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SINGULAR SETS AND NUMBER OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract: The operator equation $F(u)=f$ connected with the Dirichlet problem

$$(0.1)\qquad \begin{cases} -\Delta u + g(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is investigated. It is proved (under some assumptions) that the singular sets $S = \{f;\ (\exists u \in F^{-1}(f))\ F'(u) \text{ is not surjective}\}$ and $F^{-1}(S)$ are nowhere dense and that the number of elements of $F^{-1}(f)$ is finite, odd and locally constant for $f \notin S$. Further there are shown assumptions which guarantee that there exist right-hand sides f such that card $F^{-1}(f) = 1$.

Key words: Fredholm map of index zero, proper, eigenvalue.

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1. NOTATION AND PRELIMINARIES

We shall denote by R the set of all real numbers, by $\mu=\mu_k$ the Lebesgue measure in R^k. For $q=(q_1,\dots,q_k)\in R^k$ we define $|q| = \sum_{i=1}^{k}|q_i|$.

Let $(X, \|\cdot\|)$ be a Banach space, let $y\in X$, $M\in R$. Then $B_M(y) = \{x\in X;\ \|x-y\| \le M\}$.

Throughout the paper let Ω be a bounded domain in R^N $(N\ge 1)$

with the Lipschitz boundary (see [1] or [3]). Denote by $(X, \|\cdot\|)$ the Sobolev space $W_0^{1,2}(\Omega)$ with the norm induced by the scalar product

$$(u,v) = \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)dx .$$

Further denote by $\|\cdot\|_\alpha$ the norm in $L^\alpha(\Omega)$.

We shall write briefly $\int h$ instead of $\int_\Omega h(x)dx$.

The eigenvalues λ_k and the eigenfunctions v_k of the Dirichlet problem for the operator Δ on Ω have the following properties:

(1.1) $\quad -\Delta v_k = \lambda_k v_k \quad$ in Ω

$\qquad\qquad v_k = 0 \quad$ on $\partial\Omega$,

(1.2) $\quad 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$,

(1.3) $\quad \lambda_k \to \infty$,

(1.4) $\quad \{v_k\}$ is an orthonormal basis in X ,

(1.5) $\quad v_k$ are real analytic functions ,

(1.6) $\quad v_1 > 0 \quad$ in Ω .

Definition 1. Let X,Y be Banach spaces, $A: X \to Y$ a continuous linear mapping, $F: X \to Y$ a (nonlinear) operator of the class C^1.

The mapping A is said to be a Fredholm mapping of index 0 if $\operatorname{Im} A$ is closed and $\dim \operatorname{Ker} A = \operatorname{codim} \operatorname{Im} A < \infty$.

The operator F is said to be a Fredholm map of index 0 if $F'(x)$ is a linear Fredholm mapping of index 0 for each $x \in X$.

The operator F is said to be proper if $F^{-1}(K)$ is compact whenever $K \subset Y$ is compact.

Proposition 1. Let X,Y be real Banach spaces, let $F: X \to Y$ be a C^1 proper Fredholm map of index 0. Then the set

$\mathcal{O} = \{y \in Y;\ F'(x)$ is surjective for each $x \in F^{-1}(y)\}$ is a dense open subset of Y and for every $y \in \mathcal{O}$ the set $F^{-1}(y)$ is finite and its cardinal is locally constant on $\mathcal{O}$.

Proof. See [2] and [6].

The following proposition can be easily proved by induction.

Proposition 2. Let $\Omega \subset R^N$ be a nonempty domain, let $v: \Omega \to R$ be a real analytic function. Denote $M = \{x \in \Omega;\ v(x)=0\}$. Then either $\mu_N(M)=0$ or $M = \Omega$.

2. FORMULATION OF THE PROBLEM

An element $u \in X$ is the weak solution of (0.1) if

$$(2.1) \qquad \int \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + \int g(u)v = \int fv$$

holds for each $v \in X$.

