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REMARKS ON HAUSDORFF CONTINUOUS MULTIFUNCTION AND SELECTIONS

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Abstract. Continuity properties of multifunctions and existence of continuous selections are investigated.

Key words. Multifunctions, Hausdorff distance, selections.

Classification: 54 C 60, 54 C 65.

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1. Introduction. Let X be a metric space and let Y be a real normed space. Denote by $\mathfrak{B}$ the space of all closed convex bounded subsets of Y with nonempty interior endowed with Hausdorff distance. In this note we establish some properties of multifunctions which are used in [1] in order to study the structure of the solution set of the Cauchy problem (*) $\dot{x} \in \partial F(t,x)$, $x(0) = x_0$. In [1] it is supposed that $F : [0,1] \times Y \to \mathfrak{B}$ is Hausdorff continuous and Y is a real reflexive Banach space. The existence of solutions of (*) could be proved directly. However in [1], we establish a more precise result stating that almost all (in the sense of the Baire category) solutions of $\dot{x} \in F(t,x)$, $x(0) = x_0$ are solutions of (*). In Section 2 we introduce the terminology and review some elementary properties of Hausdorff continuous multifunctions. In Section 3 we prove the existence of (nontrivial) continuous multivalued selections for multifunctions $F : X \to \mathfrak{B}$.

2. Notations and preliminaries. Let 2^Y be the family of nonempty subsets of the real normed space Y. We shall consider the following subfamilies of 2^Y: $\mathcal{F} = \{A \in 2^Y \mid A$ is bounded$\}$, $\mathcal{K} = \{A \in 2^Y \mid A$ is closed bounded$\}$, $\mathcal{C} = \{A \in 2^Y \mid A$ is closed convex bounded$\}$, $\mathfrak{B} = \{A \in 2^Y \mid A$ is closed convex bounded with nonempty interior$\}$, $\mathfrak{Q} = \{A \in 2^Y \mid A$ is open convex bounded$\}$, $\mathcal{U} = \{A \in 2^Y \mid A$ is convex with nonempty interior$\}$. Let (X,e) be a metric space. For any set $A \subset X$ we denote by $\operatorname{int} A$, $\bar{A}$, ∂A respectively the interior, the closure, the boundary of A. If $A \subset X$ is nonempty, $\operatorname{diam} A$ stands for the diameter of A.

For any $u \in X$ we put $S(u,r) = \{x \in X \mid e(x,u) < r\}$, $r > 0$, $\bar{S}(u,r) = \{x \in X \mid e(x,u) \le r\}$, $r \ge 0$. For notational convenience the unit balls $S(0,1)$, $\bar{S}(0,1)$ in Y are denoted by S, $\bar{S}$. For any $A, B \in \mathfrak{F}$ define $h(A,B) = \inf\{t > 0 \mid A \subset B + tS,\ B \subset A + tS\}$. As is well known, h is a pseudometric in $\mathfrak{F}$, $\mathfrak{Q}$ while it is a metric (Hausdorff distance) in $\mathcal{X}$, $\mathcal{C}$, $\mathfrak{B}$. For any $u \in X$ and $A \subset X$ $(A \ne \emptyset)$, we set $d(u,A) = \inf\{e(u,a) \mid a \in A\}$. A multifunction $F : X \to 2^Y$ is said to be Hausdorff lower semicontinuous "Hausdorff l.s.c." (resp. Hausdorff upper semicontinuous "Hausdorff u.s.c.") at $x_0 \in X$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $F(x_0) \subset F(x) + \varepsilon S$ (resp. $F(x) \subset F(x_0) + \varepsilon S$) whenever $x \in S(x_0,\delta)$. F is said to be Hausdorff continuous at x_0 if is Hausdorff l.s.c. and Hausdorff u.s.c. at x_0.

<u>Proposition 2.1</u>. <u>Let</u> $F : X \to \mathcal{C}$ <u>be Hausdorff continuous</u>. <u>Then so is the multifunction</u> $F_c : X \to 2^Y$ (resp. $\partial F : X \to \mathcal{X}$) <u>given by</u> $F_c(x) = Y \setminus F(x)$ (resp. $(\partial F)(x) = \partial F(x)$), $x \in X$.

