak{M}\}$

is a sort over $\mathfrak{M}$. We shall denote this sort by $\omega_{\mathfrak{M}}$

We define, for a given model A so called σ-classes in A in such a way: $X \subseteq |A|$ is a σ-class in A iff X is a finite or countable union of definable classes of A (possibly with parameters).

The class $\{\langle A,X\rangle;\ X$ is a σ-class in $A \& A \in \mathfrak{M}\}$ is a sort over $\mathfrak{M}$; let $\sigma_{\mathfrak{M}}$ denote this sort.

Let $A,B \in \mathfrak{M}$ and let G be a mapping from $|A|$ to $|B|$. We say that G is of the sort $\mathcal{S}$ iff dom(G) and rng(G) are of the sort $\mathcal{S}$. Let $\Phi \subseteq \mathrm{Form}_L$. G is a Φ-similarity (of A and B) iff the following statement holds:

$$(\forall k \geq 1)(\forall a_1,\dots,a_k \in \mathrm{dom}(G))\ (\forall \varphi \in \Phi^{(k)})(A \models \varphi(a_1,\dots,a_k) \Leftrightarrow$$
$$\Leftrightarrow B \models \varphi(G(a_1),\dots,G(a_k))).$$

G is a $\langle \mathcal{S},\Phi\rangle$ similarity iff G is a Φ-similarity of the sort $\mathcal{S}$. The class $\mathfrak{M}$ is $\langle \mathcal{S},\Phi\rangle$-homogeneous iff every $\langle \mathcal{S},\Phi\rangle$-similarity of two models from $\mathfrak{M}$ is immediately extendable to a $\langle \mathcal{S},\Phi\rangle$-similarity of these models. (Note that a Φ-similarity G of two models A, B is immediately extendable to a Φ-similarity if

$(\forall a \in |A|)(\exists b \in |B|)(G \cup \langle b,a\rangle$ is a Φ-similarity) and

$(\forall b \in |B|)(\exists a \in |A|)(G \cup \langle b,a\rangle$ is a Φ-similarity).)

Let us agree that the letter $\Phi \subseteq L$ denotes a class Φ of formulas of L such that Φ contains the set

$\{p(x_1,\ldots,x_n);$ p is an n-ary relation symbol of L, $n \geq 1\}$

$\cup \{f(x_1,\ldots,x_n) = y;$ f is an n-ary function symbol of L, $n \geq 1\}$

$\cup \{x = \sigma;$ σ is a variable or a constant symbol of L$\}$.

Now, we shall give some consequences of our definitions.

Let $\mathfrak{M}$ be a class of models for L, $A \in \mathfrak{M} \Longrightarrow \|A\| = \aleph_1$ and let $\mathcal{S}$ be a sort over $\mathfrak{M}$, $\Phi \subseteq L$. Suppose $\mathfrak{M}$ is $\langle \mathcal{S}, \Phi \rangle$-homogeneous and let us denote $\sim = \overset{\Phi, A}{\underset{S}{=}}$. Then

[1] Every $\langle \mathcal{S}, \Phi \rangle$-similarity of two models from $\mathfrak{M}$ can be extended to an isomorphism of these models.

[2] Suppose that $A \in \mathfrak{M}$, $S \in \mathcal{S}^A$ and let $F: A \longrightarrow A$ be a function with the following property:

$(\exists c \in S)\ (|[F(c)]_\sim \cap (A - S)| \geq 2$.

Then the function F is definable in no S-expansion of A (see § 1).

Proof. Suppose $F(c) \sim d, F(c) \neq d$. The mapping $\mathrm{Id} \upharpoonright S \cup \{\langle d, F(c)\rangle\}$ is a $\langle \mathcal{S}, \Phi \rangle$-similarity of A and, consequently, it can be extended to an automorphism of the model in question. Thus, F is not definable in this model.

[3] Suppose $A \in \mathfrak{M}$, $S \in \mathcal{S}^A$ and let $U \subseteq |A|$ be such a class that

$(\exists c \in U-S)(|[c]_\sim \cap (|A| - U)| \geq 1$.

Then the predicate U is definable in no S-expansion of A.

Proof. Suppose $d \sim c$, $d \neq c$, $d \in (A-U)-S$. The mapping $\mathrm{Id} \upharpoonright S \cup \{\langle d, c\rangle\}$ is a $\langle \mathcal{S}, \Phi \rangle$-similarity of A which can be extended to an automorphism of every S-expansion of A. Thus, U is not definable in such a structure.

