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ON HIGHER EIGENVALUES OF VARIATIONAL INEQUALITIES

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Abstract: A new criterion for the existence of eigenvalues of a class of unilateral eigenvalue problems is given.

Key words: Variational inequalities, eigenvalue problems, plate buckling.

Classification: 49H05, 73H10

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1. Introduction. Let H be a real Hilbert space and $K \subset H$ a closed and convex cone with its vertex at zero, that is, a set such that $t\,u \in K$ for all $t > 0$ and for all $u \in K$. Furthermore, we assume that K is nonempty.
By $a(u,v)$ and $b(u,v)$ we denote bounded, real and symmetric bilinear forms defined on H.

In this paper we make the following assumptions:

(1) There exists a $c > 0$ such that $a(v,v) \geq c\,\|v\|^2$ for all $v \in H$, and

(2) $b(u,v)$ is completely continuous on H.

We are interested in the eigenvalue problem for the variational inequality

(3) $u \in K$: $a(u,v-u) \geq \mu\, b(u,v-u)$ for all $v \in K$,

where μ is a real eigenvalue parameter. That is, we look for nontrivial solutions u of (3) and for associated eigenvalues.

In [3,4], there was proved that the problem (3) has infinitely many eigenvalues if for the eigenvectors of the equation

(4) $u \in H$: $a(u,v) = \lambda b(u,v)$ for all $v \in H$

certain conditions are fulfilled. Particularly, one has to assume that infinitely many eigenvectors of (4) lie in the interior of the cone K.

There exist also higher eigenvelues if no eigenvector of (4) belongs to the cone, cf. [8,9]. An application to the clamped circular plate is given in [10].

For further results, applications and references with respect to (3) see [2,11]. We mention that in [5], there was first proved a result for the nonsymmetric case.

In this paper we give a variational approach for the proof of existence of eigenvalues of the inequality (3). Furthermore, one obtains lower and upper bounds of these eigenvalues if the eigenvalues of the equation (4) are known.

To simplify the presentation, we consider here the problem (3) which is linear with respect to the operators. The results stay true for nonlinear problems of type

(5) $u \in K$: $(f'(u), v-u) \geq \mu (g'(u), v-u)$ for all $v \in K$.

Here f', g' are the first Gâteaux derivatives of functionals f, g defined on H. Under certain assumptions for f, g, cf. [8], the eigenvalues which we obtain, are also points of bifurcation of associated nonlinear inequalities of type (5) if we assume that (3) is the linearization of (5).

In the case of equations it was proved in [1] under suitable assumptions that eigenvalues of the linearized problem are also points of bifurcation and vice versa.

For variational inequalities one can check easily under certain assumptions on f and g that a point of bifurcation is also an eigenvalue of the associated linear problem. But not every eigenvalue of the linear problem is a point of bifurcation, as the following easy example shows.

Set $f(x) = 1/2\, x_1^2 + x_2^2 - x_1 x_2^2$, $g(x) = 1/2\, (x_1^2 + x_2^2)$ where $x = (x_1, x_2)$ and $K = \{x \in R^2 / x_1 \geq 0\}$. The number $\mu_0 =$ is an eigenvalue of the associated linear problem (3) with $a(x,y) = x_1 y_1 + 2x_2 y_2$, $b(x,y) = x_1 y_1 + x_2 y_2$. One can easily check that μ_0 is no point of bifurcation of (5), that means, there does not exist a sequence of solutions $x^{(n)} \neq 0$ and associated eigenvalues $\mu^{(n)}$ of (5) such that $x^{(n)} \to 0$ and $\mu^{(n)} \to$ $\to 2$ as $n \to \infty$.

2. <u>Some known results</u>. We assume that the equation (4) possesses at least n+1 positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ $\dots \leq \lambda_n \leq \lambda_{n+1}$. Let $u_1, \dots, u_n$, u_{n+1} be the associated eigenvectors which are orthonormal with respect to $a(u,v)$.

Set E_n = linear hull $\{u_1, \dots, u_n\}$ and $B_n^s = \{u \in E_n / a(u,u) = s\}$, $0 < s < \infty$. By P we denote the projection operator of H onto E_n which is assumed to be orthogonal with respect to $a(u,v)$.

