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NONSTANDARD MODELS OF ARITHMETIC AS AN ALTERNATIVE BASIS FOR CONTINUUM CONSIDERATIONS

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Abstract: From the point of view of

1) nonstandard models of arithmetic: A special type of strong cuts in the sense of Kirby and Paris are considered. It is proved that the pair of such a cut and the corresponding ground model may serve as a basis for an alternative construction of real numbers. Some other set theoretical properties are proved.

2) Nonstandard analysis: Using nonstandard methods a model for real numbers is constructed in a theory much weaker than Zermelo-Fraenkel set theory.

3) Alternative set theory: Considerations in a fragment of AST are made and a contribution to the shiftings of horizon problem is given.

Key words: Nonstandard model of Peano arithmetic, Alternative set theory, strong cut, indiscernibility relation, prolongation.

Classification: Primary 03H15
Secondary 03E70, 03H05

Introduction. We consider a fragment of AST (Alternative set theory). Every nonstandard model of Peano arithmetic (PA) or Zermelo-Fraenkel set theory for finite sets (ZF_{fin}) with a new predicate R added to the language may serve as a model for our fragment if R satisfies some properties given below. We prove among others that the elements satisfying R form a strong cut in the sense of Kirby and Paris (see [KP] for the definition and properties of strong cuts). The first point of view of our paper is a consequence of this property.

In our theory we construct a model for real numbers as an Archimedean really closed field with the supreme property. Archimedean property is meant with respect to the predicate R considered as "to be a natural number" and supreme property is meant with respect to the properties described in our theory. Hence the obtained structure may be on one hand richer and on the other hand poorer than the structure of standard real numbers. Namely if there is a nonstandard number α satisfying R, then two real numbers which differ from each other only in α-th position of the dyadic expansion must also differ in our structure, but they determine maximally one standard real number. On the other hand, if the ground model is countable then our structure must be countable, too (from the external point of view). If R denotes the property "to be a standard natural number" and the standard system of the model is $P(\omega)$, then both standard and our model for real numbers are isomorphic. The mentioned construction leads to the second point of view on our paper.

Our fragment is the smallest "reasonable" fragment of AST. "Reasonable" means here that we consider infinity phenomenon using parts of formally finite sets and, in addition, we want to have the class of all small natural numbers. If we interpret R as "to be an element of X", where X is a cut different from FN, we obtain a contribution to the shiftings of horizon problem. These facts lead to the third point of view on our paper.

The technical means used in the paper are taken from nonstandard analysis and AST. The procedures are only adopted for weaker formal apparatus. This adoption, however, is not quite obvious and especially the usage of nonstandard topological technique for proving set theoretical assertions is quite remarkable.

The paper is divided into two sections. In the first one we describe our theory, give some connections to the nonstandard models of arithmetic and construct two models for real numbers. In the second one we prove some set theoretical theorems in our theory.

The contents of the paper was referred and discussed in the Prague seminar on the AST.

§ 1. <u>The construction of real numbers</u>. We now describe the mathematical theory we shall work in. The language of the theory is the language of ZF set theory with a new predicate $x \in R$. For our convenience we add to the language variables for classes. Classes are understood as suitable abbreviations. Thus e.g. $X=Y$ is an abbreviation for the formula $(\forall x)(\varphi(x) \equiv \psi(x))$, where $X = \{x; \varphi(x)\}$ and $Y = \{x; \psi(x)\}$. Formulas containing $X \in$ are not correct.

Axioms: 1) All the axioms of ZF_{fin}+ axiom of regularity, for set formulas (i.e. formulas not using the new predicate $x \in R$).

2) $R \subset N$ (where N denotes the class of all natural numbers).

3) $(\forall \alpha \in R)(\alpha + 1 \in R)$

4) $(\forall X)(\forall \alpha \in R)(\exists y)(y = X \cap \alpha)$

5) $(\forall X)(\exists y)(X \cap R = y \cap R)$

Remarks: 1) In each model of PA we can define $x \in y$ iff the x-th member of the dyadic expansion of y is 1. In this manner we obtain a model of ZF_{fin} + the axiom of regularity.

2) N is defined as the class of ordinal numbers by the usual definition.

4) This axiom is in fact an axiom scheme due to our convention about classes. The quantification $(\forall X)$ means "for every formula ...". We accept this axiom since we want to bring properties of the class R nearer to the system of the standard natural numbers having the mentioned property.