[4] Suppose $A \in \mathfrak{M}$. $S \in \mathcal{S}^A$ and let $a \in |A| - S$ be such that

$|[a]_\sim| \geq 2.$

Then a is definable in no S-expansion of A. This is an immediate consequence of [3].

Theorem. Let $A \models L$ be a model of power $\aleph_1$, $\mathcal{S}$ a sort over $\{A\}$, $\Phi \subseteq L$.

(1) Let A be $\langle \mathcal{S}, \Phi \rangle$-homogeneous. Then, for each $k \geq 1$, $S \in \mathcal{S}^A$,

$$\overset{A^k}{\underset{S}{\equiv}} = \overset{\Phi, A^k}{\underset{S}{\equiv}}$$

holds.

(2) Suppose, moreover, that A is saturated and $S \in \mathcal{S}^A \rightarrow S$ is at most countable. Then A is $\langle \mathcal{S}, \Phi \rangle$-homogeneous iff the relation $\overset{A^k}{\underset{S}{\equiv}} = \overset{\Phi, A^k}{\underset{S}{\equiv}}$ holds for each $k \geq 1$, $S \in \mathcal{S}^A$.

Proof. (1) Let $\langle a_1, \ldots, a_k \rangle \overset{\Phi, A^k}{\underset{S}{\equiv}} \langle b_1, \ldots, b_k \rangle$.

The mapping $Id \restriction S \cup \{\langle b_i, a_i \rangle;\ i = 1, \ldots, k\}$ is a $\langle \mathcal{S}, \Phi \rangle$-similarity which can be extended to an automorphism of A. Thus, the relation $\langle a_1, \ldots, a_k \rangle \overset{A^k}{\underset{S}{\equiv}} \langle b_1, \ldots, b_k \rangle$ is satisfied.

(2) We must prove the implication from right to left. Let G be a $\langle \mathcal{S}, \Phi \rangle$-similarity of A. We need to prove that G is a $Form_L$-similarity of A. Let $\psi(x_1, \ldots, x_k) \in L^{(k)}$. Then there exists a formula $\varphi \in bool(\Phi^{(k)})$ such that $A \models \varphi \leftrightarrow \psi$. We have $A \models \varphi(a_1, \ldots, a_k) \leftrightarrow \varphi(G(a_1), \ldots, (a_k))$ for each $a_1, \ldots, a_k \in$ $\in dom(G)$ and consequently,

$A \models \psi(a_1, \ldots, a_k) \leftrightarrow \psi(G(a_1), \ldots, G(a_k))$ holds for each $a_1, \ldots, a_k \in dom(G)$, too.

Now we shall give one criterion for $\langle \mathcal{S}, \Phi \rangle$-homogeneity of a class $\mathcal{M}$ of saturated models for L.

Let A, B be two saturated models for L, $\Phi \subseteq L$ and let $\mathcal{S}$

be a sort over $\{A,B\}$. Suppose that G is a $\langle \mathcal{S}, \Phi\rangle$-similarity of A, B. What possibilities exist for an immediate prolongation of G to a Φ-similarity?

Suppose that $\overline{}$ is a mapping from $\{X; \mathcal{S}(A,X)\}$ to $\{X; \mathcal{S}(A,X)\}$ and from $\{X; \mathcal{S}(B,X)\}$ to $\{X; \mathcal{S}(B,X)\}$ such that the following holds: every $\langle \mathcal{S}, \Phi\rangle$-similarity G of A, B can be extended to a Φ-similarity G^* with $\mathrm{dom}(G^*) = \overline{\mathrm{dom}(G)}$ and $\mathrm{rng}(G^*) = \overline{\mathrm{rng}(G)}$ and, moreover, $\overline{\overline{X}} = X$ is true for every X of the sort $\mathcal{S}$ in A or B. Now, we can work with G^* and our aim is the following: given $a \in |A| - \mathrm{dom}(G^*)$ we want to find $b \in |B|$ such that $G^* \cup \{\langle b,a\rangle\}$ is a Φ-similarity.