<u>Definition 1</u> [1]. The set A is said to be contractible within a set R, if there exists a homotopy $H(t,u)$, $0 \leq t \leq 1$, $u \in A$, such that $H(0,u) = u$, $H(t,u) \in R$ for all $0 \leq t \leq 1$ and for all $u \in A$, and $\{H(1,u)/u \in A\}$ consists of one element of R.

Set $K^s = \{v \in K / a(v,v) = s\}$, $0 < s < \infty$.

<u>Definition 2</u> [8]. N is the class of all compact sets $F \subset H$ such that

a) $F \subset K^1$,

b) there exists an $\alpha > 0$ such that

$$\min_{u \in F} b(u,u) \geq \lambda_{n+1}^{-1} + \alpha ,$$

where α does not depend on F,

c) F is not contractible within the set $R = \{u \in H / Pu \neq 0\}$.

This definition of N depends on n. Here we have used slightly different notations as in [8], in particular, Definition 5.1 b) in [8] was changed.

If $N \neq \emptyset$, then there exists u such that

$$b(u,u) = \sup_{F \in N} \min_{w \in F} b(w,w),$$

which is also a solution of the variational inequality, cf. [8]. The proof is based on the topological technique due to Krasnosel'skiĭ [1]. More precisely we have the

<u>Theorem</u>. Assume $N \neq \emptyset$, then there exists an eigenvalue μ_n of the inequality (3) with

$$\lambda_n \leq \mu_n \leq \frac{\lambda_{n+1}}{1 + \alpha \lambda_{n+1}} .$$

Furthermore, if the eigenspace to λ_n does not belong to the cone K, then there exists an eigenvalue μ_n of (3) such that we have $\lambda_n < \mu_n$ in the previous inequality.

The second assertion of the Theorem says that one obtains in this general case an eigenvalue of the variational inequality which is not eigenvalue of the associated equation (4). The inequality $c \geq \lambda_{n+1}^{-1} + \alpha$, where $c = b(u,u)$, follows directly from our definition of N. The estimate $c \leq \lambda_n^{-1}$ one obtains in the same manner as in [8]. Since $c = \mu_n^{-1}$, cf. [8], we get the inequality of the Theorem.

One sufficient condition for $N \neq \emptyset$ was given in [8,9]: Let $H_1 \subset H$ be a closed subspace with $H_1 \subset K$. We consider the equation

$$u \in H_1: \ a(u,v) = \varkappa\, b(u,v) \text{ for all } v \in H_1$$

and assume that there exist at least n positive eigenvalues.

Lemma 1 [7,8]. If $\varkappa_n < \lambda_{n+1}$, then $N \neq \emptyset$.

3. A new criterion for $N \neq \emptyset$. Set
$M^1 = \{u \in E_n / a(u,u) \leq 1\}$ and
$V = \{v \in E_n^{\perp} / u+v \in K \text{ for all } u \in M^1\}$
where $E_n^{\perp}$ is the orthogonal complement to E_n.

Lemma 2. We have $N \neq \emptyset$ if the following assumptions are fulfilled: 1) There exists a $v \in E_n^{\perp}$ with $u+v \in K$ for all $u \in M^1$.
2) There exists a $\eta > 0$ such that

$$\lambda_n^{-1} - \lambda_{n+1}^{-1} \geq \eta + \min_{v \in V} (\lambda_{n+1}^{-1} + \eta) a(v,v) - b(v,v)$$

Proof. Observe first that for an arbitrary fixed $v \in V$

$$F = \left\{ \frac{u+v}{\sqrt{a(u+v,u+v}} \ / \text{ for all } u \in E_n^1 \right\}, \text{ where}$$

$E_n^1 = \{u \in E_n \ / \ a(u,u)=1\}$, is not contractible within R since E_n^1 is within R homotopically removable in F by the homotopy

$$H(t,u) = \frac{u+tv}{\sqrt{a(u+tv,u+tv}}, \ 0 \leq t \leq 1.$$

It remains to show that b) of Definition 2 is fulfilled under the assumptions of Lemma 2. Set

$$w = \frac{u+v}{\sqrt{a(u+v,u+v)}}.$$

We get

$$b(w,w) = \frac{1}{1+a(v,v)}\{b(u,u) + b(v,v)\} \geq \frac{1}{1+a(v,v)}\{\lambda_n^{-1} + b(v,v)\}.$$