5) If we interpret R as standard numbers, then this axiom expresses the assumption that every definable (with the parameter R) part of R is an element of the standard system of the model.

<u>Lemma 1</u>: Every nonempty class $X \subset R$ has the minimal element.

Proof: If $x \in X$ then $(x+1) \cap X$ is a nonempty set by the Axiom 4.

<u>Theorem 2</u>: $(\forall \alpha \in R)(\alpha \subseteq R)$ $(\mathrm{Compl}(R))$.

Proof: Let us put $X = \{\alpha \in R; \alpha \nsubseteq R\}$. If $X \neq 0$ then we put $m = \min(X)$. We have $(m-1) \subset R$, hence $(m-2) \in R$ and $(m-1) \in R$ (Axiom 3). Thus $m \subset R$ - a contradiction.

<u>Theorem 3</u>: If X is a class of functions with the following property: $f''R^n \subseteq R^k$ for suitable n, k then every function G from the primitive recursive closure of X has the property, too.

<u>Corollary 4</u>: 1) R is closed on addition, multiplication, exponentiation.

2) If $f''R \subseteq R$ then $(\forall \beta \in R)(\sum_{i=0}^{\beta} f(i) \in R)$.

Before proving Theorem 3 let us remind the definition of the primitive recursive closure. A function $F: N^k \longrightarrow N^m$ is in the primitive recursive closure of the class X if it is obtained by finitely many applications of the following three operations on functions from X and four basic functions described below.

Basic functions: 1) $\pi : N^1 \longrightarrow N^0$ $\quad \pi(\alpha) = (\ )$

2) $\varsigma : N^0 \longrightarrow N^1$ $\quad \varsigma(\ ) = 0$

3) $\iota : N^1 \longrightarrow N^1$ $\quad \iota(\alpha) = \alpha$

4) $\varsigma : N^1 \longrightarrow N^1$ $\quad \varsigma(\alpha) = \alpha + 1$

Basic operations: $F \circ G(x) = F(G(x))$

$F \times G(x) = \langle F(x), G(x) \rangle$

$$F^{\#}(\alpha, x) = \underbrace{F \circ F \circ \dots \circ F(x)}_{\alpha \text{ times}}$$

Let us now prove Theorem 3: The only one nontrivival part of the proof is that one for iteration ($F^{\#}$). Let $\alpha \in R$ and $x \in R^k$. Let us put $Y = \{\beta \in \alpha + 1;\ F^{\#}(\beta, x)$ is defined and $F^{\#}(\beta, x) \notin R^k\}$. We prove that $Y \neq 0$ leads to the contradiction. If $Y \neq 0$ then let m be the minimum of Y. We have $F^{\#}(m,x) = F(F^{\#}(m-1,x)) \in R^k$ - a contradiction.

The proof of the Corollary: The first part is evident. For the proof of the second part let us note that $\sum_{i=0}^{\beta} f(i)$ is the first component of $g^{\#}(k+2,0,0,0)$ where $g(x,y,z) = \langle x+y, f(z), z+1 \rangle$.

Remarks: 1) Let us note that in the proof we have not used Axiom 5.

2) The theorem may be generalized also for class functions defined with parameter R. But in this case we can define $F^{\#}(\alpha, x)$ only for $\alpha \in R$.

Let us now give two isomorphic constructions of real numbers in our theory:

1) We construct the field of rational numbers RN using the usual construction starting from N. Thus RN is a class definable by a set formula. The class R is understood as the system of "real" natural numbers - The Archimedean property will be understood with respect to R. The support of our structure will be the class BRN $\equiv \{ r \in RN;\ (\exists \alpha \in R)(|r| < \alpha)$. On BRN we define $x \doteq y \equiv (\forall \alpha \in R)(|x-y| < 1/\alpha)$ (x is infinitely close to y). The factor structure BRN/$\doteq$ can serve as a model for real numbers as we shall prove later.

