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REPRESENTING CHORDAL GRAPHS ON $K_{1,m}$

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Abstract: Chordal graphs are precisely those graphs that can be obtained as intersection graphs of subtrees of some tree T. It is shown that when T is $K_{1,n}$ the subclass of chordal graphs so obtained is precisely the split graphs.

Key words: Chordal graphs, split graphs, intersection graphs.

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1. Introduction. We will restrict our attention to finite connected simple graphs and will, in general, use the graph theoretic terminology of [1]. A graph G is chordal if and only if G contains no induced cycles C_n for $n>3$. G is said to be represented on a tree T if and only if G is isomorphic to the intersection graph of a set of distinct subtrees of T. An elegant theorem characterizing chordal graphs is the following.

Theorem 1 (Buneman [2], Gavril [3], Walter [6,7]). G can be represented on a tree if and only if G is chordal.

This theorem only requires that there exists some representing tree, so it is natural to ask, for a specified type of tree T, what kinds of chordal graphs can be represented on T. To date, only two such types of trees have been considered. Walter [6,7] characterized those chordal graphs that can be represented on a tree homeomorphic to $K_{1,3}$. Kabell [5] characterized the chordal graphs that can be represented as intersection graphs of infinite subgraphs of $S_\infty K_{1,n}$ where $S_\infty K_{1,n}$, the <u>infinite</u> <u>n-star</u>, is the graph obtained by taking n one-way infinite paths with a common end vertex. Here we allow the representing tree to be $K_{1,n}$ and show that the graphs represented on $K_{1,n}$ are precisely the split graphs. An extension to somewhat more general trees than $K_{1,n}$ is also considered.

2. <u>Results</u>. The neighborhood $N(x)$ of vertex x in graph G consists of those vertices adjacent in G to x. A graph $G = (V,E)$ is <u>split</u> if and only if there is a partition of the vertex set as $V = I \cup K$, where I is an independent set and K is complete. Furthermore, the partition $V = I \cup K$ can always be chosen so that K is a maximum clique [4]. Henceforth we shall assume that K has been chosen in this manner.

<u>Theorem 2</u>. A graph $G = (V,E)$ is split if and only if G can be represented on $K_{1,n}$ for some n.

<u>Proof</u>. Suppose $G = (V,E)$ can be represented by the intersection of subtrees of $K_{1,n}$. Let K be the set of vertices in V that correspond to subtrees containing the "central" vertex (of degree n) in $K_{1,n}$. Let I be the set of vertices in V that correspond to subtrees not containing the central vertex. Clearly K is complete, I is independent and V is partitioned into $I \cup K$.

Now suppose $G = (V,E)$ is split, where $V = I \cup K$ and $I = \{x_1,\dots,x_r\}$. We shall construct the required $K_{1,n}$ and a representation simultaneously by adding vertices (as required) to $K_{1,r}$. First, label the end vertices (of degree 1) in $T = K_{1,r}$ by the integers $1,\ldots,r$ and the vertex of degree r by 0. Define the subtree $T(x_i)$, corresponding to vertex x_i, by $T(x_i) = \{i\}$, for $i = 1,\ldots,r$. Next, let L, initially empty, denote a collection of subsets. For each $y \in K$, we consult L to see if $N_I(y) = N(y) \cap I$ is a member of the list L. If not, we add $N_I(y)$ to L and define $T(y) = N_I(y) \cup \{0\}$. If $N_I(y) \in L$ then we add a new end vertex α to the current T (joining it to vertex 0) and define $T(y) = N_I(y) \cup \{0,\alpha\}$. This procedure is repeated for all vertices $y \in K$. Upon completion, the process yields a $K_{1,n}$ and a set of distinct subtrees that represent G. □

The method of construction in the proof above actually provides a representation of G on $K_{1,n}$ using the smallest possible n. In this regard, it is important that K be chosen as a maximum clique. Figure 1 shows a split graph G with two vertex partitions $I \cup K$. In the first case, K is not a maximum

