Jerzy Płonka
Subdirectly irreducible groupoids in some varieties

Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 4, 631--645

Persistent URL: http://dml.cz/dmlcz/106261

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
24,4 (1983)

SUBDIRECTLY IRREDUCIBLE GROUPOIDS IN SOME VARIETIES

J. PŁONKA

Abstract: In one special variety of groupoids we study free groupoids, subdirectly irreducible groupoids and the lattice of subvarieties.

Key words: Groupoid, subdirectly irreducible groupoid, variety.

Classification: 08A30

---

0. In this paper we consider only varieties of groupoids i.e. varieties of type (2) with the fundamental operation $x \cdot y$ and we accept the terminology from [2]. In [3] two varieties Σ_2 and Σ_3 of groupoids were considered where Σ_2 was defined by the identities

(1) $x \cdot x = x$

(2) $(x \cdot y)\, z = (x \cdot z) \cdot y$

(3) $x \cdot (y \cdot z) = x \cdot y$

(4) $(x \cdot y) \cdot y = x \cdot y$

and Σ_3 was defined by (1)-(3) and

(4') $(x \cdot y) \cdot y = x$ (see also [2], pp. 394-395).

In [3] it was shown that:

If a groupoid $\mathcal{G}$ belongs to Σ_2 or Σ_3 and the operation $x \cdot y$ depends on both variables in $\mathcal{G}$ then there exist in $\mathcal{G}$ exactly n n-ary polynomials depending on n variables.

In [4] all subdirectly irreducible groupoids in Σ_2 and Σ_3 were found.

In this paper we study the join $\Sigma_2 \vee \Sigma_3$. In Section 1 we prove that $\Sigma_2 \vee \Sigma_3$ is defined by the identities (1)-(3) and the identity

(5) $((x\cdot y)\cdot y)\cdot y = x\cdot y$.

We show that the only subvarieties of $\Sigma_2 \vee \Sigma_3$ are Σ_2, Σ_3, the trivial variety T i.e. the variety defined by the identity $x=y$ and the variety Σ_0 defined by the identity $x\cdot y = x$ (see Theorem 1).

In Theorem 2, Section 1 we describe the free algebras in $\Sigma_2 \vee \Sigma_3$. In Section 2 we find all subdirectly irreducible groupoids in $\Sigma_2 \vee \Sigma_3$.

For a variety K of type (2) we denote by E(K) the set of all identities of type (2) satisfied in all groupoids from K. A term φ of type (2) constructed by means of the operation $\cdot$ will be called a multiplication term. We shall use the notation $(\dots(x\cdot \underbrace{y)\cdot y\dots)\cdot y}_{n \text{ times}} = x\cdot y^n$

1. Example 1. Let X be a set such that $|X|>1$. Denote $B = \{\langle a,A\rangle : a\in A\subseteq X\}$. Consider a groupoid $\mathcal{G} = (B;\cdot)$ where $\langle a,A\rangle\cdot\langle a',A'\rangle = \langle a, A\cup\{a'\}\rangle$. Then $\mathcal{G}$ satisfies (1)-(4) so $\mathcal{G}\in \Sigma_2$, but $\mathcal{G}$ does not satisfy (4').

Example 2. Let $Z_4 = (\{0,1,2,3\};+)$ be a cyclic group with addition modulo 4. Consider a groupoid $\mathcal{G} = (\{0,1,2,3\};\cdot)$ where $x\cdot y = 3x+2y$. Then $\mathcal{G}$ satisfies (1)-(3) and (4') so $\mathcal{G}\in \Sigma_3$, but it does not satisfy (4).

Let Σ be the variety of groupoids defined by (1)-(3) and

(5). Let α be an ordinal. A multiplication term φ on variables $x_0, x_1, \dots, x_\beta, \dots$ ($\beta < \alpha$) will be called a reduced iteration if φ is of the form

(6) $x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}$ where all variables $x_{i_1}, \dots, x_{i_n}$ are different, $i_2 < i_3 < \ldots < i_n$, $0 < k_j \leq 2$ for $j = 2, \dots, n$.

Lemma 1. For any multiplication term φ there exists a reduced iteration of the form (6) such that the identity $\varphi = x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}$ belongs to $E(\Sigma)$.