2) For an arbitrary natural number $\alpha \in$ N-R we can construct a model for real numbers such that the support of the model is a part of α. For the construction we use the following intuition: We imagine α as α successive elements such that the distance between each element and its successor is $1/\sqrt{\alpha}$. The middle element is understood as 0. The support is now the class of elements having the distance from new 0 bounded by a number from R. The result of each arithmetical operation is defined as such an element of the structure which is near to the exact result (in the structure of rational numbers). The property "to be infinitely close" we define in the same way as in the case 1). Real numbers we obtain as the factor structure. Let us give the formal description: We put $\beta_0 = [\alpha/2]$ (0 in the new sense), $\gamma_1 \oplus \gamma_2 = \gamma_1 + \gamma_2 - \beta_0$ (if it is $\in \alpha$), $\ominus \gamma_1 = 2\beta_0 - \gamma_1$. We put $\beta_1 = [\sqrt{\alpha}]$. ($\beta_0 + \beta_1$ is 1 in the new sense), $\gamma_1 \odot \gamma_2 = [(\gamma_1 - \beta_0)\cdot(\gamma_2 - \beta_0)/\beta_1] + \beta_0$, $(1/\gamma) = [\beta_1^2/(\gamma - \beta_0)] + \beta_0$. We put bdn $= \{\gamma \in \alpha;\ (\exists \delta \in R)(|\gamma - \beta_0| < \beta_1 \cdot \delta)$ and we define $\gamma_1 \doteq \gamma_2 \equiv (\forall \delta \in R)(|\gamma_1 - \gamma_2| < \beta_1/\delta)$. The factor structure bdn/$\doteq$ forms a model for real numbers.

<u>Theorem 5</u>: Both the above factor structures form a really closed field in which the Archimedean property holds in the version $(\forall x)\ (\exists \beta \in R)(|x| < \beta)$ and the supreme property holds in the version $(\forall X \neq 0)((\exists y)(\forall t \in X)(t < y) \Rightarrow (\exists z)((\forall t \in X)(t < z \vee t \doteq z) \& (\forall \bar{z})((\bar{z} < z \& \neg \bar{z} \doteq z) \Rightarrow (\exists t \in X)(t > \bar{z} \& \neg t \doteq \bar{z}))))$.

Proof: The proof that $\doteq$ is a congruence on supports, arithmetical operations are defined on supports, supports are closed on arithmetical operations (with the exception of dividing by zero) and the proof of the Archimedean property do not require any special tricks. The proof of the continuity of polynomials $(x \doteq y \Rightarrow f(x) \doteq f(y))$ may be done by the induction based on the complexity of polynomials - details needed for the induction step can be found e.g. in [K]. A nonstandard proof of the intermediate property of continuous functions may be found e.g. also in [K]. From this property we deduce easily that our field is really closed. We may observe that we have not yet used Axiom 5. This axiom we need for the proof of the supreme property. Let X be a non-empty above bounded class. In view of the Archimedean property we may assume without loss of generality that $(\forall t \in X)(t < 1) \& (\exists t \in X)(t \geq 0)$. From the class X we define the class $Y \subseteq R$ by the following recursive description: $\delta \in Y \equiv (\exists t \in X)\ (t > \sum_{i \in Y \cap \delta} 2^{-(i+1)} + 2^{-(\delta+1)} \vee t \doteq \sum_{i \in Y \cap \delta} 2^{-(i+1)} + 2^{-(\delta+1)})$. Let $a \subseteq N$ be such that $Y = a \cap R$ (we use Axiom 5). In the first case we put $z = \sum_{i \in a} 2^{-(i+1)}$ and in the second one we put $z =$ = the largest element less than or equal to $\sum_{i \in a} 2^{-(i+1)}$. The proof that this z is the required supreme can be left to the reader.

Remarks: 1) Note that both the given models are isomorphic. This fact can be proved e.g. using the dyadic expansion of real numbers.

2) An equivalent axiomatic system we obtain if we change Axiom 4 by Axiom 4': R is complete $((\forall x \in R)(\forall y < x)(y \in R))$. Axiom 4' is weaker than Axiom 4, but we have proved some interesting results only from the axioms 1 - 4. Let us prove Axiom 4 using the second version of axioms. Let $X \subset \beta \in R$, hence $X \subset R$ (completeness) $\longrightarrow X = a \cap R$ (Axiom 5) $\longrightarrow X = a \cap R \cap \beta = a \cap \beta$.

§ 2. The prolongations of functions. In the second section we investigate the question, what functions are either parts of set functions or parts of functions defined by set theoretical formulas.

Definition: The class X definable by a formula not using R is called a set theoretically definable class, we use Sd(X) for denotation.