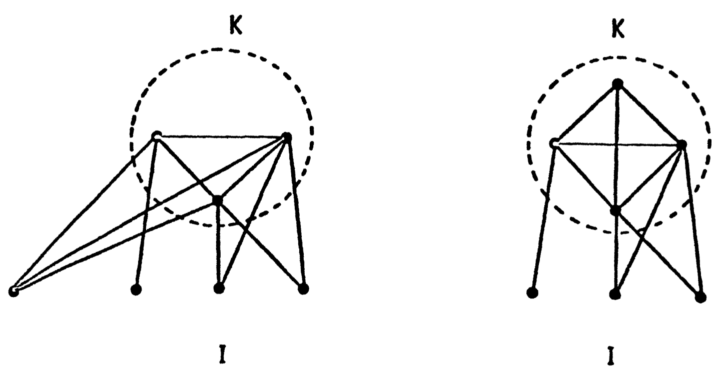

Figure 1. Two partitions of a split graph

clique and the construction above gives a representation of G on $K_{1,5}$. However, in the second case, K is a maximum clique and the construction gives a (minimal) representation on $K_{1,4}$.

Because the construction above is minimal (as is easily demonstrated), we have the following result.

Proposition. If $G = (V,E)$ is a split graph with $V = I \cup K$ and $K = \{y_1,\ldots,y_m\}$ a maximum clique, then the smallest n such that G can be represented on $K_{1,n}$ is given by

$$n = |I| + (|K| - |\{N_I(y_1),\ldots,N_I(y_m)\}|).$$

In the expression for n above, the last indicated cardinality just counts the number of distinct sets $N_I(y_j)$, so the quantity in parentheses is the number of vertices α added in the construction process.

We now turn our attention to representing graphs on a somewhat more general type of tree, namely a diameter three caterpillar T. That is, T is obtained from a single edge xy by joining a number of vertices to x and a number of vertices to y. For obvious reasons, such a tree is called a dumbbell.

A graph G is 3-split if and only if G is constructed by taking two split graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ with $V_1 = I^1 \cup K^1$ and $V_2 = I^2 \cup K^2$, and then adjoining a complete graph K as follows:

$$V(G) = V_1 \cup V_2 \cup V(K),\ E(G) = E_1 \cup E_2 \cup E(K) \cup E,$$

where E consists of all edges between K and $K^1 \cup K^2$ together with any arbitrary collection of edges between K and $I^1 \cup I^2$; see Figure 2. Observe that if G is 3-split, then the graph $G - X$ where X is K^1, K^2 or K is either split or the disjoint union of two split graphs.

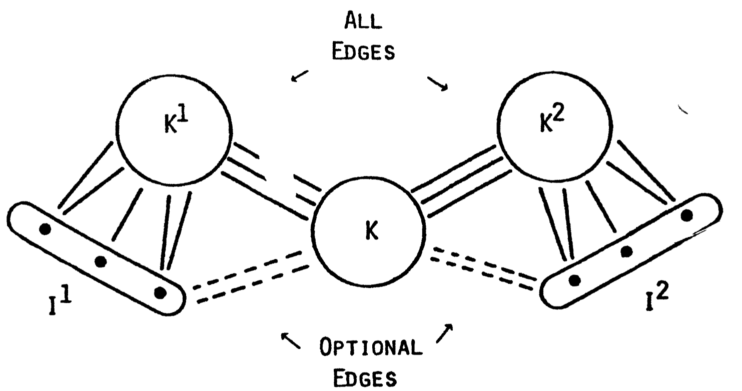

Figure 2. A schematic diagram of a 3-split graph

<u>Theorem 3</u>. A graph $G = (V,E)$ is 3-split if and only if G can be represented on a dumbbell.

<u>Proof</u>. The proof is a straightforward modification of the previous theorem. In this case, the appropriate identification is made between (a) vertices in K^1 and subtrees containing x but not y in the dumbbell, (b) vertices in K^2 and subtrees containing y but not x, and (c) vertices in K and subtrees containing both x and y. Also, vertices in I^1 and I^2 correspond respectively to end vertices joined to x and y in the dumbbell. The remaining argument parallels that given in the proof of Theorem 2. □

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