We shall suppose that $g: R \to R$ is a continuous function satisfying (for $N \geq 2$) the condition

$$(2.2) \qquad |g(t)| \leq c(1+|t|^{\varkappa}) ,$$

where c and $\varkappa$ are positive constants, $\varkappa(N-2) \leq N+2$.

Using the imbedding theorems (see [1,3]) and the continuity of the operator of Nemyckij (see [8]) we get that the mapping $v \mapsto \int g(u)v$ is a continuous linear functional on X. By the Riesz theorem it can be represented by an element $G(u) \in X$, i.e. $(G(u),v) = \int g(u)v$ for each $v \in X$.

Similarly for $f \in W^{-1,2}(\Omega)$ (= the dual space to X) we find a representative $\tilde{f} \in X$; $(\tilde{f},v) = \int fv$ for each $v \in X$. In what follows we deal only with $\tilde{f}$ (as an element of X) so that we shall write only f instead of $\tilde{f}$.

Clearly, the problem (2.1) is equivalent to the equation

$$F(u) = f \quad , \tag{2.3}$$

where the operator $F: X \to X$ is defined by $F(u) = u + G(u)$.

3. PROPERTIES OF OPERATOR F

Using the imbedding theorems and the continuity of the operator of Nemyckij it can be proved the following assertion.

Lemma 1. Let i be a natural number, let $g \in C^i(R)$ and let (for $N \geq 2$)

$$|g^{(i)}(t)| \leq c(1+|t|^{\alpha}) \quad , \tag{3.1}$$

where $\alpha \geq 0$ and $(\alpha+i)(N-2) < N+2$.

Then G is a compact operator of the class C^i and

$(G^{(i)}(u)(u_1,\dots,u_i),v) = \int g^{(i)}(u)u_1\dots u_i v$.

Corollary. Let the assumptions of Lemma 1 be fulfilled. Then F is a Fredholm map of index 0.

Proof. $F'(u)$ is a compact perturbation of the identity for any $u \in X$.

Lemma 2. Let $\liminf\limits_{|t|\to\infty} \frac{g(t)}{t} > -\lambda_1$. Then F is coercive.

Proof. There exist $\varepsilon > 0$ ($\varepsilon < \lambda_1$) and $K > 0$ such that $\frac{g(t)}{t} \geq -\lambda_1 + \varepsilon$ for $|t| \geq K$. Since $|g(t)| \leq M$ on $\langle -K,K\rangle$, we get

$$(F(u),u) = \|u\|^2 + \int g(u)u = \|u\|^2 + \int_{|u|<K} g(u)u + \int_{|u|\geq K} g(u)u \geq$$

$$\geq \|u\|^2 - MK\mu(\Omega) + (-\lambda_1+\varepsilon)\int u^2 \geq \frac{\varepsilon}{\lambda_1}\|u\|^2 - MK\mu(\Omega) ,$$

hence F is coercive.

Lemma 3. Let the assumptions of Lemmas 1 and 2 be fulfilled. Then F is proper.

Proof. Let $K \subset X$ be compact. Choose a sequence $\{u_n\} \subseteq F^{-1}(K)$. Since F is coercive, $\{u_n\}$ is bounded and we may assume $G(u_n) \to h$. Further $F(u_n) \in K$ so that we may assume $F(u_n) \to f$. Then $u_n = F(u_n) - G(u_n) \to f-h$, i.e. $F^{-1}(K)$ is relatively compact. $F^{-1}(K)$ is closed, since F is continuous.

In case that $F \in C^1(X)$ we shall denote
$B = \{u \in X;\ F'(u) \text{ is not surjective}\}$, $S = F(B)$, $\mathcal{O} = X - S$.
The elements of the set $\mathcal{O}$ are called regular values of F.