Let us note that in view of our definition of classes we do not admit the quantification of class variables in formulas defining classes.

Definition: A Sd class S is called the generating system of a totally disconnected indiscernibility relation iff it has the following properties: 1) $\mathrm{dom}(S) \supseteq R$

2) $(\forall \alpha \in R)(S''\{\alpha\}$ is an equivalence relation on the universal class V $(=\{x; x=x\})$ having a small number of factors $((\exists \beta \in R)\ (\forall x)((\forall t,s \in x)(t \neq s \Longrightarrow \neg \langle t,s\rangle \in S''\{\alpha\}) \Longrightarrow$
$\Longrightarrow \mathrm{card}(x) \leq \beta))$

3) $(\forall \alpha \in R)(S''\{\alpha + 1\} \subseteq S''\{\alpha\})$

Remark: Using overspill we can restrict S on a suitable

$\beta \in N-R$ such that the property 3) holds for all elements of β except the last one and $S''\beta$ is an equivalence on V for all elements of β. Further we suppose that generating systems are adopted in the given manner.

Definition: A relation T is called a totally disconnected indiscernibility relation iff there is a generating system S such that $\langle x,y\rangle \in T \equiv (\forall \beta \in R)(\langle x,y\rangle \in S''\{\beta\})$.

We use notations $\underset{1}{=}$, $\underset{2}{=}$, $\overset{*}{=}$, ... for totally disconnected indiscernibility relations. We shall omit further the words totally disconnected, as we shall not use other types of indiscernibility relations.

Definition: A class X is called a figure with respect to $\overset{*}{=}$ iff it has the property $(\forall x,y)(x \in X \,\&\, y \overset{*}{=} x \Rightarrow y \in X)$.

Theorem 6: Every class is a figure w.r.t. a suitable indiscernibility relation.

Proof: It suffices to prove that R is a figure, Sd classes are figures and that figures are closed on the Gödelian operations. R is a figure in the indiscernibility relation with the generating system having the following description: $S''\{\alpha\} = \mathrm{Id} \upharpoonright \alpha \cup (V-\alpha)\times(V-\alpha)$ where Id denotes the identity mapping. Every Sd class X is a figure in the indiscernibility relation with the generating system $S''\{\alpha\} = (X\times X) \cup (V-X)\times(V-X)$. For the binary operations it is useful to observe the following fact: If $\underset{1}{=}$, $\underset{2}{=}$ are indiscernibility relations with the generating systems S_1, S_2 respectively, then $\underset{1}{=} \cap \underset{2}{=}$ is an indiscernibility relation with the generating system S described by $S''\{\alpha\} = S_1''\{\alpha\} \cap S_2''\{\alpha\}$, where $\mathrm{dom}(S) = \mathrm{dom}(S_1) \cap \mathrm{dom}(S_2)$. Hence if X_1, X_2 are figures then we may assume that they are figures in the same indiscernibility relation. The operation $-X$: If X is a figure w.r.t. $\doteq_1$ then $V-X$ is also a figure w.r.t. $\doteq_1$. The operation $X \cap Y$: If X, Y are figures w.r.t. $\doteq_1$ then $X \cap Y$ is also a figure w.r.t. $\doteq_1$. The operation $X-Y$: $X-Y = X \cap (V-Y)$. The operations $V \times X$ and $X \times V$: If X is a figure w.r.t. $\doteq_1$ then $V \times X$ is a figure in the indiscernibility relation $\doteq_2$ described by $x \doteq_2 y \equiv$ $\equiv (x \in V-V^2 \,\&\, y \in V-V^2) \vee (\exists x_1, x_2, y_1, y_2)(x = \langle x_1, x_2 \rangle \,\&\, y = \langle y_1, y_2 \rangle \,\&$ $\&\, x_2 \doteq_1 y_2)$. The proof for $X \times V$ is analogous. The operation $X \times Y$: $X \times Y = (X \times V) \cap (V \times Y)$. The operation $X \wedge Y$: $X \wedge Y = X \cap (V \times Y)$. The operation dom(X): If X is a figure w.r.t. $\doteq_1$ with the generating system S_1 then dom(X) is a figure w.r.t. $\doteq_2$ with the generating system S_2 described by the following manner: $S_2''\{\alpha\}$ is the equivalence generated by the partition having as elements Boolean combinations of domains of elements of partition generated by $S_1''\{\alpha\}$. To prove that S_2 is a generating system, it suffices to note that if $S_1''\{\alpha\}$ has β equivalence classes then $S_2''\{\alpha\}$ has less than 2^{β} equivalence classes and R is closed on 2^{x}. The operation E: $E = \{\langle x, y \rangle ; x \in y\}$ is a Sd class. The proofs for the operations of conversions may be left to the reader.