Proof. In fact by (3) we can reduce all open parentheses standing after a variable in φ . Then we get $\varphi = (\ldots(x_{s_1} \cdot x_{s_2}) \ldots) x_{s_r}$ belongs to $E(\Sigma)$. By (2) the order of variables $x_{s_2}, \dots, x_{s_r}$ is arbitrary and we get $\varphi = (\ldots(x_{i_1} \cdot x_{i_1}) \cdot \ldots) \cdot x_{i_1}) \cdot x_{i_2}) \cdot \ldots) x_{i_2} \cdot \ldots)\ x_{i_n} \cdot \ldots) \cdot x_{i_n}$ belongs to $E(\Sigma)$ where $i_1 = s_1$ and $i_2 < i_3 < \ldots < i_n$. Now by (1) and (5) we get the statement of the Lemma.

Lemma 2. If two reduced iterations $x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}$ and $x_{j_1} \cdot x_{j_2}^{q_2} \cdot \ldots \cdot x_{j_m}^{q_m}$ are different then the identities (1)-(3) together with the identity

$$(7) \qquad x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n} = x_{j_1} \cdot x_{j_2}^{q_2} \cdot \ldots \cdot x_{j_m}^{q_m}$$

imply one of the following identities: (4), (4'), $x \cdot y = x$.

Proof. If $i_1 \neq j_1$ then multiplying (7) on left by x_{i_1} we get by (3) $x_{i_1} \cdot x_{j_1} = x_{i_1}$. If $i_1 = j_1$ but there exists i_r, $r \in \{2, \dots, n\}$ such that $i_r \notin \{j_2, \dots, j_m\}$ then putting in (7) x_{i_1} for all variables different from x_{i_r} we get by (1)-(3) $x_{i_1} \cdot x_{i_r} = x_{i_1}$ or $x_{i_1} \cdot (x_{i_r})^2 = x_{i_1}$. If the variables on both sides of (7) are the same but $k_r \neq q_r$ for some $1 < r \leq n$ then putting

x_{i_1} for all variables different from x_{i_r} we get $x_{i_1} \cdot x_{i_r} =$ $= x_{i_1} \cdot (x_{i_r})^2$. Thus anyway we get one of the identities from the lemma.

<u>Theorem 1</u>. The lattice of subvarieties of Σ consists of the varieties T, Σ_0, Σ_2, Σ_3 and Σ where $T \subset \Sigma_0$, $\Sigma_0 \subset \Sigma_2$, $\Sigma_0 \subset \Sigma_3$, Σ_2 and Σ_3 are incomparable and $\Sigma = \Sigma_2 \vee \Sigma_3$.

Proof. The varieties Σ_2 and Σ_3 are incomparable (see Examples 1 and 2). Obviously any of the varieties T, Σ_0, Σ_2, Σ_3 is a subvariety of Σ since any of the identities $x=y$, $x \cdot y = x$, (4),(4') implies (5). Obviously $T \subset \Sigma_0$, $\Sigma_0 \subset \Sigma_2$, $\Sigma_0 \subset \Sigma_3$. On the other hand, J. Dudek proved in [1] that T and Σ_0 are the only subvarieties of Σ_2 and Σ_3 and all are different. Thus to complete the proof it is enough to show that if K is a proper subvariety of Σ then K is a subvariety of Σ_2 or Σ_3. Let

(8) $\quad (\varphi = \psi) \in E(K) \setminus E(\Sigma)$.

By Lemma 1, $\varphi = x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}$ and $\psi = x_{j_1} \cdot x_{j_2}^{q_2} \cdot \ldots \cdot x_{j_m}^{q_m}$ so $\varphi =$ $= \psi$ implies (7) where by (8) the sides of (7) are different. Now by Lemma 2, K is a subvariety of Σ_2 or Σ_3.

Example 3. In the set $\{0,1,2\}$ let us define an operation $\oplus$ putting

$$(9) \qquad x \oplus y = \begin{cases} x+y & \text{if } x+y \leq 2 \\ x+y-2 & \text{otherwise} \end{cases}$$

Let us consider a groupoid $\mathcal{G} = (\{0,1,2\} \times \{0,1,2\}; \cdot)$ where

$$\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = \begin{cases} \langle x_1, y_1 \rangle & \text{if } x_1 = x_2 \\ \langle x_1, y_1 \oplus x_2 \rangle & \text{otherwise.} \end{cases}$$

Then $\mathcal{G}$ satisfies (1)-(3) and (5) so $\mathcal{G}$ belongs to Σ and $\mathcal{G}$ satisfies neither (4) nor (4').