<u>Proof</u>. It is routine to see that F_c is continuous. To prove that ∂F is continuous take $x_0 \in X$ and let $\varepsilon > 0$. There is a $\delta > 0$ such that for each $x \in S(x_0,\delta)$ we have $h(F(x), F(x_0)) < \varepsilon$, $h(F_c(x), F_c(x_0)) < \varepsilon$. Since $\partial F(x) = F(x) \cap \overline{F_c(x)} \subset (F(x_0) + \varepsilon S) \cap (F_c(x_0) + \varepsilon S) = \partial F(x_0) + \varepsilon S$, and $\partial F(x_0) = F(x_0) \cap \overline{F_c(x_0)} \subset (F(x) + \varepsilon S) \cap (F_c(x) + \varepsilon S) = \partial F(x) + \varepsilon S$ it follows that ∂F is continuous.

<u>Lemma 2.2</u>. <u>Let</u> $A, B \in \mathfrak{B}$ <u>satisfy</u> $A \cap B \supset \bar{S}(y_0,r)$, $r > 0$. <u>Let</u> $\varepsilon > 0$. <u>Then</u> $A \cap (B + \sigma S) \subset A \cap B + \varepsilon S$ <u>where</u> $\sigma = \varepsilon r / \operatorname{diam} A$.

<u>Proof</u>. Let $y \in A \cap (B + \sigma S)$ and take $\tilde{y} \in B$ such that $|y - \tilde{y}| < \sigma$. Suppose $\tilde{y} \ne y$ (the case $\tilde{y} = y$ is trivial) and set $u = y_0 + r(\tilde{y} - y)/|\tilde{y} - y|$. Clearly $u \in \bar{S}(y_0,r) \subset A$. Since y and u lie in the convex set A, also $v(t) = ty + (1-t)u$ $(t \in [0,1])$ is in A. Analogously $\tilde{v}(t) = t\tilde{y} + (1-t)y_0$ $(t \in [0,1])$ is in B. An easy computation shows that $v(t^*) = \tilde{v}(t^*)$ for $t^* = r/(r + |\tilde{y} - y|)$. Hence, denoting by y^* the point $v(t^*) = \tilde{v}(t^*)$, we have $y^* \in A \cap B$; furthermore

$$|y - y^*| = (1 - t^*)|u - y| = |u - y|\ |\tilde{y} - y|/(r + |\tilde{y} - y|) < (\operatorname{diam} A)|\tilde{y} - y|/r < \varepsilon .$$

Thus $y = y^* + (y - y^*) \in y^* + \varepsilon S \subset A \cap B + \varepsilon S$ and the lemma is proved.

<u>Proposition 2.3</u>. <u>Let</u> $F : X \to \mathfrak{B}$ <u>and</u> $G : X \to \mathfrak{B}$ <u>be Hausdorff continuous</u>

<u>multifunctions such that</u> $F(x) \cap G(x)$ $(x \in X)$ <u>has nonempty interior. Then the multifunction</u> $F \cap G : X \to \mathcal{B}$ <u>given by</u> $(F \cap G)(x) = F(x) \cap G(x)$, $x \in X$, <u>is Hausdorff continuous</u>.

<u>Proof</u>. Fix $x_0 \in X$, $0 < \varepsilon < 1$, and take $k = \operatorname{diam}(F(x_0) \cup G(x_0))$. From the hypotheses it follows that there is a $\delta > 0$ such that for each $x \in S(x_0,\delta)$ we have: $F(x) \cap G(x) \supset S(y_0,r)$ (for some $y_0 \in Y$ and $r > 0$), and $h(F(x), F(x_0)) < \sigma$, $h(G(x), G(x_0)) < \sigma$, where $\sigma = \varepsilon r/(k+1)$. Hence, by virtue of Lemma 2.2, we have

$$F(x) \cap G(x) \subset (F(x_0) + \sigma S) \cap (G(x_0) + \sigma S)$$
$$\subset (F(x_0) + \sigma S) \cap G(x_0) + \varepsilon S \subset F(x_0) \cap G(x_0) + 2\varepsilon S, \quad x \in S(x_0,\delta).$$

Analogously $F(x_0) \cap G(x_0) \subset F(x) \cap G(x) + 2\varepsilon S$, and the proof is complete.