Let us note that in the given theory, there is a natural definition of a one-one mapping $F: N \longleftrightarrow V$. This mapping is defined by a set formula without parameters. The definition of F can be found in [V]. Here we give only an instructive example: $324 = 2^8 + 2^6 + 2^2$; $8 = 2^3$, $6 = 2^2 + 2^1$, $2 = 2^1$; $3 = 2^1 + 2^0$, $1 = 2^0$; $F(0) = \emptyset$; $F(1) = \{0\}$, $F(2) = \{\{0\}\}$, $F(3) = \{\{0\},0\}$; $F(6) = \{\{\{0\}\},\{0\}\}$, $F(8) = \{\{\{0\},0\}\}$; $F(324) =$ $= \{\{\{\{0\},0\}\}, \{\{\{0\}\},\{0\}\}, \{\{0\}\}\}$. The natural ordering on V may be defined using the given function. If we speak about an ordering on V we shall keep in our minds the given natural ordering.

Theorem 7: Any function F is a part of a "tube" consisting of R Sd functions. Formally: $(\forall F)(\exists T, Sd(T))((\forall \alpha \in R)(T''\{\alpha\}$ is a function) & $F \subseteq T''R)$.

Corollary 8: For $\beta \in N-R$ and $\alpha \in N$ there is no function F such that $F''\alpha = \alpha \cdot \beta$.

Remark: It is consistent that $(\exists F)(F: \alpha \longleftrightarrow \beta) \equiv$ $\equiv (\alpha/\beta) \doteq 1$, where $\doteq$ is the relation of nearness defined in the first section.

An unsolved problem: Is it consistent also the negation?, i.e. $(\exists F)(F: \alpha \longleftrightarrow \beta) \& \neg(\alpha/\beta) \doteq 1$.

Proof of Theorem 7: Let F be a figure w.r.t. $\doteqdot$ and let S_1 be a generating system for $\doteqdot$. We prove $(\forall x \in F)(\exists \gamma \in R)((S_1''\{\gamma\})''\{x\}$ is a function). Let us note that $(S_1''\{\gamma\})''\{x\}$ is the equivalence class containing x. We have $(\forall x \in F)(\forall \gamma \in dom(S_1)-R)((S_1''\{\gamma\})''\{x\} \subseteq F)$ since F is a figure. Hence $(S_1''\{\gamma\})''\{x\}$ is a function. Thus there must be $\gamma \in R$ such that $(S_1''\{\gamma\})''\{x\}$ is a function as R is not Sd (cooverspill). We construct T by taking for increasing γ successively these equivalence classes of $S_1''\{\gamma\}$ which are functions. The possibility of indexing of the equivalence classes of $S_1''\{\gamma\}$ $(\gamma \in R)$ by elements of R is a consequence of Corollary 4.2) from § 1. Let us proceed in the construction of T more formally. In the following two theorems we shall need analogous constructions but we shall proceed no more formally. Let us define a set relation s_1 by the following description: $x \in s_1''\{\gamma\} \equiv x$ is the least element of $(S_1''\{\gamma\})''\{x\}$. (A coding of the equivalence classes.) Let

$h: card(s_1) \longleftrightarrow s_1$ be the one-one mapping given by the successive ordering of extensions of s_1 ($h(\alpha)$ = the least element of the least extension of $s_1 - h''\alpha$). Corollary 4.2) from § 1 gives that $(\forall \alpha \in R)((h^{-1})''(s_1'' \alpha) \subseteq R)$. Let $a = \{\beta \in card(s_1); (S_1''\{\alpha\})''\{h(\alpha)\}$ is a function$\}$, where α is such that $h(\beta) \in s_1''\{\alpha\}$. Let $g: card(a) \longleftrightarrow a$ is the numbering of elements of a in the increasing manner. T is now described as follows: $T''\{\beta\} = (S_1''\{\alpha\})''\{h(g(\beta))\}$ where α is such that $h(g(\beta)) \in S_1''\{\alpha\}$.