Let α be an ordinal. If $a \in \{0,1,2\}^{\alpha}$ we shall denote by $a(k)$ the k'th coordinate of a. Let us denote by p_k the element of $\{0,1,2\}^{\alpha}$ for which $p_k(k) = 1$ and $p_k(i) = 0$ for $i \neq k$. We denote by B the set of all $a \in \{0,1,2\}^{\alpha}$ having a finite number of coordinates different from 0. Finally let $B_{\alpha} = \{\langle p_k, a\rangle : k < \alpha, a \in B, a(k) = 0\}$.

We define a groupoid $\mathcal{L}_{\alpha} = (B_{\alpha}; \cdot)$ where

$$\langle p_k, a\rangle \cdot \langle p_{k_1}, a_1\rangle = \begin{cases} \langle p_k, a\rangle & \text{if } k = k_1 \\ \langle p_k, a'\rangle & \text{otherwise} \end{cases}$$

where $a'(i) = a(i) \oplus p_{k_1}(i)$; $\oplus$ is defined by (9).

<u>Theorem 2</u>. A free groupoid in the variety Σ with α free generators is isomorphic to $\mathcal{L}_{\alpha}$.

Proof. Let $\mathbb{F}_{\alpha}$ be the set of all multiplication terms on variables $x_0, x_1, \ldots, x_{\beta}, \ldots$, $\beta < \alpha$. Let $\sim$ be a relation in $\mathbb{F}_{\alpha}$ defined by the formula $\varphi \sim \psi \Longleftrightarrow (\varphi = \psi) \in E(\Sigma)$. A free algebra with α free generators in Σ is isomorphic to the algebra $\mathcal{F}_{\alpha} = (\{[\varphi]_{\sim}\}_{\varphi \in \mathbb{F}_{\alpha}}; \cdot)$. By Lemma 1 any term φ has a representation in the form $\varphi = x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}$. But this representation is unique. In fact if $\varphi = x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}$ and $\varphi = x_{j_1} \cdot x_{j_2}^{q_2} \cdot \ldots \cdot x_{j_m}^{q_m}$ where the right sides of the last identities are different then by Lemma 2 one of the identities (4), (4') or $x \cdot y = x$ belongs to $E(\Sigma)$, which contradicts Example 3.

Now the mapping h defined by the formula $h([x_{i_1} \cdot x_{i_2}^{k_2} \cdot \ldots \cdot x_{i_n}^{k_n}]_{\sim}) = \langle p_{i_1}, b\rangle$, where $b(i_j) = k_j$ for $2 \leq j \leq n$ and $b(r) = 0$ for $r \notin \{i_2, \ldots, i_n\}$ - sets up an isomorphism of $\mathcal{F}_{\alpha}$ onto $\mathcal{L}_{\alpha}$. In fact h is 1-1 since the representation from

Lemma 1 is unique and h is a homomorphism by (1)-(3) and (5).

2. For a class K of groupoids we shall denote by P(K), S(K), H(K) and I(K) the classes of all products, subgroupoids, homomorphic images and isomorphic copies of groupoids from K, respectively. If $\{X_i\}_{i\in I}$ is a partition of a set X we shall denote by $e(\{X_i\}_{i\in I})$ the equivalence relation induced by this partition.

Let us consider the following 6 groupoids

$\mathcal{G}_1 = (\{a\};\cdot)$.

$\mathcal{G}_2 = (\{a,b\};\cdot)$ where $x\cdot y = x$ for any $x,y\in\{a,b\}$.

$\mathcal{G}_3 = (\{a,b,\varkappa_1\};\cdot)$ where $a\cdot\varkappa_1=b$, $b\cdot\varkappa_1=a$ and $x\cdot y=x$ otherwise.

$\mathcal{G}_4 = (\{a,b,c,\varkappa_1\};\cdot)$ where $a\cdot\varkappa_1=b$, $b\cdot\varkappa_1=a$ and $x\cdot y=x$ otherwise.

$\mathcal{G}_5 = (\{a,c,\varkappa_2\};\cdot)$ where $a\cdot\varkappa_2=c$, and $x\cdot y=x$ otherwise.

$\mathcal{G}_6 = (\{a,b,c,\varkappa_1,\varkappa_2\};\cdot)$ where $a\cdot\varkappa_1=b$, $b\cdot\varkappa_1=a$, $a\cdot\varkappa_2=$
$= b\cdot\varkappa_2=c$ and $x\cdot y=x$ otherwise.

It was proved in [4] that

(i) a groupoid $\mathcal{G}$ belongs to Σ_2 and it is subdirectly irreducible iff $\mathcal{G}$ is isomorphic to one of the groupoids $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_5$.