<u>Proposition 2.4</u>. <u>Let</u> $F : X \to \mathcal{B}$ <u>and</u> $G : X \to \mathcal{C}$ <u>be Hausdorff continuous and satisfy</u> $G(x) + rS \subset F(x)$, $x \in X$, <u>for some</u> $r > 0$. <u>Then the multifunction</u> $F \setminus G : X \to \mathcal{F}$ <u>given by</u> $(F\setminus G)(x) = F(x)\setminus G(x)$, $x \in X$, <u>is Hausdorff continuous</u>.

<u>Proof</u>. Let $x_0 \in X$ and take $0 < \varepsilon < r/2$. Take $\delta > 0$ such that $h(F(x), F(x_0)) < \varepsilon$, $h(G(x), G(x_0)) < \varepsilon$ for each $x \in S(x_0,\delta)$. From this and the fact that $G(x_0) + rS \subset F(x_0)$, $G(x) + rS \subset F(x)$ it is not difficult to obtain $h(F(x)\setminus G(x), F(x_0)\setminus G(x_0)) < 2\varepsilon$.

<u>Remark 2.5</u>. The statement of Proposition 2.1 fails if $\mathcal{C}$ is replaced by $\mathcal{K}$. If in the Proposition 2.3 the assumption that $F(x) \cap G(x)$ have nonempty interior is replaced by $F(x) \cap G(x) \neq 0$ $(x \in X)$, the conclusion is no longer true. If in the Proposition 2.4 the hypothesis $G(x) + rS \subset F(x)$, $x \in X$, is replaced by $G(x) \subset F(x)$, the conclusion is not true in general.

3. <u>Multivalued selections of multifunctions</u>. For each $A \in \mathcal{B}$ let $\sigma_A = \sup\{r > 0 \mid \text{there is } a \in A \text{ such that } S(a,r) \subset A\}$. Evidently, $\sigma_A > 0$.

<u>Lemma 3.1</u>. Let $F : X \to \mathcal{B}$ be Hausdorff l.s.c. (resp. u.s.c.). Then the function $\sigma_F : X \to \mathbb{R}$ given by $\sigma_F(x) = \sigma_{F(x)}$, $x \in X$, is l.s.c. (resp. u.s.c.). In particular σ_F is continuous whenever F is Hausdorff continuous.

<u>Proof</u>. Let F be Hausdorff l.s.c. and, for a contradiction, suppose that σ_F is not l.s.c.. Then there are $x_0 \in X$, $\varepsilon > 0$, and a sequence $\{x_n\} \subset X$ converging to x_0 such that $\sigma_F(x_n) < \sigma_F(x_0) - \varepsilon$, $n \in \mathbb{N}$. Since F is Hausdorff

l.s.c., there is $n_0 \in \mathbb{N}$ such that $F(x_0) \subset F(x_{n_0}) + (\varepsilon/2)S$. We have $\sigma_F(x_{n_0}) + \varepsilon < \sigma_F(x_0)$, thus there are $y \in F(x_0)$ and $r \in \mathbb{R}$, $\sigma_F(x_{n_0}) + \varepsilon < r \leq \sigma_F(x_0)$, such that $S(y,r) \subset F(x_0)$. Therefore $S(y,\sigma_F(x_{n_0}) + \varepsilon/2) + (\varepsilon/2)S \subset S(y,r) \subset F(x_0) \subset F(x_{n_0}) + (\varepsilon/2)S$ and so $S(y,\sigma_F(x_{n_0}) + \varepsilon/2) \subset F(x_{n_0})$. Hence $\sigma_F(x_{n_0}) + \varepsilon/2 \leq \sigma_F(x_{n_0})$, a contradiction, and σ_F is l.s.c.. If F is Hausdorff u.s.c. the proof is similar. The last statement is obvious.

