Angel Rafael Larotonda; Ignacio M. Zalduendo

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CONTINUITY OF CHARACTERS IN PRODUCTS OF ALGEBRAS

Angel R. LAROTONDA, Ignacio M. ZALDUENDO

Abstract: A topological m-convex algebra A over the complex field is called functionally continuous if every character of A is continuous. In this note we prove: a product of a family A_i ($i \in I$) of functionally continuous algebras is functionally continuous if and only if the cardinal of the set I is non measurable.

Key words: Topological algebras, character, functionally continuous, non measurable cardinals.

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§ 1. One of the automatic continuity problems is to find conditions on a topological m-convex algebra, A, which ensure that every character $h: A \to \mathbb{C}$ is continuous. Such an algebra is called functionally continuous. It is well known that Banach algebras are functionally continuous, whereas the same problem for Fréchet algebras remains unsolved.

In this note we are concerned with the stability of the property of functional continuity by products. In other words, given a family $(A_i)_{i\in I}$ of functionally continuous algebras, is the product algebra functionally continuous? We prove that this problem is related with the existence of measurable cardinals. More precisely, the product algebra is functionally continuous if and only if the cardinal of the set I is non-measurable.

Throughout, $(A_i)_{i\in I}$ will denote a family of algebras over the field of complex numbers, $\mathbb{C}$, and $A = \prod(A_i : i \in I)$. $\rho_i : A \to A_i$ will be the projection mappings, and we will write $\rho_i(x) = x_i$, and $x = (x_i)_{i\in I}$. We do not suppose the A_i to be unitary, but when they are, e_i will denote the idempotent element of A defined by $e_i = (\delta_{ij})_{j\in I}$, where δ_{ij} is Kronecker's delta. A character is a non-zero algebra homomorphism h from the algebra to the complex numbers. Note that when the algebra is unitary, $h(1) = 1$.

§ 2. We will prove now that every character h of A factors through some A_k (there is a character φ of A_k such that $h = \varphi\rho_k$) if and only if the cardinal of I is non-measurable. We will need the following lemma.

Lemma 1: If the cardinal of I is that of the continuum, every character of A factors through some A_k.

Proof: Let $h : A \to \mathbb{C}$ be a character, and $t : I \to \mathbb{C} - \{0\}$ a bijection. We suppose first that all the A_i are unitary.

The element $x = (t(i))_{i\in I}$ of A is invertible, so that $h(x) \neq 0$. Hence there is a unique $k \in I$ for which $t(k) = h(x)$. Let $y = x(1-e_k)-h(x).1.y$ is also invertible, because,

$$y_i = \begin{cases} t(i) - t(k), & \text{if } i \neq k \\ -t(i), & \text{if } i = k. \end{cases}$$

If $h(e_k)$ were zero, we would have a contradiction, for then $h(y) = h(x)\ (1-h(e_k))-h(x)=h(x)-h(x) = 0$. Therefore $h(e_k) \neq 0$; in fact, $h(e_k) = 1$, because e_k is idempotent. Then h factors through A_k, because Ker ρ_k is contained in Ker h: if $\rho_k(a) = 0$,

$$h(a) = h(a).h(e_k) = h(a.e_k) = h(0) = 0.$$

If the A_i are not unitary, we may consider $\Pi(A_i : i \in I)$ as a bilateral ideal of $\Pi\{A_i^+ : i \in I\}$ where A_i^+ is obtained by adjoining a unit to A_i, and apply Theorem C.1 of [2].

We shall also need the following:

Lemma 2: If the cardinal of I is lesser than, or equal to that of the continuum, then every character of A factors through some A_k.

Proof: Consider a disjoint union $J = I \cup I'$ so that the cardinal of J is the cardinal of the continuum, and $A_i = \mathbb{C}$ for all $i \in I'$. Then A may be identified with a bilateral ideal of $\Pi(A_j : j \in J)$, and we apply once again Theorem C.1 of [2], and Lemma 1.

We recall now that a cardinal α is said to be measurable if there is a set I whose cardinal is α , and a σ-additive measure μ defined on the σ-algebra of all subsets of I verifying:

i) For all $J \subset I$, $\mu(J)$ is either 0 or 1.

ii) For all $i \in I$, $\mu(\{i\}) = 0$.

iii) $\mu(I) = 1$; in other words, μ is non-trivial.

Obviously, all numerable cardinals are non-measurable. In fact, all known cardinals are non-measurable, but we will only use this in the case of the cardinal of the continuum. (See [3].)

We are now ready to prove:

Theorem 1: Let I be a set. Then the following are equivalent.

a) The cardinal of I is non-measurable.

b) For every family $(A_i)_{i\in I}$ of algebras, every character of the product $A = \prod\{A_i : i\in I\}$ factors through some A_k.

<u>Proof:</u> a) $\Longrightarrow$ b). Let h be a character of A. As in Lemma 1, we may suppose all A_i to be unitary, and it will be enough to see that $h(e_k) = 1$ for some $k\in I$.