Construction. Let g satisfy the assumptions of Lemma 1, let $g'(t) > -\lambda_{k+1}$ for each $t \in R$ and let $\liminf_{|t| \to \infty} \frac{g(t)}{t} > -\lambda_{k+1}$. Put $\tilde{X} = \{u \in X;\ u \perp v_i \text{ for } i=1,\dots,k\}$ and denote $P: X \to \tilde{X}$ the orthogonal projection. Let us consider the problem

$$\tilde{u} + PG(\tilde{u} + \sum_{i=1}^{k} s_i v_i) = \tilde{f}, \tag{3.2}$$

where s_i are fixed real numbers, $\tilde{f} \in \tilde{X}$ and $\tilde{u} \in \tilde{X}$ is an unknown. Denote $\tilde{G}(\tilde{u}) = PG(\tilde{u} + \sum_{i=1}^{k} s_i v_i)$, $\tilde{F}(\tilde{u}) = \tilde{u} + \tilde{G}(\tilde{u})$.
Then $\tilde{G}: \tilde{X} \to \tilde{X}$ is a compact operator of the class C^1 and similarly as for F, we get that $\tilde{F}: \tilde{X} \to \tilde{X}$ is a proper Fredholm map of index 0. The set $\tilde{B} = \{\tilde{u} \in \tilde{X};\ \tilde{F}'(\tilde{u}) \text{ is not surjective}\}$ is empty, since for $\tilde{u}, \tilde{v} \in \tilde{X}$, $\tilde{v} \neq 0$ we have
$(\tilde{F}'(\tilde{u})\tilde{v}, \tilde{v}) > \|\tilde{v}\|^2 - \lambda_{k+1} \int \tilde{v}^2 \geq 0$.
By [5] we get that $\tilde{F}: \tilde{X} \to \tilde{X}$ is a global diffeomorphism so that the solution $\tilde{u}$ of (3.2) can be written in the form

$$\tilde{u} = h(s_1, \dots, s_k, \tilde{f})$$

where h is of the class C^1 (by the implicit function theorem and for fixed $s_1,\dots,s_k$ h is a diffeomorphism of $\tilde{X}$ onto $\tilde{X}$.

Thus the problem $F(u)=f$ (for $u=\tilde{u}+\sum_{i=1}^{k} s_i v_i$, $f=\tilde{f}+\sum_{i=1}^{k} t_i v_i$) is equivalent to the problem

$$\begin{cases} PF(u) = Pf \\ (F(u),v_i) = (f,v_i) \qquad i=1,\dots,k \end{cases}$$

or

$$(3.3) \qquad \begin{cases} \tilde{u} = h(s_1,\dots,s_k,\tilde{f}) \\ t_i = F_i(s_1,\dots,s_k) \qquad i=1,\dots,k \end{cases}$$

where

$$F_i(s_1,\dots,s_k) = F_i(s_1,\dots,s_k,\tilde{f}) \equiv$$

$$\equiv s_i + (G(\sum_{j=1}^{k} s_j v_j + h(s_1,\dots,s_k,\tilde{f})),v_i) .$$

Further

$$u \in B \iff \det\left(\frac{\partial F_i}{\partial s_j}\right) = 0 .$$

In what follows we shall observe the notation introduced above.

4. THE STRUCTURE OF THE SOLUTION SET FOR THE COERCIVE OPERATOR F

Theorem 1. Let $g \in C^1(R)$, $\liminf_{|t|\to\infty} \frac{g(t)}{t} > -\lambda_1$ and let (for $N\geq 2$) $|g'(t)| \leq c(1+|t|^{\alpha})$, where $\alpha \geq 0$, $(\alpha+1)(N-2) < N+2$.

(i) Then $\mathcal{O}$ is a dense open subset of X, for every $f \in \mathcal{O}$ the set $F^{-1}(f)$ is finite, its number of elements is odd and locally constant.

(ii) If $U \subseteq \mathcal{O}$ is a domain, then $F^{-1}(U) = G_1 \cup \dots \cup G_k$, where G_i $(i=1,\dots,k)$ are pairwise disjoint domains, $F(G_i) = U$ and $\operatorname{card}(F^{-1}(f_1) \cap G_i) = \operatorname{card}(F^{-1}(f_2) \cap G_i)$ for any $f_1,f_2 \in U$.

If U is simply connected, then F/G_i is a homeomorphism.

Proof.