__Lemma 3.2.__ Let $A \in \mathcal{B}$. For each $0 < \mu < \sigma_A$ put $A_\mu = \{a \in A \mid S(a,\mu) \subset A\}$ and let $A_0 = A$ if $\mu = 0$. Then $A_\mu \in \mathcal{B}$ and, furthermore, we have

$$A_\mu = \{a \in A \mid d(a, \partial A) \geq \mu\} \tag{3.1}$$

$$\partial A_\mu = \{a \in A \mid d(a, \partial A) = \mu\} . \tag{3.2}$$

__Proof.__ When $\mu = 0$ we have $A_0 \in \mathcal{B}$ and (3.1), (3.2) are true. Suppose $0 < \mu < \sigma_A$. From the definition of σ_A there is $a \in A$ and $\mu < r \leq \sigma_A$ such that $S(a,r) \subset A$. Since $S(a,r-\mu) + \mu S = S(a,r) \subset A$ it follows that $S(a,r-\mu) \subset A_\mu$ and so A_μ has nonempty interior. Let us prove that A_μ is convex.
To this end let $a_1, a_2 \in A_\mu$ that is $S(a_1,\mu) \subset A$, $S(a_2,\mu) \subset A$. Since A is convex, for each $t \in [0,1]$ we have $tS(a_1,\mu) + (1-t)S(a_2,\mu) = S(ta_1 + (1-t)a_2,\mu) \subset A$ and hence $ta_1 + (1-t)a_2 \in A_\mu$. Clearly A_μ is bounded and, as one can easily verify, also closed. Therefore $A_\mu \in \mathcal{B}$. Consider now (3.1). Let $a \in A_\mu$. Then $S(a,\mu) \subset A$ and hence $d(a, \partial A) \geq \mu$. Conversely, if $a \in A$ satisfies $d(a, \partial A) \geq \mu$, we have $S(a,\mu) \subset A$ thus $a \in A_\mu$. Therefore (3.1) is true.
Let us prove (3.2). Denote by B_μ the set on the right hand side of (3.2). Let $a \in \partial A_\mu$. Since $a \in A_\mu$, from (3.1) we have $d(a, \partial A) \geq \mu$. For a contradiction, suppose $d(a, \partial A) > r > \mu$. Evidently $S(a,r-\mu) + \mu S = S(a,r) \subset A$ which implies that $a \in \operatorname{int} A_\mu$, a contradiction. Hence $d(a, \partial A) = \mu$ and $a \in B_\mu$. Conversely, let $a \in B_\mu$. We have $a \in A_\mu$ for $B_\mu \subset A_\mu$. Suppose that $a \in \operatorname{int} A_\mu$ that is $S(a,r) \subset A_\mu$ for some $r > 0$. Then $S(a, \mu+r) = S(a,r) + \mu S \subset A$ from which we obtain $d(a, \partial A) \geq \mu + r$, a contradiction. Therefore $a \in \partial A_\mu$ and also (3,2) is true.

__Remark 3.3.__ Let $A \in \mathcal{B}$. For any $0 \leq \mu < \sigma_A$, put $A^0_\mu = \{a \in A \mid d(a,\partial A)>\mu\}$. Evidently $A^0_\mu = \operatorname{int} A_\mu$ thus A^0_μ is nonempty open convex bounded, that is $A^0_\mu \in \mathcal{Q}$.

Remark 3.4. If $A \in \mathcal{B}$ and $0 < \mu < \sigma_A$, we have $A_\mu + \mu S \subset A$. The inclusion can be strict. In fact simple examples show that $A \setminus (A_\mu + \mu S)$ can have nonempty interior.

Lemma 3.5. Let $A \in \mathcal{B}$. Let $0 < \mu < \sigma_A/2$ and take $0 < \varepsilon < \operatorname{diam} A$. There is then $\delta_0 > 0$, given by $\delta_0 = \varepsilon(\sigma_A/2 - \mu)/\operatorname{diam} A$ (resp. $\delta_0 = \min\{\mu, \varepsilon(\sigma_A/2 - \mu)/\operatorname{diam} A\}$) such that, whenever $0 \le \delta \le \delta_0$, we have $A_\mu \subset A_{\mu+\delta} + \varepsilon S$ (resp. $A_{\mu-\delta} \subset A_\mu + \varepsilon S$). Moreover, if $0 < \mu < \sigma_A/4$, we have $h(A_\mu, A) \le (\mu \operatorname{diam} A)/(\sigma_A/2 - \mu)$.