Let $\tau(\Phi^{(1)}(\mathrm{dom}(G^*)),a)$ be the class

$$\{\varphi(x) \in \Phi^{(1)}(\mathrm{dom}(G^*)); A \models \varphi(a)\}.$$

The class $\tau = \tau(\Phi^{(1)}(\mathrm{dom}(G^*)),a)$ is a type in the structure $\langle A,c\rangle_{c\in\mathrm{dom}(G^*)}$. We denote by τ^{G^*} the set

$$\{\varphi^{G^*}(x); \varphi(x) \in \tau\},$$

where $\varphi^{G^*}(x)$ is the formula of the language $L(\mathrm{rng}(G^*))$ of the form $\varphi(x,G^*(a_1),\dots,G^*(a_k))$, where $a_1,\dots,a_k$ is the list of all parameters of the formula φ from $\mathrm{dom}(G^*)$. We shall give some conditions for $\overline{}$ and Φ which assure that τ^{G^*} is realized in B.

At first, we must give a few necessary notions. Let $\mathcal{M}$ be a class of models for L and let $\mathcal{S}$ be a sort over $\mathcal{M}$. A mapping $\overline{}$ is a <u>closure on</u> $\mathcal{S}$ (in $\mathcal{M}$) iff for every $A \in \mathcal{M}$ it holds:

$$\overline{}: \mathcal{S}^A \longrightarrow \mathcal{S}^A \text{ (i.e. } X \in \mathcal{S}^A \Rightarrow \overline{X} \in \mathcal{S}^A) \text{ and}$$

$$S \in \mathcal{S}^A \Rightarrow \{c^A; \text{ c is a constant of } L\} \subseteq \overline{S} \;\&\; S \subseteq \overline{S} = \overline{\overline{S}}.$$

A <u>set</u> $S \in \mathcal{S}^A$ is <u>$\overline{}$-closed</u> iff $S = \overline{S}$ holds. A <u>mapping</u> G between two models from $\mathcal{M}$ <u>is</u> <u>$\overline{}$-closed</u> iff dom(G) and rng(G) are $^{-}$-closed. Let $\Phi \subseteq L$, $\Psi \subseteq L$. The <u>closure</u> $^{-}$ <u>on</u> $\mathcal{S}$ <u>respects</u> Φ <u>-similarities</u> iff every $\langle \mathcal{S}, \Phi\rangle$-similarity of two models from $\mathfrak{M}$ can be extended to a closed Φ-similarity of these models. The <u>closure</u> $^{-}$ <u>on</u> $\mathcal{S}$ Φ<u>-respects types over</u> Ψ iff the following condition holds:

Let G be a closed Φ-similarity of models $A, B \in \mathfrak{M}$. If $\tau \subseteq \Psi^{(1)}(\mathrm{dom}(G))$ is a type in A then τ^{G} is a type in B and if $\tau \subseteq \Psi^{(1)}(\mathrm{rng}(G))$ is a type in B then $\tau^{G^{-1}}$ is a type in A.

Let $\Phi_0, \Phi_1, \nabla \subseteq \mathrm{Form}_L$. A closure $^{-}$ on $\mathcal{S}$ (in $\mathfrak{M}$) is $\langle \Phi_0, \Phi_1, \nabla\rangle$ <u>-stable</u> iff $^{-}$ respects $(\Phi_0 \cup \Phi_1)$-similarities and $(\Phi_0 \cup \Phi_1)$-respects types over $(\Phi_1 \cup \nabla)$.

Let $\mathfrak{M}$ be a class of saturated models for L. Suppose that $\mathcal{S}$ is a sort over $\mathfrak{M}$ such that $\mathcal{S} \subseteq \omega_{\mathfrak{M}}$. Let $\Phi = \Phi_0 \cup \Phi_1$ and suppose that Φ_0, Φ_1 are closed under $\neg$. If $^{-}$ is a $\langle \Phi_0, \Phi_1, \Phi_0\rangle$-stable closure on $\mathcal{S}$ in $\mathfrak{M}$ then $\mathfrak{M}$ is $\langle \mathcal{S}, \Phi\rangle$-homogeneous. This is a trivial fact. Our aim is now the following: to describe conditions on $\Phi_0, \Phi_1, \nabla \subseteq L$ and $^{-}$ such that the $\langle \Phi_0, \Phi_1, \nabla\rangle$ -stability of $^{-}$ (on $\mathcal{S}$ in $\mathfrak{M}$) assures the $\langle \mathcal{S}, \Phi_0 \cup \Phi_1\rangle$ -homogeneity of $\mathfrak{M}$ without the assumption that $\mathcal{S} \subseteq \omega_{\mathfrak{M}}$ and $\nabla = \Phi_0$.