(i) According to Proposition 1, Lemmas 1,2 and 3 it remains to prove that $\operatorname{card} F^{-1}(f)$ is odd for $f \in \mathcal{O}$.

Choose $f \in \mathcal{O}$. For $\nu \in \langle 0,1 \rangle$ we define $F_\nu : X \to X : u \mapsto u + \nu G(u)$. Analogously as in Lemma 2 we get that there exist positive constants δ and C such that $(F_\nu(u),u) \geq \delta \|u\|^2 - C$ for each $\nu \in \langle 0,1 \rangle$ and $u \in X$. Consequently, there exists $P > \|f\|$ such that $F^{-1}(f) \subseteq B_P(0)$ and $f \notin F_\nu(\partial B_P(0))$ for any $\nu \in \langle 0,1 \rangle$. By the homotopy invariance property of the Leray-Schauder degree we get $1 = \deg(F_0, B_P(0), f) = \deg(F_1, B_P(0), f) = \deg(F, B_P(0), f)$. Let $F^{-1}(f) = \{u_1, \dots, u_k\}$. Since $1 = \deg(F, B_P(0), f) = \sum_{j=1}^{k} i(u_j)$, where $i(u_j) = \pm 1$, k has to be an odd number.

(ii) Let $U \subseteq \mathcal{O}$ be a (nonempty) domain. Then $F^{-1}(U) = \bigcup_{i=1}^{\infty} G_i$, where G_i are pairwise disjoint domains.

First we show that $F(G_i)$ is closed and open in U. By the implicit function theorem F/G_i is a local homeomorphism, hence $F(G_i)$ is open. Choose $f \in \overline{F(G_i)} \cap U$. Then there exist $u_n \in G_i$, $F(u_n) \to f$. Since F is proper, we may assume $u_n \to u$. Then $F(u) = f$, G_i is closed in $F^{-1}(U)$, thus $u \in G_i$, $f \in F(G_i)$, i.e. $F(G_i)$ is closed in U.

Consequently, $F(G_i) = U$ for any $G_i \neq \emptyset$ so that $F^{-1}(U) = \bigcup_{i=1}^{k} G_i$ (since $F^{-1}(f)$ is finite for $f \in \mathcal{O}$).

Using the implicit function theorem and the properness of F one can easily prove that $\operatorname{card}(F^{-1}(f) \cap G_i)$ is a continuous function on U so that $\operatorname{card}(F^{-1}(f) \cap G_i)$ is locally constant.

If U is simply connected, then F/G_i is a homeomorphism by [5,7].

Remark 1. Let the assumptions of Theorem 1 be fulfilled and let, moreover, $g'(t) > -\lambda_{k+1}$ for each $t \in R$. Then

$X = \operatorname{Im} F'(u) + \{\sum_{i=1}^{k} c_i v_i;\ c_i \in R\}$ for any $u \in X$.

Applying [6] (Théoreme 1.1) to the mapping

$$\varphi: R^k \times X \to X : ((c_1,\dots,c_k),u) \mapsto F(u) + \sum_{i=1}^{k} c_i v_i$$

we get that the set $\mathcal{O}^{k,f} = \{(c_1,\dots,c_k) \in R^k;\ f + \sum_{i=1}^{k} c_i v_i \in \mathcal{O}\}$ is dense and open in R^k (for any $f \in X$).

Remark 2. Let g satisfy the assumptions of Theorem 1, let $g'(t) \geq -\lambda_1$ for each $t \in R$ and suppose there exist $t_n \nearrow 0$, $s_n \searrow 0$ such that $g'(t_n) > -\lambda_1$, $g'(s_n) > -\lambda_1$. Then $B \subseteq \{0\}$ so that the function F_1 (from Construction in §3 with $k=1$) is a homeomorphism (since $F_1(R)=R$ and $F_1'(s) \neq 0$ for $s \neq 0$). Thus $F: X \to X$ is a global homeomorphism (cf. [5]).

