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THE TOPOLOGICAL PROOF OF THE NACHBIN-SHIROTA'S THEOREM

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Abstract: This paper is concerned with the topological proof of the wellknown Nachbin-Shirota's theorem about barrelledness of the continuous function space C(X) in compact-open topology.

Key words: The space of function, barrelledness, boundedness, compactness, σ-barrelledness, infrabarrelledness.

Classification: 54C35

In this paper we give a pure topological proof for the next important theorem. The proof is based on a method of A.V. Arhangel'skiĭ [1].

Theorem 1 (L. Nachbin [2], T. Shirota [3]). $C_c(X)$ is barrelled if and only if every bounded set $A \subseteq X$ is relatively compact.

Here $C_c(X)$ means the space of all continuous functions on X with compact-open topology. The set $A \subseteq X$ is called bounded in X if every function $f \in C(X)$ is bounded on A.

The necessity of the condition in Theorem 1 can be proved by a simple way. The sufficiency of the condition on the space X is more difficult. The main difficulty lies in the construction of such a compact set $K(V) \subseteq X$ for every barrel $V \in C_c(X)$ which satisfies the following condition:

(*) if $f \in C(X)$ and $f(x) = 0$ for every $x \in H$, where H is some neighbourhood of $K(V)$, then $f \in V$.

When constructing the set $K(V)$, it is usually necessary to use some deep and nontrivial theorems of functional analysis [4],[5]. Here we give a simple pure topological description of this set. Moreover, our method allows to give a simple proof for the Schmets's theorem [6] characterizing σ-barrelledness and the Warner's theorem [7] characterizing infrabarrelledness of $C_c(X)$.

Remind that a closed convex balanced absorbent subset of a topological vector space is called a barrel. The space in which every barrel is a neighbourhood of zero is called barrelled.

In this paper, the space X is supposed to be completely regular. If $f \in C(X)$, P compact and $P \subseteq X$, $\varepsilon > 0$, then $\langle f,P,\varepsilon \rangle = \{g \in C(X): |f(x) - g(x)| < \varepsilon$ for every $x \in P\}$ is a basic neighbourhood of f in $C_c(X)$ and we denote $\langle P,\varepsilon \rangle = \langle g,P,\varepsilon \rangle$ where $g(x) = 0$ for every $x \in X$.

For $V \subseteq C_c(X)$ put $K(V) = \{x \in X$: for every neighbourhood $\mathcal{O}x$ there is $f \in C(X)$ such that $f(X) \setminus \mathcal{O}x) = 0$ and $f \notin V\}$. (In the case of infrabarrelledness $K(V) = \{x \in X$: for every neighbourhood $\mathcal{O}x$ there is $f \in C(X)$, $f: X \to [0,1]$ such that $f(X \setminus \mathcal{O}x) = 0$ and $f \notin V\}$.)

The main role is played by the next lemma.

<u>Lemma</u>. If V is a barrel in $C_c(X)$, then $K(V)$ is bounded in X.

<u>Proof</u>. Suppose that $K(V)$ is unbounded in X. Then there exists an infinite discrete family T of sets open in X such that $W \cap K(V) \neq \emptyset$ for every $W \in T$. Define inductively sequences of sets $\{W_n\}$, $W_n \in T$, of functions $\{f_n\}$, $f_n \in C(X)$, of compact

sets $\{P_n\}$, $P_n \subseteq X$, and of numbers $\{\varepsilon_n\}$, $\varepsilon_n > 0$, such that

(1) $W_{n+1} \cap (\bigcup_{i=1}^{m} P_i) = \emptyset$,

(2) $f_n \notin V$ and $f_n(x) = 0$ for every $x \in X \setminus W_n$,

(3) $\langle f_n, P_n, \varepsilon_n \rangle \cap V = \emptyset$.

The procedure is simple. W_1 is an arbitrary element of the family T, the function f_1 corresponds to the definition of the set K(V), the choice of the compact set P_1 and of the number ε_1 follows from the fact that V is closed; there is $W_2 \in T$, $W_2 \cap P_1 = \emptyset$, and so on.