Remarks: 1) We can observe from the proof that if F is a one-one function then the extensions of T may be chosen also in such a way that they are one-one functions.

2) Note that we have not used Axiom 5 in the proof.

We prove now that if dom(F) = R then F may be prolonged to a set function. (Compare with the axiom of prolongation from [V]$_\circ$.)

<u>Lemma 9</u>: Let f be a not increasing function $f: \alpha \longrightarrow N$. If $(\forall \beta \in \alpha - R)$ $(f(\alpha) \in R)$ then there are $\gamma \in R$ and $\beta \in \alpha - R$ such that $f(\gamma) = f(\beta)$.

Proof: Let $\delta = \min\{\gamma; f(\gamma) < \gamma\}$. Obviously $\delta \in R$. If $Y = \{\gamma \in \delta + 1; (\exists \beta \in R)(f(\beta) = \gamma)\}$ then $Y \subseteq \delta \in R$ and hence Y is a set by Axiom 4. Let m = min(Y). We put $\gamma = \min\{\delta; f(\delta) = m\}$ and $\beta = \max\{\delta; f(\delta) = m\}$.

<u>Theorem 10</u>. If F is a class function such that dom(F) $\subseteq$ R then there is a set function g such that $F = g \wedge dom(F)$.

Proof: Let F be a figure w.r.t. $\mp$ with generating system S_1. We prove that $(\forall x \in F)(\exists \gamma \in R)((S_1''\{\gamma\})''\{x\} \subseteq F)$. For

an arbitrary $x \in F$ we define the function $f(\gamma) =$
$= \mathrm{card}((S_1''\{\gamma\})''\{x\})$. Using Lemma 9 for f we obtain $\gamma \in R$ and $\beta \in N-R$ such that $f(\gamma) = f(\beta)$, hence $(S_1''\{\gamma\})''\{x\} =$
$=(S_1''\{\beta\})''\{x\} \subseteq F$. Let us numerate all the equivalence classes of equivalences from S similarly as in the proof of Theorem 7. Let $X \subseteq R$ be the class of those numbers such that the corresponding equivalence classes are parts of F. (Note that X is definable from F and R.) Using Axiom 5 we obtain a set b such that $X = b \cap R$. The required function g we obtain as the union of all the equivalence classes corresponding to elements of $b \cap \gamma$ for a suitable γ . In fact, let $\{f_\delta;\ \delta \in b\}$ be the system of the equivalence classes corresponding to elements of b. We may assume f_δ to be a set function as this property is fulfilled for $\delta \in R$ (overspill). Now for every $\beta \in R$ we have that $\bigcup\{f_\delta;\ \delta \in b \cap \beta\}$ is a function since $\bigcup\{f_\delta;\ \delta \in b \cap \beta\} \subseteq F$. Hence we have that $g = \bigcup\{f_\delta;\ \delta \in b \cap \gamma\}$ is a function for a suitable $\gamma \in N-R$ by overspill. For proving $F \subseteq g$ it suffices to turn to the definition of X. Hence $F = g \restriction \mathrm{dom}(F)$.

<u>Corollary 11</u>: R is a strong cut in N (in the sense of Kirby and Paris [KP]).

Proof: For proving the weak regularity we do not use Axiom 5. For a function f such that $\mathrm{dom}(f) \in R$ we put $X =$
$= \mathrm{dom}(f \cap (R \times R))$. X is a set as $X \subseteq \mathrm{dom}(f) \in R$ (Axiom 4). We put $\gamma = \max(\{f(t);\ t \in X\}) + 1$. Hence $\gamma \in R$ and $\mathrm{rng}(f) \cap R \subseteq \gamma$. Thus the weak regularity is proved.

Let f be a function such that $\mathrm{dom}(f) \supseteq R$. Let $F = f \cap$
$\cap ((N-R) \times R)$. Let g be a prolongation of F. We may assume that $\mathrm{dom}(F) = \mathrm{dom}(g) \cap R$ (if not, we improve g using Axiom 5 on

dom(F)). If we put $\gamma = \max(\{\delta \in \mathrm{dom}(g);\ \delta < \min(g''(\delta + 1))\})$ then for every $\beta \in R$ such that $f(\beta) \notin R$ we have $f(\beta) > \gamma$.