5. THE SINGULAR SET B

Example 1. Let $N=1$, $\Omega=(a,b)$, let g satisfy the assumptions of Theorem 1 and, moreover, $g(t) = -\lambda_k t$ for $|t| \leq M$. Then $\{u \in X;\ |u| \leq M \text{ in } \Omega\} \subseteq B$. Since the imbedding $X \subseteq L^{\infty}(\Omega)$ is continuous, B contains a neighbourhood of 0 in X.

Theorem 2. Let i and g satisfy the assumptions of Lemma 1, let $u_0 \in B$. Denote $V = \operatorname{Ker} F'(u_0)$, $V_0 = V - \{0\}$.

(i) Let $i \geq 2$ (so that $F \in C^2(X)$) and let

$(\exists u \in X)(\forall v \in V_0) \qquad (F''(u_0)(v,u),v) \neq 0$.

Then there exists $\varepsilon > 0$ such that $\{u_0 + tu;\ |t| < \varepsilon\} \cap B = \{u_0\}$.

(ii) Let $i \geq 3$, let $F''(u_0)(v,v)=0$ for each $v \in V$ and let

$$(\exists u_1 \in X)(\exists u_2 \in X)(\forall v \in V_0) \qquad (F'''(u_0)(v,v,u_1),u_2) \neq 0 .$$

Then $u_0 \notin \operatorname{int} B$.

Proof.

(i) Suppose the contrary, i.e. there exist $s_n \in R$ and $w_n \in X$ such that $s_n \to 0$, $\|w_n\| = 1$, $F'(u_0+s_n u)w_n = 0$.

Then $F'(u_0)w_n = (F'(u_0)-F'(u_0+s_n u))w_n = O(s_n)$

(i.e. $\|F'(u_0)w_n\| \leq Cs_n$), thus $w_n = z_n + O(s_n)$, where $z_n \in V$, $\|z_n\|=1$. Since $\dim V < \infty$, we may assume $w_n \to z \in V_0$.

Define $t(s) = (F'(u_0+su)z,z)$, then $t'(0) = \int g''(u_0)uz^2 \neq 0$.

On the other hand,

$$t(s_n) = (F'(u_0+s_n u)z,z) = (F'(u_0+s_n u)(z-w_n),z) =$$

$$= ((F'(u_0+s_n u) - F'(u_0))(z-w_n),z) = O(s_n)\|z-w_n\| = o(s_n),$$

which gives us a contradiction.

(ii) Suppose there exist $w_n \in X$ and $s_n \in R$ such that $s_n \to 0$, $w_n \in \operatorname{Ker} F'(u_0+s_n u_1)$, $\|w_n\| = 1$, $(F''(u_0+s_n u_1)(w_n,w_n),u_2) = 0$.

Then again $w_n = z_n + O(s_n)$, $z_n \in V_0$ and we may assume $w_n \to z \in V_0$.

Define $T(s) = (F''(u_0+su_1)(z,z),u_2)$, then $T'(0) \neq 0$. Nevertheless, $T(s_n) = ((F''(u_0+s_n u_1)-F''(u_0))((z,z)-(w_n,w_n)),u_2) -$

$$- (F''(u_0)(w_n,w_n),u_2) = o(s_n),$$

since $\|F''(u_0+s_n u_1)-F''(u_0)\| = O(s_n)$, $\|z-w_n\| = o(1)$ and

$$(F''(u_0)(w_n,w_n),u_2) = (F''(u_0)(z_n+O(s_n),z_n+O(s_n)),u_2) =$$

$$= (F''(u_0)(z_n,z_n),u_2) + (F''(u_0)(z_n,u_2),O(s_n)) + o(s_n) = o(s_n).$$

Thus we have a contradiction and therefore in each neighbourhood U of u_0 there exists $\tilde{u}_0$ such that

$(F''(u_0)(w,w),u_2) \neq 0$ for each $w \in \operatorname{Ker} F'(\tilde{u}_0) - \{0\}$.

Using (i) we get $\tilde{u}_0 \notin \operatorname{int} B$ so that also $u_0 \notin \operatorname{int} B$.

Example 2. Let $N \leq 3$, $g(t) = \alpha \operatorname{arctg}(t)$, $\alpha \in R$. We shall prove that the set B is nowhere dense.