(ii) A groupoid $\mathcal{G}$ belongs to Σ_3 and is subdirectly irreducible iff $\mathcal{G}$ is isomorphic to one of the groupoids $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$.

<u>Lemma 3</u>. The groupoid $\mathcal{G}_6$ belongs to Σ, moreover $\mathcal{G}_6 \in$ $\in$ HSP $\{\mathcal{G}_3, \mathcal{G}_5\}$.

In fact the set $S = (\{a,b,\varkappa_1\}\times\{a,b,\varkappa_2\})\setminus\{\langle\varkappa_1,\varkappa_2\rangle\}$ is a subalgebra of $\mathcal{G}_3\times\mathcal{G}_5$. So the algebra $\mathcal{T} = (S;\cdot)$ belongs to SP$\{\mathcal{G}_3, \mathcal{G}_5\}$. Further, a relation

$e=e(\{\{\langle a,a\rangle\},\{\langle b,a\rangle\},\{\langle a,c\rangle,\langle b,c\rangle\},\{\langle \varkappa_1,a\rangle,$
$\langle \varkappa_1,c\rangle\},\{\langle a,\varkappa_2\rangle,\langle b,\varkappa_2\rangle\}\})$
is a congruence in $\mathcal{V}$. Finally, the algebra $\mathcal{V}/e$ is isomorphic to $\mathcal{G}_6$.

Lemma 4. The groupoid $\mathcal{G}_6$ is subdirectly irreducible.

Proof. It is enough to show that if R is a congruency in $\mathcal{G}_6$ such that $[a]_R \neq [b]_R$ then $R = \omega$ where ω is the diagonal. We shall write $[x]$ instead of $[x]_R$. In fact, let $c \in [a]$. Then $b = a \cdot \varkappa_1 R\ c \cdot \varkappa_1 = cR\ a$. So bRa - a contradiction. The same contradiction gives the assumption that $c \in [b]$. If $c \in [\varkappa_1]$ then $a = a \cdot c\ R\ a \cdot \varkappa_1 = b$ - a contradiction. If $c \in [\varkappa_2]$ then $a = a \cdot c\ R\ a \cdot \varkappa_2 = c$ - a contradiction (see the first case). So $[c] = \{c\}$. If $\varkappa_1 \in [a]$ then $b = a \cdot \varkappa_1 R\ a \cdot a = a$ - a contradiction. The same contradiction gives the assumption $\varkappa_1 \in [b]$. If $\varkappa_1 \in [\varkappa_2]$ then $a = b \cdot \varkappa_1 R b \cdot \varkappa_2 = c$ - a contradiction. So $[\varkappa_1] = \{\varkappa_1\}$. If $\varkappa_2 \in [a]$ then $a = a \cdot a\ R\ a \cdot \varkappa_2 = c$ - a contradiction. Analogously $\varkappa_2 \notin [b]$. Thus $R = \omega$.

Theorem 3. A groupoid $\mathcal{G}$ belongs to Σ and it is subdirectly irreducible iff $\mathcal{G}$ is isomorphic to one of the groupoids $\mathcal{G}_1,\dots,\mathcal{G}_6$.

Proof. $\Leftarrow$. For the groupoids $\mathcal{G}_1,\dots,\mathcal{G}_5$ the statement holds by Theorem 1, (i) and (ii). For the groupoid $\mathcal{G}_6$ the statement holds by Theorem 1, Lemma 3 and Lemma 4.

Before we prove the necessity we have to show some properties.

Let $\mathcal{G} = (G;\cdot)$.

(iii) $\mathcal{G} \in \Sigma$ iff the following conditions $1^\circ, 2^\circ$ and 3° are satisfied.

1° There exists a partition $\{G_i\}_{i\in I}$ of G such that for any $i\in I$ the set $\{h_i^j\}_{j\in I}$ of mappings from G_i into G_i is given.

2° The mappings h_i^j satisfy the following conditions:

$$\forall_{i\in I}\ h_i^i = id;\quad \forall_{i,j,s\in I}\ h_i^j \circ h_i^s = h_i^s \circ h_i^j;$$
$$\forall_{i,j\in I}\ h_i^j \circ h_i^j \circ h_i^j = h_i^j.$$

3° If $a\in G_i$, $b\in G_j$ then $a\cdot b = h_i^j(a)$.

The proof is analogous to that of Theorem 3 from [3].

(iv) If $\mathcal{G}$ is of the form from (iii), $a\in G_k$ then for any $i\in I$ one of the following cases holds.