Proof. Let A, μ, ε and $0 \le \delta \le \delta_0 = \varepsilon(\sigma_A/2 - \mu)/\operatorname{diam} A$ be as in the statement. From the definition of σ_A, there is $a \in A$ such that $S(a, \sigma_A/2) \subset A$. Since A_μ and $A_{\mu+\delta}$ are in $\mathcal{B}$ (in fact $0 < \mu \le \mu + \delta < \sigma_A/2$) the inclusion $A_\mu \subset A_{\mu+\delta} + \varepsilon S$ $(0 \le \delta \le \delta_0)$ is true if we show that $\partial A_\mu \subset A_{\mu+\delta} + \varepsilon S$. To this end, let $y \in \partial A_\mu$ and suppose that $|y - a| \le \varepsilon$. Since $S(a, \mu+\delta) \subset S(a, \mu + (\sigma_A/2 - \mu)) = S(a, \sigma_A/2) \subset A$, we have $a \in A_{\mu+\delta}$ and hence $y = a + (y-a) \in A_{\mu+\delta} + \varepsilon S$. Now, suppose that $y \in \partial A_\mu$ is such that $|y - a| > \varepsilon$. Let $y^* = (1-t^*)y + t^* a$, where $t^* = \varepsilon / |y - a|$, and observe that $|y^* - y| = \varepsilon$. Observe that $S(a, \sigma_A/2 - \mu) + \mu S = S(a, \sigma_A/2) \subset A$ whence $S(a, \sigma_A/2 - \mu) = a + (\sigma_A/2 - \mu)S \subset A_\mu$. Also $y \in A_\mu$ thus, since A_μ is convex, we have

$$(3.3) \qquad A_\mu \supset (1 - t^*)y + t^*[a + (\sigma_A/2 - \mu)S] = y^* + t^*(\sigma_A/2 - \mu)S .$$

This implies that

$$d(y^*, \partial A_\mu) \ge t^*\left(\frac{\sigma_A}{2} - \mu\right) = \frac{\varepsilon}{|y - a|}\left(\frac{\sigma_A}{2} - \mu\right) \ge \frac{\varepsilon(\sigma_A/2 - \mu)}{\operatorname{diam} A} = \delta_0 .$$

Let $v \in \partial A$ be arbitrary. From (3.3), $y^* \in \operatorname{int} A_\mu$ whence the segment $[y^*, v]$ meets ∂A_μ in a point u and we have $|y^* - v| = |y^* - u| + |u - v|$. Evidently, $|y^* - u| \ge d(y^*, \partial A_\mu) \ge \delta_0 \ge \delta$. On the other hand $v \in \partial A$ and $u \in \partial A_\mu$ thus $|u - v| \ge \mu$. Hence $|y^* - v| \ge \delta + \mu$ and, since $v \in \partial A$ is arbitrary, we have $d(y^*, \partial A) \ge \mu + \delta$, thus $y^* \in A_{\mu+\delta}$. Since $y = y^* + (y - y^*) \in A_{\mu+\delta} + \varepsilon S$, the proof of the inclusion $A_\mu \subset A_{\mu+\delta} + \varepsilon S$ $(0 \le \delta \le \delta_0)$ is complete. The argument to prove that $A_{\mu-\delta} \subset A_\mu + \varepsilon S$ $(0 \le \delta \le \delta_0)$ is similar. It can be obtained (with few minor modifications) by replacing A_μ and $A_{\mu+\delta}$ in the above proof by $A_{\mu-\delta}$ and A_μ respectively. To prove the last statement of the proposition, suppose $0 < \mu < \sigma_A/4$. Let $\varepsilon = (\mu \operatorname{diam} A)/(\sigma_A/2 - \mu)$ and observe that $0 < \varepsilon < \operatorname{diam} A$. Moreover, $\delta_0 =$

min $\{\mu, \varepsilon(\sigma_A/2-\mu)/\text{diam } A\} = \mu$, thus we have $A_{\mu-\delta_0} \subset A_\mu + \varepsilon S$, that is $A \subset A_\mu + \varepsilon S$. Evidently $A_\mu \subset A$ and so $h(A_\mu, A) \le \varepsilon = (\mu \text{ diam } A)/(\sigma_A/2-\mu)$. This completes the proof.

Lemma 3.6. [2, p. 170]. Let $p_1 : X \to \mathbb{R}$ and $p_2 : X \to \mathbb{R}$ be an u.s.c. and a l.s.c. function such that $p_1(x) < p_2(x)$, $x \in X$. Then there exists a continuous function $p : X \to \mathbb{R}$ such that $p_1(x) < p(x) < p_2(x)$, $x \in X$.