Define now a sequence of numbers $\{c_n\}$. Define $c_1 = 1$ and take c_{n+1} so that $|c_{n+1}(f_1(x) + \frac{1}{c_i} \cdot f_i(x) + \ldots + \frac{1}{c_n} \cdot f_n(x))| < \varepsilon_{n+1}$ for every $x \in P_{n+1}$ and $0 < c_{n+1} < \frac{1}{n+1}$. It is possible since the function $\sum_{i=1}^{m} \frac{1}{c_i} \cdot f_i$ is bounded on P_{n+1}.

Let $g = \sum_{i=1}^{\infty} \frac{1}{c_i} f_i$. The continuity of the function g follows from the discreteness of the family $\{W_n\}$ and the condition (2).

Let us prove that $c_{n+1} \cdot g \notin V$. If $x \in P_{n+1}$, then it follows from the conditions (1) and (2) that $f_k(x) = 0$ for every $k > n+1$. Consequently, $c_{n+1} \cdot g(x) = c_{n+1} \sum_{i=1}^{m} \frac{1}{c_i} \cdot f_i(x) + f_{n+1}(x)$, hence $|c_{n+1} g(x) - f_{n+1}(x)| < \varepsilon_{n+1}$ by the definition of c_{n+1}, which means that $c_{n+1} \cdot g \in \langle f_{n+1}, P_{n+1}, \varepsilon_{n+1} \rangle$ and, by the condition (3), $c_{n+1} \cdot g \notin V$. So, $c_{n+1} \cdot g \notin V$ for all c_{n+1} and $c_{n+1} \to 0$, thus V does not absorb the function g, which contradicts the barrelledness of V. The lemma is proved.

<u>Proof of the theorem 1.</u> Let every set bounded in X be relatively compact and V be a barrel in $C_c(X)$. We shall prove that there is $\delta > 0$ such that $\langle K(V), \delta \rangle \subseteq V$. Since K(V) is

closed, it follows from Lemma that V is a neighbourhood of zero in $C_c(X)$, which implies that $C_c(X)$ is barrelled.

First of all we shall show that if $f \in C(X)$ and $f(x) = 0$ for every $x \in H$, where H is some open neighbourhood of K(V), then $f \in V$. Suppose the contrary. Find a compact set $P \subseteq X$ and a number $\varepsilon > 0$ such that $\langle f, P, \varepsilon \rangle \cap V = \emptyset$. Let $F = P \setminus H$. For every $x \in F$ there exists a neighbourhood $\mathcal{O}x$ which satisfies the following condition: for every function $g \in C(X)$ with $g(X \setminus \mathcal{O}x) = 0$ it follows that $g \in V$. The family $\{\mathcal{O}x\}_{x \in F}$ covers F and has a finite subcover $\{\mathcal{O}x_i\}_{i=1}^{n}$. Find a partition of unity subordinated to it, i.e. a family of functions $g_1, g_2, \dots, g_n \in C(X)$ such that for every $i = 1,2,\dots,n$ $g_i(X \setminus \mathcal{O}x_i) = 0$ and $\sum_{i=1}^{m} g_i(x) = 1$ for every $x \in F$. It follows that $a \cdot g_i \in V$ for every $i = 1,2,\dots,n$ and every $a \in R$ and, moreover, $g_i \cdot f \in V$. Denote $\tilde{g}_i = g_i \cdot f$, $\tilde{g} = \sum_{i=1}^{m} \tilde{g}_i$. Then $\tilde{g} \in V$ since $\tilde{g} = \frac{1}{n}(n \cdot \tilde{g}_1 + \dots + n \cdot \tilde{g}_n)$ and the set V is convex. For $x \in F$, $|\tilde{g}(x) - f(x)| = |\sum_{i=1}^{m} g_i(x) \cdot f(x) - f(x)| = |f(x)(\sum_{i=1}^{m} g_i(x) - 1)| = 0$. If $x \in P \setminus F$, then $f(x) = 0$ and $\tilde{g}(x) = 0$. As a result, $f(x) = \tilde{g}(x)$ for every $x \in P$ and consequently $\tilde{g} \in \langle f, P, \varepsilon \rangle$ and $\tilde{g} \notin V$, which is a contradiction.

Thus K(V) satisfies the condition (*). The next part of our proof is standard (see [4]); we shall present it here for the sake of completeness.