(10) $h_k^i(a) = a$

(11) $h_k^i(a) = b \neq a$, $h_k^i(b) = b$

(12) $h_k^i(a) = b$, $h_k^i(b) = a$, $a\neq b$

(13) $h_k^i(a) = b$, $h_k^i(b) = c$, $h_k^i(c) = b$, $a\neq b$, $a\neq c$, $b\neq c$

If $\{R_s\}_{s\in S}$ is a set of nontrivial congruences in a groupoid $\mathcal{G}$ such that $\bigcap_{s\in S} R_s = \omega$ then the set $\{R_s\}_{s\in S}$ will be called a decomposition of $\mathcal{G}$. Obviously, if such a decomposition exists then $\mathcal{G}$ is subdirectly reducible.

For a set A we shall denote by D(A) the set of all 1-element subsets of A.

From now on we assume that a groupoid $\mathcal{G} = (G;\cdot)$ belongs to Σ, is subdirectly irreducible and is of the form from (iii)

Similarly like in [4] (Lemma 1) we can prove

Lemma 5. If for any $i,j\in I$, $h_i^j = id$ then $\mathcal{G}$ is isomorphic to $\mathcal{G}_1$ or to $\mathcal{G}_2$.

In view of Lemma 5 in the sequel we shall assume that

(14) $$\exists_{i,j\in I}\ h_i^j \neq id$$

Let us put $J = \{j\in I: |G_j| > 1\}$.

Lemma 6. $|J| = 1$.

Proof. By (14) we have $|J| \geq 1$. Similarly like in [4] (Lemma 2) we can prove $|J| \leq 1$.

By Lemma 6 we can denote by k the unique element of J.

Put $I' = I \setminus \{k\}$. So for any $i \in I'$ we have $|G_i| = 1$. Thus only mappings h_k^j for $j \in I'$ can be different from the identity.

Lemma 7. If $i, j \in I'$ and $i \neq j$ then $h_k^i \neq h_k^j$.

The proof is analogous to that of Lemma 3 from [4].

Let $I_o = \{i \in I' : h_k^i \neq id\}$. By (14) we have $I_o \neq \emptyset$.

For any $i \in I_o$ we define two relations R_i and R^i as follows:
$a R_i b$ iff $a = b$ or $a, b \in G_k$, $b = h_k^i(a)$, and $a = h_k^i(b)$;
$a R^i b$ iff $a = b$ or $a, b \in G_k$ and $h_k^i(a) = h_k^i(b)$.

Similarly like in [4] we can prove that any of R_i and R^i is a congruence of $\mathcal{G}$.

Lemma 8. For any $i \in I_o$ we have $R_i \neq \omega$ or $R^i \neq \omega$.

In fact, since $|G_j| = 1$ for $j \in I'$, so it must exist $a \in G_k$ such that $h_k^i(a) \neq a$. Consequently one of the cases (11),(12) or (13) holds and $|[a]_{R_i}| > 1$ or $|[a]_{R^i}| > 1$.

Lemma 9. For any $i \in I_o$ we have $R_i = \omega$ or $R^i = \omega$.

In fact, $R_i \cap R^i = \omega$ since if $a R_i \cap R^i b$ then $a = b$ or $a, b \in G_k$ and $a = h_k^i(b) = h_k^i(a) = b$. Thus if both R_i and R^i are different from ω then $\{R_i, R^i\}$ is a decomposition of $\mathcal{G}$ - a contradiction.

Lemma 10. If for some $i \in I_o$ we have $R_i = \omega$, then for $a \in G_k$ exactly one of the cases (10) or (11) holds. If for some $i \in I_o$ we have $R^i = \omega$, then for $a \in G_k$ exactly one of the cases

(10) or (12) holds.

In fact, the case (13) is impossible by Lemma 9. If $R_i = \omega$ then (12) is impossible. If $R^i = \omega$ then (11) is impossible.

We denote $I_o^2 = \{i \in I_o : R_i = \omega\}$, $I_o^3 = \{i \in I_o : R^i = \omega\}$. By Lemma 8 and 9 we have $I_o = I_o^2 \cup I_o^3$ and $I_o^2 \cap I_o^3 = \emptyset$.

Lemma 11. If $I_o^3 = \emptyset$, then $\mathcal{G}$ is isomorphic to $\mathcal{G}_5$. If $I_o^2 = \emptyset$ then $\mathcal{G}$ is isomorphic to $\mathcal{G}_3$ or $\mathcal{G}_4$.