Let $F : X \to \mathcal{B}$ be Hausdorff l.s.c.. By Lemma 3.1, σ_F is l.s.c. and positive and by Lemma 3.6 there is a continuous function $\mu : X \to \mathbb{R}$ satisfying $0 < \mu(x) < \sigma_F(x)/2$, $x \in X$. For each $x \in X$ set $F_{\mu(x)}(x) = \{y \in F(x) \mid d(y, \partial F(x)) \ge \mu(x)\}$, $x \in X$. Evidently, $F_{\mu(x)}(x) \in \mathcal{B}$ thus the multifunction $F_\mu : X \to \mathcal{B}$ given by $F_\mu(x) = F_{\mu(x)}(x)$, $x \in X$, is a multivalued selection of F.

Proposition 3.7. Let $F : X \to \mathcal{B}$ be Hausdorff l.s.c. (resp. continuous) and let $\mu : X \to \mathbb{R}$ be continuous and satisfy $0 < \mu(x) < \sigma_F(x)/2$, $x \in X$. Then the multifunction $F_\mu : X \to \mathcal{B}$ given by $F_\mu(x) = F_{\mu(x)}(x)$, $x \in X$, is also Hausdorff l.s.c. (resp. continuous). Moreover if $0 < \mu(x) < \sigma_F(x)/4$, $x \in X$, we have $h(F_\mu(x), F(x)) \le (\mu(x) \text{ diam } F(x))/(\sigma_F(x)/2 - \mu(x))$.

Proof. Let F be Hausdorff l.s.c. and suppose, for a contradiction, that F_μ is not so. Then there are $x_0 \in X$, $0 < \varepsilon < \text{diam } F(x_0)$, and a sequence $\{x_n\} \subset X$ converging to x_0 such that $F_{\mu(x_0)}(x_0) \not\subset F_{\mu(x_n)}(x_n) + \varepsilon S$, $n \in \mathbb{N}$. Let $\{y_n\} \subset Y$ be such that

$$(3.4) \qquad y_n \in F_{\mu(x_0)}(x_0) \qquad y_n \notin F_{\mu(x_n)}(x_n) + \varepsilon S, \qquad n \in \mathbb{N}.$$

By Lemma 3.5 we have $F_{\mu(x_0)}(x_0) \subset F_{\mu(x_0)+\delta_0}(x_0) + \varepsilon S$ where $\delta_0 = \varepsilon(\sigma_F(x_0)/2 - \mu(x_0))/\text{diam } F(x_0)$. Hence, for each $n \in \mathbb{N}$, $y_n \in F_{\mu(x_0)+\delta_0}(x_0) + \varepsilon S$ and so there is $z_n \in F_{\mu(x_0)+\delta_0}(x_0)$ satisfying $|y_n - z_n| < \varepsilon$. Moreover, since μ is continuous, there is $k \in \mathbb{N}$ such that whenever $n \ge k$ we have $\mu(x_0) - \delta_0/2 < \mu(x_n) < \mu(x_0) + \delta_0/2$ and, in particular, $\mu(x_0) > \mu(x_n) - \delta_0/2$. Consequently, $\mu(x_0) + \delta_0 > \mu(x_n) + \delta_0/2$ which implies that $F_{\mu(x_0)+\delta_0}(x_0) \subset F_{\mu(x_n)+\delta_0/2}(x_0)$, $n \ge k$. Since F is Hausdorff l.s.c. there is $k_1 \ge k$ such that $F(x_0) \subset F(x_n) + (\delta_0/2)S$ for all $n \ge k_1$. Furthermore, for each $n \ge k_1$ we have $z_n \in F_{\mu(x_n)+\delta_0/2}(x_0)$ which implies that $S(z_n, \mu(x_n) + \delta_0/2) \subset F(x_0)$. Hence, $z_n + (\mu(x_n) + \delta_0/2)S \subset F(x_0) \subset F(x_n) + (\delta_0/2)S$ thus $z_n + \mu(x_n)S \subset F(x_n)$, that is $z_n \in F_{\mu(x_n)}(x_n)$, if $n \ge k_1$. Then $y_n = z_n + (y_n - z_n) \in F_{\mu(x_n)}(x_n) + \varepsilon S$ for each $n \geq k_1$, in contradiction to (3.4). Therefore F_μ is Hausdorff l.s.c.