Since B is empty for $\alpha \geq 0$, we may assume $\alpha < 0$. Let $u_0 \in B$. Denote $V = \operatorname{Ker} F'(u_0)$, $V_0 = V - \{0\}$.

If $\int g''(u_0)v^2 u_0 \neq 0$ for each $v \in V_0$, then $u_0 \notin \operatorname{int} B$.

Suppose $\int g''(u_0)v^2 u_0 = 0$ for some $v \in V_0$. Since $g''(u_0) = -\frac{2u_0}{(1+u_0^2)^2}$, we get $u_0 v \equiv 0$. For any $w \in X$ we have

$$0 = (F'(u_0)v,w) = (v,w) + \int \frac{\alpha vw}{1+u_0^2} = (v,w) + \int \alpha vw ,$$

thus $\alpha = -\lambda_k$, $v = v_k$. Using Proposition 2 we get $u_0 \equiv 0$.

Hence $F''(u_0)(z,z) = 0$ and $(F'''(u_0)(z,z,v),v) = -\int 2\alpha z^2 v^2 \neq 0$ for any $z \in V_0$, thus $u_0 \notin \operatorname{int} B$.

Remark 3. If B is nowhere dense, then the set $F^{-1}(S)$ is nowhere dense.

6. EXISTENCE OF RIGHT-HAND SIDES WITH A UNIQUE SOLUTION

Lemma 4. Let X be a real Banach space, let $G: X \to X$ be a compact C^1 map, $\|G(x)\| \leq K$ for each $x \in X$. Put $F = Id + G$, $B = \{x \in X;\ F'(x) \text{ is not surjective}\}$ and $\mathcal{O} = X - F(B)$. Let B be bounded. Then $y \in \mathcal{O}$ and $\operatorname{card} F^{-1}(y) = 1$ for each $y \in X$ whose norm is sufficiently large.

Proof. It is clear that for F the assertions of Theorem 1 are valid. Since B is bounded, we have $F(B)\subset B_M(0)$. Choose $y\in X$, $\|y\|>M+4K$. We shall prove that $\operatorname{card} F^{-1}(y) = 1$.

Denote $U = \operatorname{int}(B_{4K}(y))$ and choose $x_0\in F^{-1}(y)$. If $x\in F^{-1}(y)$ then $\|x-x_0\| \leq 2K$ and $F(B_{2K}(x_0))\subset U$, thus $F^{-1}(U)$ is a domain. Since U is simply connected, F is a homeomorphism of $F^{-1}(U)$ onto U. Consequently, $\operatorname{card} F^{-1}(y) = 1$.

<u>Theorem 3.</u> Let $g\in C^1(R)$, let g,g' be bounded, $g'(t)> -\lambda_{k+1}$ for each $t\in R$ and let $\liminf\limits_{|t|\to\infty} g'(t) > -\lambda_1$. Then

$$(\exists K,\varepsilon>0)(\forall f\in X)\quad (\|f\|>K\ \&\ \|Pf\|<\varepsilon\|f\|) \Rightarrow f\in\mathcal{O},\ \operatorname{card} F^{-1}(f) = 1.$$

Proof.

1. We show $\sum_{i=1}^{k} t_i v_i\in\mathcal{O}$ for $t = (t_1,\dots,t_k)\in R^k$ sufficiently large.

Suppose there exist $u_n = \sum_{i=1}^{k} s_i^{(n)} v_i + \tilde{u}_n \in B$ such that $F(u_n) = \sum_{i=1}^{k} t_i^{(n)} v_i$, $|t^{(n)}|\to\infty$. Then $\|u_n\|\to\infty$.

Since $\tilde{u}_n + PG(u_n) = 0$ and g is bounded, the sequence $\{\tilde{u}_n\}$ is bounded and hence $|s^{(n)}|\to\infty$.

Choose $w_n\in \operatorname{Ker} F'(u_n)$, $\|w_n\|= 1$. We may assume $w_n \rightharpoonup w$ (so that $w_n\to w$ in $L^2(\Omega)$) and $\frac{s_i^{(n)}}{|s^{(n)}|}\to s_i$, $i=1,\dots,k$.