Proof. If $I_o^3 = \emptyset$ then by Lemma 10 and (iii) we infer that $\mathcal{G}$ satisfies (4) and by (i) and (14), $\mathcal{G}$ is isomorphic to $\mathcal{G}_5$. If $I_o^2 = \emptyset$ then by Lemma 10 and (iii) we infer that $\mathcal{G}$ satisfies (4') and by (ii) and (14), $\mathcal{G}$ is isomorphic to $\mathcal{G}_3$ or $\mathcal{G}_4$.

Q.E.D.

In view of Lemma 11 from now on we assume that

(15) $$I_o^2 \neq \emptyset \neq I_o^3.$$

Denote $R_\cap = (\bigcap_{i \in I_o^3} R_i) \cap (\bigcap_{i \in I_o^2} R^i)$.

Lemma 12. Any congruence class $[a]_{R_\cap}$ is either 1-element or is of the form $[a]_{R_\cap} = \{a,b\}$ where $a \neq b$, for any $i \in I_o^3$ we have $h_k^i(a) = b$ and $h_k^i(b) = a$ and for any $i \in I_o^2$ we have $h_k^i(a) = h_k^i(b) \notin [a]_{R_\cap}$.

Proof. For $i \in I_o^3$ any congruence class $[a]_{R_i}$ is at most 2-element. So if $|[a]_{R_\cap}| > 1$ then it mus be $[a]_{R_i} \subseteq [a]_{R_\cap}$. Consequently if $|[a]_{R_\cap}| > 1$ then $[a]_{R_\cap} = [a]_{R_i} = \{a,b\}$ where $a,b \in G_k$. Moreover for any $i \in I_o^3$ we have $h_k^i(a) = b$ and $h_k^i(b) = a$. Let $j \in I_o^2$, $|[a]_{R_\cap}| > 1$ and $[a]_{R_\cap} = \{a,b\}$. So

(16) $$h_k^j(a) = h_k^j(b).$$

By (15) and by the first part of the proof there exists $i \in I_o^3$ such that

(17) $$h_k^i(a) = b \text{ and } h_k^i(b) = a.$$

Let us assume that $h_k^j(a) \in [a]_{R_\cap}$ and e.g. $h_k^j(a) = b$. Then by (16) and (17) we get $h_k^j h_k^i(a) = b$, $h_k^i h_k^j(a) = a$, which contradicts 2^o. Analogously $h_k^j(a) \neq a$.

Let us denote $R(2) = \{R^i\}_{i \in I_o^2}$ and $R(3) = \{R_i\}_{i \in I_o^3}$

Lemma 13. The set G_k contains exactly one 2-element class of the congruence $R_\cap$ and exactly one 1-element class of the congruence $R_\cap$.

Proof. If $R_\cap = \omega$ then obviously we have a decomposition of $\mathcal{G}$, namely $\{R_i\}_{i \in I_o^3} \cup \{R^i\}_{i \in I_o^2}$, since any of these congruences is not trivial. If $R_\cap \neq \omega$ then by Lemma 12 there exists a 2-element class of the congruence $R_\cap$. If there exist two different 2-element classes $[a]_{R_\cap}$ and $[a']_{R_\cap}$ included in G_k then two congruences $e(\{[a]_{R_\cap}\} \cup D(G \setminus [a]_{R_\cap}))$ and $e(\{[a']_{R_\cap}\} \cup D(G \setminus [a']_{R_\cap}))$ form a decomposition of $\mathcal{G}$ - a contradiction. Denote $Q = [a]_{R_\cap}$. By Lemma 12 it is easy to check that the relation $e_Q = e(\{G_k \setminus Q\} \cup D(Q) \cup D(G \setminus G_k))$ is a congruence of $\mathcal{G}$. We shall show that

$$|G_k \setminus Q| = 1$$

In fact it cannot be $G_k \setminus Q = \emptyset$ since $I_o^2 \neq \emptyset$ and by Lemma 12 it must be for $j \in I_o^2$, $h_k^j(a) \notin Q$.

If $|G_k \setminus Q| > 1$ then e_Q is nontrivial and $R(2) \cup R(3) \cup \{e_Q\}$ is a decomposition of $\mathcal{G}$.