The last theorem may be also generalized if we suppose the Axiom of weak choice: $(\forall X, \mathrm{dom}(X) = R)(\exists F \text{ a function})(F \subseteq X \ \&\ \mathrm{dom}(F) = R)$.

<u>Theorem 12</u>: The following properties are equivalent for a class function F.

1) There is a Sd function G such that $F = G \wedge \mathrm{dom}(F)$.

2) $(\forall x \subseteq \mathrm{dom}(F))(F \wedge x$ is a set function).

Proof: 1) $\Longrightarrow$ 2) is obvious.

For 2) $\Longrightarrow$ 1) it suffices to prove the following assertion: If F having the property 2) is a figure in $\neq$ with the generating system S_1 then $(\forall x \in F)(\exists \gamma \in R)((S_1''\{\gamma\})''\{x\} \wedge \mathrm{dom}(F) = F \wedge \mathrm{dom}((S_1''\{\gamma\})''\{x\}))$. The prolonging function can be constructed from classes $(S_1''\{\gamma\})''\{x\}$ analogously as in the proof of the preceding theorem. Let us prove by contradiction the assertion. Suppose that there is $x \in F$ such that for every $\gamma \in R$ we have $X''\{\gamma\} = (F \wedge \mathrm{dom}((S_1''\{\gamma\})''\{x\}) - ((S_1''\{\gamma\})''\{x\} \wedge \wedge \mathrm{dom}(F))) \neq 0$. Due to the Axiom of weak choice there is a function G such that dom(G) = R and $G \subseteq X$. From the last theorem the existence of a function g prolonging G follows. We prove that there is $\beta \in \mathrm{dom}(g) - R$ such that $g''\beta \subseteq F$. At first let us observe that there is $\alpha \in \mathrm{dom}(g) - R$ such that $\mathrm{dom}(g''\alpha) \subseteq \mathrm{dom}(F)$. For $\gamma \in R$ we have $g(\gamma) \in F$ by the definition of g. We also have $\mathrm{dom}(g(\gamma)) \subseteq \mathrm{dom}((S_1''\{\gamma\})''\{x\})$, hence by overspill we have that for all $\gamma < \alpha$ (where α is suitably chosen) this formula holds. But for $\gamma \in N - R$ we have $(S_1''\{\gamma\})''\{x\} \subseteq F$ since F is a figure. We have proved that $\mathrm{dom}(g''\alpha) \subseteq \mathrm{dom}(F)$. We know that

$F \upharpoonright dom(g''\alpha)$ is a set function by the assumption. For every $\gamma \in R$ we have $g''\gamma \subseteq F \upharpoonright dom(g''\alpha)$ hence by overspill we have $g''\beta \subseteq F$ for a suitable $\beta \in dom(g)-R$. For every $\gamma \in R$ we have $g(\gamma) \notin (S_1''\{\gamma\})''\{x\}$ hence by overspill the formula holds also for $\gamma \in \beta - R$. This is a contradiction with $dom(g(\gamma)) \subseteq$ $\subseteq dom((S_1''\{\gamma\})''\{x\})$, $g(\gamma) \in F$ and $(S_1''\{\gamma\})''\{x\} \subseteq F$. Thus we have proved the assertion.

Remarks: 1) There is a model of PA in which ω is not a strong cut (see [MC]). Hence Axiom 5 is independent on other axioms. The positive answer on this independence problem I obtained also from Professor Wilkie. A direct method for construction of models of PA with the negation of Axiom 5 for ω was developed by A. Kučera and the author. This method is based on Arithmetical hierarchy.

2) In AST can be proved the theorem saying that every real semiset function can be prolonged to a set function iff the restriction on an arbitrary subset of its domain is a set. The proof is analogous to that one of the last theorem. For the notion of a real class see [ČV].

Unsolved problems: 1) Is there a model of PA with a strong cut R not being a model of our theory?

2) Is our theory consistent with the negation of the Axiom of weak choice? There are some troubles with this axiom as it is not known whether AST without AC and with the negation of the Axiom of weak choice is consistent.

3) Recall the problem mentioned in this paper: Is our theory consistent with $(\exists \alpha, \beta)((\exists F)(F: \alpha \longleftrightarrow \beta) \,\&\, \neg$ $\neg(\alpha / \beta) \doteq 1)$.

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