Denote $v = \sum_{i=1}^{k} s_i v_i$. By Proposition 2 the set $\{x\in\Omega;\ v(x)=0\}$ has measure zero.

Since $\int g'(u_n)w_n^2 = -\|w_n\|^2 = -1$ and g' is bounded, we have

$$(6.1)\qquad \int g'(u_n)w^2 \to -1 .$$

Further $u_n = |s^{(n)}|(v+z_n)$, where $z_n = \sum_{i=1}^{k}\left(\frac{s_i^{(n)}}{|s^{(n)}|} - s_i\right)v_i + \frac{\tilde{u}_n}{|s^{(n)}|} \to 0$.

Since $\liminf_{|t|\to\infty} g'(t) > -\lambda_1$, there exist $\eta > 0$ ($\eta < \lambda_1$) and $M > 0$ such that $g'(t) > -\lambda_1 + \eta$ for $|t| \geq M$.

There exists $\delta > 0$ such that $\int_N w^2 < \frac{\eta}{2\lambda_1\lambda_{k+1}}$ for any $N \subset \Omega$ measurable, $\mu N < \delta$, and there exists $\vartheta > 0$ such that the measure of the set $A_1 = \{x; |v(x)| < 2\vartheta\}$ is less than $\frac{\delta}{2}$.

The measure of the set $A_2 = \{x; |z_n(x)| \geq \vartheta\}$ is also less than $\frac{\delta}{2}$ for $n \geq n_0$. For $|s^{(n)}| > \frac{M}{\vartheta}$ and $x \notin A_1 \cup A_2$ we have $|u_n(x)| \geq M$, hence

$$\int g'(u_n)w^2 \geq -\lambda_{k+1}\int_{A_1\cup A_2} w^2 + (-\lambda_1+\eta)\int w^2 > -\frac{\eta}{2\lambda_1} + \frac{-\lambda_1+\eta}{\lambda_1}\|w\|^2 \geq$$

$$\geq -1 + \frac{\eta}{2\lambda_1},$$

which gives us a contradiction (according to (6.1)).

2. We show that card $F^{-1}(\sum_{i=1}^{k} t_i v_i) = 1$ for t sufficiently large.

Define $H: R^k \to R^k: s \mapsto (F_1(s,0),\dots,F_k(s,0))$

(F_i are functions from Construction in §3).

Then H is a C^1 map, $H = Id + D$, where D is compact and bounded (on R^k). The set $B_H = \{s;\ H'(s) \text{ is not surjective}\}$ is bounded (since $H(B_H)$ is bounded). Using Lemma 4 we get our assertion.

3. We prove the assertion of the theorem.

Suppose there exist $f_n \in X$, $\|f_n\| \to \infty$, $\frac{\|Pf_n\|}{\|f_n\|} \to 0$ such that $f_n \notin \mathcal{O}$ or card $F^{-1}(f_n) \neq 1$.

We may assume $f_n \notin \mathcal{O}$ (otherwise we choose $f_n^* \in S\cap(f_n, (Id-P)f_n)$).

Then there exist $u_n \in B$, $F(u_n) = f_n$. We have

$$f_n = \tilde{f}_n + \sum_{i=1}^{k} t_i^{(n)} v_i , \qquad u_n = \tilde{u}_n + \sum_{i=1}^{k} s_i^{(n)} v_i .$$

Since $\frac{\|\tilde{f}_n\|}{\|f_n\|} \to 0$ and $\|f_n\| \to \infty$, we get $\frac{\|\tilde{f}_n\|}{|t^{(n)}|} \to 0$,

$|t^{(n)}| \to \infty$, $|s^{(n)}| \to \infty$, $\frac{\|\tilde{u}_n\|}{|s^{(n)}|} \to 0$ (g is bounded).

Now we get a contradiction analogously as in the first part of the proof.