We say that $\nabla, \Phi_0 \subseteq L$ <u>are conjugated by a closure</u> $^{-}$ (on $\mathcal{S}$ in $\mathfrak{M}$) iff we have, for every $A \in \mathfrak{M}$ and every closed set $S \in \mathcal{S}^{A}$: S is dense in A w.r.t.

$\underset{\overline{\overline{S}}}{\nabla, A}$ and $\underset{\overline{\overline{S}}}{\nabla, A} = \underset{\overline{\overline{S}}}{\Phi_0, A}$. Assume $\Psi \subseteq L$. A closure $^{-}$ (on $\mathcal{S}$ in $\mathfrak{M}$) <u>countably determines</u> Ψ iff we have, for every $A \in \mathfrak{M}$ and every closed $S \in \mathcal{S}^{A}$:

$$(\forall a \in |A| - S)(\exists \{\psi_i\}_{\omega} \subseteq \Psi^{(1)}(S))([a]_{\underset{\overline{\overline{S}}}{\Psi^{(1)}, A}} = \bigcap_{\omega} \psi_i[A]).$$

Note that if $\mathcal{S} \subseteq \omega_{\mathfrak{M}}$ then every closure $^{-}$ on $\mathcal{S}$ in $\mathfrak{M}$ countably determines each $\Psi \subseteq L$.

<u>Theorem</u> (Criterion of homogeneity). Let $\mathfrak{M}$ be a class of saturated models for $L, \nabla, \Phi_0, \Phi_1 \subseteq L$ such that $x = y \in \nabla \subseteq \subseteq \Phi_0 \cup \Phi_1$ and let ∇, Φ_0, Φ_1 be closed under $\neg$. Suppose, moreover, that $\mathcal{S}$ is a sort over $\mathfrak{M}$ and let $^{-}$ be a closure on $\mathcal{S}$ in $\mathfrak{M}$ such that

(a) $^{-}$ is $\langle \Phi_0, \Phi_1, \nabla \rangle$-stable,

(b) ∇, Φ_0 are conjugated by $^{-}$ and

(c) $^{-}$ countable determines both ∇ and Φ_1.

Then $\mathfrak{M}$ is $\langle \mathcal{S}, \Phi_0 \cup \Phi_1 \rangle$-homogeneous.

Proof. It suffices to prove: Let $A, B \in \mathfrak{M}$ and let G be a closed $(\Phi_0 \cup \Phi_1)$-similarity of the sort $\mathcal{S}$ between A and B. Then, for each $a \in A$,

$\tau^G(\Phi^{(1)}(\mathrm{dom}(G)), a)$ is realized in B, where $\Phi = \Phi_0 \cup \Phi_1$.

Let us denote $\Psi = \nabla \cup \Phi_1$ and $S = \mathrm{dom}(G)$. Suppose $a \in \in |A| - S$. Choose a sequence $\{\psi_i\}_\omega \subseteq \Psi^{(1)}(S)$ such that

$$[a]_{\underline{\underline{\Psi, A}}_{S}} = \bigcap_\omega \psi_i[A].$$

The mapping G respects types over Ψ, thus, there is a $b \in B$ which realizes the type $\{\psi_i^G\}_\omega$ in B. We shall prove that b realizes $\tau^G((\Phi_0 \cup \Phi_1)^{(1)}(S), a)$. At first, let us prove two lemmas.

<u>Lemma (a)</u>. Let M be a saturated model for L, $k \geq 1$.

(1) Suppose that $U_i, U \in \mathfrak{D}_M^k$, $i \in \omega$, and let $\bigcap_\omega U_i \subseteq U$ hold. Then there exists $m \in \omega$ such that $\bigcap_{i<m} U_i \subseteq U$ holds, too.

(2) Let $\Psi \subseteq L$, $S \subseteq |M|$ and suppose $\psi \in \Psi^{(k)}(S)$. Then $\psi[M^k]$ is a $\underline{\underline{\Psi, M^k}}_{S}$-neighbourhood of each element of $\psi[M^k]$.

The proof is easy and we omit it. We shall use this lemma (a) frequently during our proof.