The lemma is proved if there exists a $v \in V$ such that

$$\frac{1}{1+a(v,v)}\left\{\lambda_n^{-1} + b(v,v)\right\} \geq \lambda_{n+1}^{-1} + \eta \cdot$$

'he last inequality can be written as

$$\lambda_n^{-1} - \lambda_{n+1}^{-1} \geq \eta + (\lambda_{n+1}^{-1} + \eta)a(v,v) - b(v,v).$$

There exists a solution of the minimum problem in b) since V is closed and convex and since we have

$$b(v,v) \leq \lambda_{n+1}^{-1}\, a(v,v) \text{ for all } v \in E_n^{\perp}$$

relying on

$$\lambda_{n+1}^{-1} = \max_{v \in E_m^{\perp} \setminus \{0\}} \frac{b(v,v)}{a(v,v)} \cdot \qquad \text{Q.E.D.}$$

<u>Corollary</u>. Assume that λ_{n+1} has the multiplicity p, that is, we have $\lambda_n < \lambda_{n+1} = \dots = \lambda_{n+p} < \lambda_{n+p+1}$. Further we assume that for an eigenvector v_o from the linear hull $\{u_{n+1},\dots$ $\dots,u_{n+p}\}$ the inclusion $u+v_o \in K$ is true for all $u \in M^1$. Then $N \neq \emptyset$.

<u>Proof</u>. By setting $v = v_o$ into the right hand side of the inequality of Lemma 2 we obtain

$$\eta + (\lambda_{n+1}^{-1} + \eta)a(v_o,v_o) - \lambda_{n+1}^{-1}\, a(v_o,v_o) = \eta + \eta\, a(v_o,v_o).$$

The inequality in 2) of Lemma 2 is fulfilled for a sufficiently small $\eta > 0$ since we have assumed that $\lambda_n < \lambda_{n+1}$.

Q.E.D.

The corollary covers some results of M. Kučera [2,3,4] concerning the symmetric case.

It follows from this corollary that there exist eigenvalues of the variational inequality (3) which are not eigenvalues of the corresponding equation, if eigenvectors lie in the interior $K^{\bullet}$ of the cone K.

Dr. M. Kučera pointed out to me that the assumptions of Lemma 2 are also satisfied if there exists an eigenvector $v_0 \in \partial K$ (the boundary of K) corresponding to λ_{n+1} and there exists $v_1 \in K^0 \cap E_n^{\perp}$. This follows since $(1-t)v_0 + tv_1$, $0 < t < 1$, satisfies 1) and for t small it satisfies also 2) of Lemma 2.

In order to illustrate the Theorem we consider a problem for the clamped circular plate [10]. Let

$$\Omega_R = \{x \in R^2 \ / \ x_1^2 + x_2^2 < R^2\}, \ 0 < R < \infty ,$$

and K the cone

$$K = \{v \in H_{2,2}(\Omega_R) \ / \ v(x) \geq 0 \text{ for } x \in A, \ v(x) \leq 0 \text{ for } x \in B\}.$$

By A, B we denote subsets of the annulus

$$\{x \ / R_1 < (x_1^2 + x_2^2)^{1/2} < R\}, \ 0 < R_1 < R.$$

The compressive forces are acting in the inner normal direction. The inequality modelling this problem is given by

$$u \in K: \int_{\Omega_R} \Delta u \Delta (v-u) dx \geq \lambda \int_{\Omega_R} \{ u_{x_1}(v-u)_{x_1} + u_{x_2}(v-u)_{x_2} \} dx$$

for all $v \in K$.

Let τ_n be the zeros of the Bessel functions $J_{\mu}(x)$, $\mu = 1,2,\dots$, which are ordered according to their magnitudes $\tau_1 < \tau_2 < \dots$ $\dots$ ($\tau_1 = 3.832$, $\tau_2 = 5.136$, $\tau_3 = 6.380, \dots$).

From the Theorem and Lemma 1 it follows, provided $A \cup B \neq \emptyset$ and $\tau_n / \tau_{n+1} < R_1/R$, that there exists an eigenvalue λ_n of the above variational inequality with $\tau_n^2/R^2 < \lambda_n \leq \tau_n^2/R_1^2$.

From Lemma 2 it follows that there exist infinitely many eigenvalues which are not eigenvalues of the associated equation, if A B is a nonempty set of finitely many points of Ω_R.

This result obtained also M. Kučera [3,4] from his theory of variational inequalities based on a penalty technique.

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