Let $C^*(X)$ mean the space of all continuous bounded functions in the topology of uniform convergence on X. Since $C^*(X)$ is barrelled and $V \cap C^*(X)$ is a barrel in $C^*(X)$, there is $\sigma > 0$ such that $f \in V$ whenever $f \in C^*(X)$ and $|f(x)| < \sigma$ for every $x \in X$. We can put $\delta = \frac{\sigma}{2}$. Then if $f \in \langle K(V), \delta \rangle$ there is a neighbourhood H of the set K(V) such that $|f(x)| < \delta$ for every $x \in H$. Let $g(x) = \max\{f(x), \delta\} + \min\{f(x), -\delta\}$. Then $2g(x) = 0$ for every $x \in H$. It means that $2g \in V$. Moreover, $|2(f(x) - g(x))| < \sigma$

for every $x \in X$. Hence $2(f - g) \in V$. The equality $f = \frac{1}{2}(2g) + \frac{1}{2}(2(f - g))$ shows that $f \in V$.

The proof of the theorem is complete.

From Lemma it follows also the following characterisation of σ-barrelledness of $C_c(X)$.

Remind that a topological vector space is called σ-barrelled if every barrel being an intersection of the countable number of zero neighbourhoods is a zero neighbourhood.

Theorem 2 (I. Schmets [6]). $C_c(X)$ is σ-barrelled if and only if every bounded σ-compact subset in X is relatively compact.

Proof. Again it is necessary to prove sufficiency only. Let V be a barrel in $C_c(X)$ and $V = \bigcap_{n=1}^{\infty} \langle K_n, \varepsilon_n \rangle$. Then $\bigcup_{n=1}^{\infty} K_n \subseteq K(V)$. Indeed, if $x_o \in \bigcup_{n=1}^{\infty} K_n \setminus K(V)$, there is $\mathcal{O}x_o$ such that $f \in V$ if $f(X \setminus \mathcal{O}x_o) = 0$ and (if $x_o \in K_n$) there is $f \in C(X)$ such that $f(x_o) > \varepsilon_n$ and $f(x) = 0$ for $x \in X \setminus \mathcal{O}x_o$. Therefore $f \notin \langle K_n, \varepsilon_n \rangle$ and hence $f \notin V$, which is a contradiction. It means that $\overline{\bigcup_{n=1}^{\infty} K_n} \subseteq K(V)$ and $\bigcup_{n=1}^{\infty} K_n$ is bounded in X from Lemma. Therefore $\langle \overline{\bigcup_{n=1}^{\infty} K_n}, \varepsilon_o \rangle \subseteq V$ where $\varepsilon_o = \inf \{\varepsilon_n\}$ (it is obvious that $\varepsilon_o > 0$). It follows that V is a zero neighbourhood in $C_c(X)$. The theorem is proved.

Remind that a topological vector space is called infrabarrelled if every barrel which absorbs bounded sets is a zero neighbourhood. A subset $A \subseteq X$ is called semibounded if every lower semi-continuous nonnegative function f which is bounded on every compact is bounded on A. We give a simple proof of the following characterization of infrabarrelledness of $C_c(X)$.

Theorem 3 (S. Warner [7]). $C_c(X)$ is infrabarrelled if and only if every semibounded set in X is relatively compact.

Proof. We shall show that if V is a barrel in $C_c(X)$ which absorbs bounded sets then K(V) is semibounded in X. Suppose the contrary. Let g be a lower semi-continuous mapping bounded on all compact sets, $g \geq 0$ and g be unbounded on K(V). Find $x_n \in K(V)$ such that $g(x_n) > n$ for every $n \in N$. There is a neighbourhood $\mathcal{O}x_n$ of the point x_n such that $g(y) > n$ for every $y \in \mathcal{O}x_n$. Let $f_n \in C(X)$ and $f_n(X \setminus \mathcal{O}x_n) = 0$, $f_n \notin V$, $f_n : X \to [0,1]$. It follows that $n \cdot f_n(x) \leq g(x)$ for every $x \in X$ and $n \cdot f_n \notin V$. The set $\{n \cdot f_n : n \in N\} \subseteq C_c(X)$ is bounded in $C_c(X)$ (because g is bounded on every compact set) and not absorbed by V because $\frac{1}{n}(n \cdot f_n) \notin V$, which contradicts our assumption. Therefore, K(V) is semibounded in X. The remaining part of our proof of Theorem 3 coincides with the proof of Theorem 1.

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