Now, suppose F Hausdorff continuous. To show that so is F_μ it is sufficient to prove that F_μ is u.s.c.. Arguing by contradiction one finds $x_0 \in X$, $0 < \varepsilon < \operatorname{diam} F(x_0)$, and a sequence $\{x_n\} \subset X$ converging to x_0 such that $F_{\mu(x_n)}(x_n) \not\subset F_{\mu(x_0)}(x_0) + \varepsilon S$, $n \in \mathbb{N}$. Let $\{y_n\} \subset Y$ be such that

$$(3.5) \qquad y_n \in F_{\mu(x_n)}(x_n) \qquad\qquad y_n \notin F_{\mu(x_0)}(x_0) + \varepsilon S \;, \qquad n \in \mathbb{N}.$$

By Lemma 3.5 there is $\delta_0 > 0$ given by $\delta_0 = \min\{\mu(x_0),\ \varepsilon(\sigma_F(x_0)/2 - \mu(x_0))/\operatorname{diam} F(x_0)\}$ such that $F_{\mu(x_0)-\delta_0}(x_0) \subset F_{\mu(x_0)}(x_0) + \varepsilon S$. By the continuity of μ there is $k \in \mathbb{N}$ such that $\mu(x_0) - \delta_0/2 < \mu(x_n) < \mu(x_0) + \delta_0/2$ if $n \geq k$. Thus, for each $n \geq k$, $\mu(x_0) > \mu(x_n) - \delta_0/2 > \mu(x_0) - \delta_0 \geq 0$ and hence $F_{\mu(x_n)-\delta_0/2}(x_0) \subset F_{\mu(x_0)-\delta_0}(x_0)$. On the other hand by the Hausdorff continuity of F there is $k_1 \geq k$ such that $F(x_n) \subset F(x_0) + (\delta_0/2)S$ if $n \geq k_1$. Since $y_n + \mu(x_n)S \subset F(x_n) \subset F(x_0) + (\delta_0/2)S$, it follows that $y_n + (\mu(x_n) - \delta_0/2)S \subset F(x_0)$. Hence for each $n \geq k_1$ we have $y_n \in F_{\mu(x_n)-\delta_0/2}(x_0) \subset F_{\mu(x_0)-\delta_0}(x_0) \subset F_{\mu(x_0)}(x_0) + \varepsilon S$, which contradicts (3.5). Therefore F_μ is Hausdorff u.s.c. . The last statement of the proposition follows from Lemma 3.5. This completes the proof.

Remark 3.8. Let $F : X \to \mathcal{B}$ be Hausdorff continuous. Let $\mu : X \to \mathbb{R}$ be continuous and satisfy $0 < \mu(x) < \sigma_F(x)/2$, $x \in X$. For each $x \in X$, put $F^0_{\mu(x)} = \{y \in F(x) \mid d(y, \partial F(x)) > \mu(x)\}$. From Remark 3.3 it follows that $F^0_{\mu(x)}(x) \in \mathcal{Q}$, thus the multifunction F^0_μ given by $F^0_\mu(x) = F^0_{\mu(x)}(x)$, $x \in X$, maps X into $\mathcal{Q}$. Since $F^0_\mu(x) = \operatorname{int} F_\mu(x)$, by virtue of Propositions 3.7 and 2.15, it follows that F^0_μ is Hausdorff continuous. Observe that also the multifunction $\partial F^0_\mu : X \to \mathcal{K}$ given by $(\partial F^0_\mu)(x) = \partial F^0_\mu(x)$, $x \in X$, is Hausdorff continuous since, by Proposition 2.5, $x \to \partial F_\mu(x)$ is so and $\partial F_\mu(x) = \partial F^0_\mu(x)$, $x \in X$.

Proposition 3.9. Let $F : X \to \mathcal{U}$ be Hausdorff l.s.c.. Then there exists a Hausdorff continuous multifunction $G : X \to \mathcal{B}$ and a positive continuous function $t : X \to \mathbb{R}$, satisfying $G(x) + t(x)S \subset F(x)$, $x \in X$.