Lemma (b). $[b]_{\underset{G"S}{\overset{\Psi,A}{=}}} = \bigcap_{\omega} \psi_i^G[B].$

Proof. Note that the following statement holds:
Let $\{\chi, \chi_0, \dots, \chi_n\} \subseteq \Psi^{(1)}(S)$. Then

$$(*) \qquad \bigcap_{i<m} \chi_i[A] \subseteq \chi[A] \Rightarrow \bigcap_{i<m} \chi_i^G[B] \subseteq \chi^G[B]$$

This follows immediately from the fact that G $(\Phi_0 \cup \Phi_1)$-respects types over $(\Phi_1 \cup \nabla)$ and Ψ is closed under $\neg$. To simplify our designation we put $T = G"S$. The inclusion $\subseteq$ in question is clear. Assume that there is $b' \in \bigcap_{\omega} \psi_i^G[B] - [b]_{\underset{T}{\overset{\Psi,B}{=}}}$.

We deduce from this that there exists $\psi \in \Psi^{(1)}(T)$ such that $b \in \psi[B]$, $b' \notin \psi[B]$. Suppose $A \models \neg \psi^{G-1}(a)$. Then $\models \psi^{G^{-1}}[A]$ is a $\underset{S}{\overset{\Psi,A}{=}}$ -neighbourhood of a and, consequently, there is m such that $\bigcap_{i<m} \psi_i[A] \subseteq \neg \psi^{G^{-1}}[A]$. From this fact and from our assumption on b' it follows $b' \in \neg \psi[B]$. But this is a contradiction. Therefore $A \models \psi^{G^{-1}}(a)$ holds. Then there exists m such that $\bigcap_{i<m} \psi_i[A] \subseteq \psi^{G^{-1}}[A]$ and, by using the statement $(*)$ above we obtain $\bigcap_{i<m} \psi_i^G[B] \subseteq \psi[B]$. But this is a contradiction with our assumption $b' \notin \psi[B]$.

Now, we clear up the statement: b realises $\tau^G(\Psi^{(1)}(S), a)$ in B.

Let $\psi \in \tau(\Psi^{(1)}(S), a)$. It is enough to prove: $B \models \psi^G(b)$. Suppose $B \models \neg \psi^G(b)$. $\neg \psi^G B$ is a $\underset{T}{\overset{\Psi,B}{=}}$ -neighbourhood of b. Thus, there exists m such that $\bigcap_{i<m} \psi_i^G[B] \subseteq \neg \psi^G B$. We deduce from the statement $(*)$ that $\bigcap_{i<m} \psi_i[A] \subseteq \neg \psi[A]$, which is a

contradiction.

To prove b realizes $\tau^G((\Phi_0 \cup \Phi_1)^{(1)}(S),a)$ in B it remains to clear up the following implication:

$$(\varphi \in \Phi_0^{(1)}(S) \,\&\, A \models \varphi(a)) \Longrightarrow B \models \varphi^G(b).$$

Suppose the presumption is true and the consequence is false. The set $\varphi[A]$ is a $\overset{\nabla,A}{\underset{S}{=}}$-neighbourhood of a, because $\varphi[A]$ is a $\overset{\Phi_0,A}{\underset{S}{=}}$-neighbourhood of a and $\overset{\Phi_0,A}{\underset{S}{=}} = \overset{\nabla,A}{\underset{S}{=}}$. Similarly, we can see that $\neg\varphi^G[B]$ is a $\overset{\nabla,B}{\underset{T}{=}}$-neighbourhood of b.

Let $\{\nu_i\}_\omega \subseteq \nabla^{(1)}(S)$ be a sequence with the property:

$$[a]_{\overset{\nabla,A}{\underset{S}{=}}} = \bigcap_\omega \nu_i[A].$$

It is clear that b realizes the type $\{\nu_i^G(x)\}_\omega$ and, by using the lemma (b), we obtain $[b]_{\overset{\nabla,B}{\underset{T}{=}}} = \bigcap_\omega \nu_i^G[B]$. From this, the facts mentioned above and the lemma (a), we deduce that there exists m such that

$$\bigcap_{i\le m} \nu_i[A] \subseteq \varphi[A]$$

and

$$\bigcap_{i\le m} \nu_i^G[B] \subseteq \neg\varphi[B].$$

The class S is dense in A w.r.t. $\overset{\nabla,A}{\underset{S}{=}}$. Thus, there exists $c \in \bigcap_{i<m} \nu_i[A] \cap S$. It is clear that $A \models \varphi(c)$ and $G(c) \in \bigcap_{i<m} \nu_i^G[B]$. We deduce from this that both $A \models \varphi(c)$ and $B \models \neg\varphi^G(G(c))$ hold – which is a contradiction with the assumption that G is a Φ_0-similarity.

R e f e r e n c e s

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