If $J\subset I$, let e_J denote the idempotent element of A such that

$$p_i(e_J) = \begin{cases} 1, & \text{if } i\in J \\ 0, & \text{if } i\notin J. \end{cases}$$

For all $J\subset I$, $h(e_J)$ is either 0 or 1, so we may define

$$\mu : P(I) \to \{0,1\},\ \mu(J) = h(e_J)$$

μ is non-trivial, for $\mu(I) = h(1) = 1$. μ is readily seen to be finitely additive: if $J_1\cap J_2 = \emptyset$, e_{J_1} and e_{J_2} are orthogonal idempotents, so $\mu(J_1)$ and $\mu(J_2)$ cannot both be 1. To prove that μ is σ-additive, let $(J_n)_{n\in N}$ be a sequence of disjoint subsets of I. We must consider two cases:

First, if some J_m has measure 1, it is the only element in the sequence of measure 1, for if the same were true for J_n,

$$\mu(J_m\cup J_n) = h(e_{J_m\cup J_n}) = h(e_{J_m} + e_{J_n}) = h(e_{J_m}) + h(e_{J_n}) = 2,$$

absurd.

Hence, $1\geq \mu(\bigcup_m J_n)\geq \mu(J_m) = 1 = \sum_m \mu(J_n)$, so μ is σ-additive in this case.

On the other hand, if $\mu(J_n) = 0$ for all n, we must prove that $\mu(\ ) = 0$ where $J = \bigcup_m J_n$. If not, $\mu(J) = h(e_J) = 1$, so that h factors through $A_J = \prod\{A_i : i\in J\}$. But

$$A_J = \prod_m A_{J_n},\ \text{where } A_{J_n} = \prod\{A_i : i\in J_n\}.$$

By Lemma 2, h finally factors through some A_{J_n}, so there is an $n \in N$ for which $h(e_{J_n}) = \mu(J_n) = 1$, a contradiction.

Since the cardinal of I is non-measurable, the measure cannot be zero on the singleton $\{i\}$ for all $i \in I$. Therefore there is a $k \in I$ such that $\mu(\{k\}) = h(e_k) = 1$.

b) $\Rightarrow$ a). Let $\mu : P(I) \to \{0,1\}$ be a non-trivial measure. We will prove that $\mu(\{k\}) = 1$ for some $k \in I$. Every character h, of $A = \mathbb{C}^I$ factors through $\mathbb{C}$ ($h(x) = x_k$ for some $k \in I$, and all $x \in A$).

Let $M = \{x \in A: \mu(x^{-1}(0)) = 1\}$. M is an ideal of A:

$0 \in M$, for μ is non-trivial.

If $x \in M$ and $y \in M$, then $x+y \in M$, because

$$\mu(x^{-1}(0) \cup y^{-1}(0)) + \mu(x^{-1}(0) \cap y^{-1}(0)) = \mu(x^{-1}(0)) + \mu(y^{-1}(0))$$

so that $\mu(x^{-1}(0)) \cap y^{-1}(0))) = 1$, and therefore $\mu((x+y)^{-1}(0))) = 1$, because $x^{-1}(0) \cap y^{-1}(0) \subset (x+y)^{-1}(0)$.

If $x \in M$, and $a \in A$, $ax \in M$:

$1 = \mu(x^{-1}(0) \leq \mu((ax)^{-1}(0)) \leq 1$, because $x^{-1}(0) \subset (ax)^{-1}(0)$.

M has codimension one. To prove this we will see that all $x \in A$ may be written as

(*) $x = y + \lambda \,.\, 1$, where $y \in M$, and $\lambda \in \mathbb{C}$.

Let $\nu : P(\mathbb{C}) \to \{0,1\}$, $\nu(s) = \mu(x^{-1}(s))$. ν is then a non-trivial σ-additive measure, but the cardinal of the continuum is non-measurable, so $\nu(\{\lambda\}) = 1$, for some $\lambda \in \mathbb{C}$. Now write $y = x - \lambda.1$. Then $y \in M$, because

$\mu((x - \lambda 1)^{-1}(0)) = \mu(x^{-1}(\{\lambda\})) = \nu(\{\lambda\}) = 1$, which proves (*).

$h: A \to \mathbb{C}$ defined by $h(x) = \lambda$ if $x = y + \lambda.1$ ($y \in M$, $\lambda \in \mathbb{C}$), is a character, so $h(x) = x_k$, for some $k \in I$, and all

$x \in A$. Therefore $x - x_k.1 \in M$ for all $x \in A$. Then $x = e_k$, we have $e_k - 1 \in M$, so that $\mu(\{k\}) = \mu((e_k-1)^{-1}(0)) = 1$. The cardinal of I is therefore non-measurable.

§ 3. Now, if the algebras considered are topological algebras, we apply Theorem 1 to obtain immediately the following:

Theorem 2: Let I be a set. Then the following are equivalent:

a) The cardinal of I is non-measurable.

b) Every product $A = \prod\{A_i : i \in I\}$ of functionally continuous algebras is functionally continuous.

Note that when the cardinal of I is non-measurable, Theorem 2 says that every maximal ideal of codimension one of A is closed, but this does not mean that every maximal ideal is closed (as in the case of Banach algebras). For example any maximal ideal of the product $\mathbb{C}^N$ containing the direct sum $\mathbb{C}^{(N)}$ is necessarily dense, and therefore not closed, but of infinite codimension. Actually, in Theorem 2, the algebras need not be topological algebras (that is, with continuous sum and products); algebras with topologies making all characters continuous will do.

R e f e r e n c e s

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Departamento de Matemática , Facultad de Ciencias Exactas y
Naturales, Avenida Intendente Güiraldes 2620.
1428 Buenos Aires, Argentina

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