Proof. Let $z \in X$. Since $F(z)$ has nonempty interior there are $G_z \in \mathcal{B}$ and $t_z > 0$ such that $G_z + 2t_z S \subset F(z)$. Furthermore, F is Hausdorff l.s.c. thus there is $\delta_z > 0$ such that $G_z + t_z S \subset F(x)$ for each $x \in S_z = \{u \in X \mid e(u,z) < \delta_z\}$. As $\{S_z\}_{z \in X}$ is an open covering of the metric space X, there is a partition of unity subordinated to $\{S_z\}_{z\in X}$. Hence there is a family $\mathfrak{P}$ of continuous functions $p_z : X \to [0,1]$, whose supports form a neighborhood finite closed covering of X; furthermore the support of each p_z lies in S_z, and $\sum_{z\in X} p_z(x) = 1$, $x \in X$. Set

$$t(x) = \sum_{z\in X} p_z(x)\, t_z \qquad G_0(x) = \sum_{z\in X} p_z(x)\, G_z , \qquad x \in X .$$

Observe that $t : X \to \mathbb{R}$ is continuous and positive while, as we shall see, G_0 is Hausdorff continuous. To this end, fix $x_0 \in X$ and $\varepsilon > 0$. For $r_0 > 0$ small enough there is only a finite number of functions $p_{z_i} \in \mathfrak{P}$ $(i = 1,2,\dots,k)$ whose supports meet $S(x_0,r_0)$. By the continuity of p_{z_i} there is $0 < r < r_0$ such that

$$|p_{z_i}(x) - p_{z_i}(x_0)| < \varepsilon \Big[\sum_{i=1}^{k} h(G_{z_i}, 0)\Big]^{-1}, \qquad x \in S(x_0,r) ,$$

where $i = 1,2,\dots,k$. Then, for each $x \in S(x_0,r)$, we have

$$h(G(x), G(x_0)) = h\Big(\sum_{i=1}^{k} p_{z_i}(x) G_{z_i}, \sum_{i=1}^{k} p_{z_i}(x_0) G_{z_i}\Big)$$

$$\le \sum_{i=1}^{k} h(p_{z_i}(x) G_{z_i}, p_{z_i}(x_0) G_{z_i}) \le \sum_{i=1}^{k} |p_{z_i}(x) - p_{z_i}(x_0)|\, h(G_{z_i}, 0) < \varepsilon$$

and G_0 is Hausdorff continuous at x_0. Moreover, we have $G_0(x) + t(x)S \subset F(x)$, $x \in X$. In fact, take any $x_0 \in X$ and denote by p_{z_i} $(i = 1,2,\dots,k)$ those functions in $\mathfrak{P}$ whose supports contain x_0. Since $x_0 \in S_{z_i}$ we have $G_{z_i} + t_{z_i} S \subset F(x_0)$, $i = 1,2,\dots,k$, and thus

$$G_0(x_0) + t(x_0)S = \sum_{i=1}^{k} p_{z_i}(x_0) G_{z_i} + \Big(\sum_{i=1}^{k} p_{z_i}(x_0) t_{z_i}\Big) S$$

$$= \sum_{i=1}^{k} p_{z_i}(x_0)(G_{z_i} + t_{z_i} S) \subset \sum_{i=1}^{k} p_{z_i}(x_0) F(x_0) = F(x_0) .$$

Then the multifunction G defined by $G(x) = \overline{G_0(x)}$, $x \in X$, maps X into $\mathfrak{B}$, is Hausdorff continuous, and satisfies $G(x) + t(x)S \subset F(x)$, $x \in X$. This completes the proof.

Remark 3.10. The above argument shows that, if we retain the hypotheses

(and notations) of Proposition 3.9, then $F : X \to \mathcal{U}$ admits a continuous single valued selection $g : X \to Y$ satisfying $g(x) + t(x)S \subset F(x)$, $x \in X$.
For continuous F, from X to the nonempty open convex subsets of Y, the existence of continuous single valued selections follows from Michael [4, Theorem 8.5]. Observe that if in Proposition 3.9 F is supposed to be lower semicontinuous (that is, whenever $V \subset Y$ is open in Y then the set $\{x \in X \mid F(x) \cap V \neq \emptyset\}$ is open in X), the existence of continuous single valued selections may fail. In fact, as shown by Michael [3, Example 6.3], there exists a lower semicontinuous multifunction, from [0,1] to the nonempty open convex subsets of a Banach space, which has no single valued continuous selections. This pathology is ruled out under the stronger hypothesis that F be Hausdorff l.s.c.

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