Proof. $\Longrightarrow$ of Theorem 3. If any h_k^i is the identity, then by Lemma 5, $\mathcal{G}$ is isomorphic to $\mathcal{G}_1$ or $\mathcal{G}_2$. Otherwise by Lemma 6 there exists exactly one $k \in I$ such that $|G_k| > 1$ and (14) holds. If $I_o^3 = \emptyset$, then by Lemma 11, $\mathcal{G}$ is isomorphic to $\mathcal{G}_5$. If $I_o^2 = \emptyset$ then by Lemma 11, $\mathcal{G}$ is isomorphic to $\mathcal{G}_3$ or $\mathcal{G}_4$. If (15) holds then by Lemma 13 we can denote by a,b,c the elements of G_k where $[a]_{R_\cap} = [b]_{R_\cap} = \{a,b\}$ and $[c]_{R_\cap} = \{c\}$. By Lemma 12 for any $i \in I_o^3$ we have $h_k^i(a) = b$, $h_k^i(b) = a$ and $h_k^i(c) = c$. So by Lemma 7 we have $|I_o^3| = 1$. Let us put $I_o^3 = \{i_o\}$ and denote by $\varkappa_1$ the only element of G_{i_o}. Analogously for any $j \in I_o^2$ we have by Lemma 12: $h_k^j(a) = h_k^j(b) = h_k^j(c) = c$. So by Lemma 7 we have $|I_o^2| = 1$. Put $I_o^2 = \{j_o\}$ and denote by $\varkappa_2$ the only element of G_{j_o}.

It must be $I_o' \setminus I_o = \emptyset$. In fact, if $m \in I_o' \setminus I_o$ and d is the only element of G_m, then two congruences $e(\{\{d\}, G\setminus\{d\}\})$, $e(\{\{c,d\}, D(G)\setminus\{c,d\}\})$ form a decomposition of $\mathcal{G}$. So $G_k = \{a,b,c\}$, $G \setminus G_k = \{\varkappa_1, \varkappa_2\}$ and G satisfies formulas of multiplication in $\mathcal{G}_6$. Thus $\mathcal{G}$ is isomorphic to $\mathcal{G}_6$ where the isomorphism is defined by denoting elements of G in the above way.

Q.E.D.

By Birkhoff theorem (see [2], p. 124), we have

<u>Corollary 1</u>. A groupoid $\mathcal{G}$ belongs to Σ iff $\mathcal{G}$ is isomorphic to a subdirect product of a family of groupoids $\mathcal{G}_2 - \mathcal{G}_6$

<u>Corollary 2</u>. A groupoid $\mathcal{G}$ belongs to Σ iff $\mathcal{G}$ can be embedded into some cartesian power of $\mathcal{G}_6$.

In fact, any of the groupoids $\mathcal{G}_1 - \mathcal{G}_5$ is a subalgebra of $\mathcal{G}_6$.

The groupoid $\mathcal{G}_6$ has 5 elements and generates Σ.

One can ask if there exist groupoids having less elements and generating Σ. The answer is "yes".

Let us consider two groupoids $\mathcal{G}_7$ and $\mathcal{G}_8$ defined as follows:

$\mathcal{G}_7 = (\{a,b,c,d\};\cdot)$ where $a\cdot d=b$, $b\cdot d=c$, $c\cdot d=b$, and $x\cdot y=x$ otherwise.

$\mathcal{G}_8 = (\{a,b,c,d\};\cdot)$ where $a\cdot c=a\cdot d=b$, $b\cdot c=b\cdot d=a$, $a=c\cdot b=d=d\cdot a=d\cdot b$, and $x\cdot y=x$ otherwise.

Theorem 4. $\mathcal{G}$ is a 4-element groupoid such that $HSP\{\mathcal{G}\} = \Sigma$ iff $\mathcal{G}$ is isomorphic to $\mathcal{G}_7$ or $\mathcal{G}_8$.

The number 4 is the least number of elements of groupoids generating Σ.

Proof. Consider in $\mathcal{G}_7$ two congruences R_1 and R_2 where $R_1 = e(\{\{a,c\},\{b\},\{d\}\})$, $R_2 = e(\{\{a\},\{b,c\},\{d\}\})$. Then $\mathcal{G}_7/R_1$ is isomorphic to $\mathcal{G}_3$ and $\mathcal{G}_7/R_2$ is isomorphic to $\mathcal{G}_5$. But $R_1 \cap R_2 = \omega$ so $\mathcal{G}_7$ is isomorphic to a subdirect product of $\mathcal{G}_3$ and $\mathcal{G}_5$. Consequently $\{\mathcal{G}_3, \mathcal{G}_5\} \subseteq HSP\{\mathcal{G}_7\}$ and by Lemma 3 and Corollary 2 we have $HSP\{\mathcal{G}_7\} = \Sigma$. The proof that $HSP\{\mathcal{G}_8\} = \Sigma$ is similar - it is enough to consider two congruences $R_3 = e(\{\{a\},\{b\},\{c,d\}\})$ and $R_4 = e(\{\{a,b\},\{c\},\{d\}\})$.

To prove that $\mathcal{G}_7$ and $\mathcal{G}_8$ are the only 4-element groupoids generating Σ let us assume that $\mathcal{G} = (\{a,b,c,d\};\cdot) \in \Sigma$. By (iii) we have $1 \leq |I| \leq 4$. If $|I| = 4$, then any G_i is one element and by (iii) $x\cdot y = x$ for any $x,y \in \{a,b,c,d\}$. Thus $\mathcal{G}$ belongs to Σ_0 and does not generate Σ by Theorem 1. The same case holds if $|I| = 1$.

In general, if $\mathcal{G}$ satisfies $x\cdot y = x$, then it cannot generate Σ . Excluding this case we have the following possibilities for $\mathcal{G}$, up to permutations of the elements a,b,c,d:

(c_1) $\mathcal{G}$ is isomorphic to $\mathcal{G}_7$ or $\mathcal{G}_8$.

For $I = \{1,2\}$, $G_1 = \{a,b,c\}$, $G_2 = \{d\}$ we have possibilities:

(c_2) $a \cdot d = b$, $b \cdot d = a$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_3$.

(c_3) $a \cdot d = c$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_2$.

(c_4) $a \cdot d = b \cdot d = c$ and $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_2$.

For $I = \{1,2\}$, $G_1 = \{a,b\}$, $G_2 = \{c,d\}$ we have possibilities:

(c_5) $a \cdot c = a \cdot d = b$, $b \cdot c = b \cdot d = a$, $c \cdot a = c \cdot b = d$, $d \cdot a = d \cdot b = c$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_3$.

(c_6) $a \cdot c = a \cdot d = b$, $c \cdot a = c \cdot b = d$ and $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_2$.

(c_7) $a \cdot c = a \cdot d = b$, $b \cdot c = b \cdot d = a$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_3$.

(c_8) $a \cdot c = a \cdot d = b$ and $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_2$.

For $I = \{1,2,3\}$, $G_1 = \{a,b\}$, $G_2 = \{c\}$, $G_3 = \{d\}$ we have possibilities:

(c_9) $a \cdot c = b$ and $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_2$.

(c_{10}) $a \cdot c = b$, $b \cdot c = a$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_3$.

(c_{11}) $a \cdot c = a \cdot d = b$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_2$.

(c_{12}) $a \cdot c = a \cdot d = b$, $b \cdot c = b \cdot d = a$, $x \cdot y = x$ otherwise. Then $\mathcal{G} \in \Sigma_3$.

However, by Theorem 1 only in the case (c_1), $\mathcal{G}$ generates Σ

Finally, if $\mathcal{G}$ has less than 4 elements and belongs to Σ, then in its decomposition into subdirect product of subdirectly irreducible groupoids from Σ, $\mathcal{G}_4$ and $\mathcal{G}_6$ cannot occur.

If only $\mathcal{G}_2$ or $\mathcal{G}_3$ occur, then $\mathcal{G} \in \Sigma_3$ and does not generate Σ.

If only $\mathcal{G}_2$ or $\mathcal{G}_5$ occur, then $\mathcal{G} \in \Sigma_2$ and does not generate Σ.

If $\mathcal{G}_3$ and $\mathcal{G}_5$ occur, then $\mathcal{G}$ is isomorphic both to $\mathcal{G}_3$

and to $\mathcal{G}_5$ by projections,which is a contradiction since $\mathcal{G}_3$ is not isomorphic to $\mathcal{G}_5$.

References

[1] J. DUDEK: A new characterization of groupoids with at most n essentially n-ary polynomials, Bull. de la Soc. Royale des Sci. de Liège, 48, n°9-10(1980),390-392.

[2] G. GRÄTZER: Universal algebra, Springer-Verlag, 1979.

[3] J. PŁONKA: On algebras with n distinct essentially n-ary operations, Algebra Universalis,vol.1,fasc.1(1971), 73-79.

[4] J. PŁONKA: Subdirectly irreducible algebras with exactly n essentially n-ary polynomials, to appear.

Institute of the Polish Academy of Sciences, ul. Kopernika 18,
Wrocław, Poland

(Oblatum 12.9.1983)