Example 3. Let $N \leq 3$, $g(t) = \alpha \operatorname{arctg}(t)$, $\alpha \in R$.
Using Remark 2 we get that the operator $F(\cdot) = F(\alpha, \cdot)$ is a global homeomorphism for $\alpha \geq -\lambda_1$. From Ljusternik-Schnirelmann theory it follows that card $F^{-1}(0) \geq 2k+1$ for $\alpha \in (-\lambda_{k+1}, -\lambda_k)$. Nevertheless, by Theorem 3 there exists f such that card $F^{-1}(f) = 1$.

Further suppose $\alpha > -\lambda_2$.
Let us consider $\tilde{f} = 0$ in Construction (§3) and denote $K(s,\alpha) = F_1'(s)$. Then $K(0,-\lambda_1) = 0$, $\frac{\partial K}{\partial \alpha}(0,\alpha) = \int v_1^2 > 0$.
By the implicit function theorem for each s in a neighbourhood of 0 there exists an unique $\alpha(s)$ in a neighbourhood of $-\lambda_1$ such that $K(s,\alpha(s)) = 0$. We get $\alpha'(0) = 0$, $\alpha''(0) < 0$.
In a way analogous to that in the first part of the proof of Theorem 3 one can prove that assumptions $\alpha_n \nearrow -\lambda_1$, $K(s_n,\alpha_n)=0$ imply $s_n \to 0$. Thus for $\alpha \in (-\lambda_1-\varepsilon, -\lambda_1)$ there exist exactly 2 solutions $s_1(\alpha) < 0 < s_2(\alpha)$ of the equation $F_1'(s) \equiv K(s,\alpha)=0$.
Since card $F_1^{-1}(0) \geq 3$, there exist $t_1(\alpha) < 0 < t_2(\alpha)$ such that the equation $F(u) = tv_1$ (which is equivalent to the equation $F_1(s) = t$) has exactly

(i) 3 solutions for $t \in (t_1(\alpha), t_2(\alpha))$

(ii) 2 solutions for $t \in \{t_1(\alpha), t_2(\alpha)\}$

(iii) 1 solution for $t \notin \langle t_1(\alpha), t_2(\alpha)\rangle$.

Further $tv_1 \in \mathcal{O}$ iff $t \notin \{t_1(\alpha), t_2(\alpha)\}$.

7. PROBLEM IN RESONANCE

Let $g(t) = -\lambda_m t + g_1(t)$ satisfy the assumptions of Lemma 1. Let $\lambda_{m-1} < \lambda_m = \lambda_{m+1} = \dots = \lambda_{m+p} < \lambda_{m+p+1}$ (where $p \geq 0$ and $\lambda_0 = 0$ for $m=1$). Denote W the linear hull of $v_m, \dots, v_{m+p}$; let $Q: X \to W$ be the orthogonal projection. Put
$V_{\varkappa} = \{ f \in X;\ |(f,w)| < \varkappa \int |w| \text{ for each } 0 \neq w \in W \}$.
Then $V_{\varkappa} = W^{\perp} + \varkappa W_0$, where W_0 is an open neighbourhood of 0 in W; $W_0 = \{f \in W;\ |(f,w)| < \int |w| \text{ for each } 0 \neq w \in W\}$.
The following assertion can be proved.

<u>Theorem 4</u>. Let g_1 be bounded and g_1' lower bounded. Let $\liminf_{|t| \to \infty} g_1(t)t > 0$ or $\limsup_{|t| \to \infty} g_1(t)t < 0$.

(i) For each $M > 0$ there exists $\rho > 0$ such that for any $f \in X$ with $\|f\| \leq M$ and $\|Qf\| \leq \rho$ there exists a solution of the problem $F(u) = f$.

(ii) Let $\liminf_{|t| \to \infty} |g_1(t)| = \varkappa > 0$. Then for any $f \in V_{\varkappa}$ there exists a solution of the problem $F(u) = f$; the set $\mathcal{O}_{\varkappa} = \mathcal{O} \cap V_{\varkappa}$ is dense and open in $V_{\varkappa}$ and for $f \in \mathcal{O}_{\varkappa}$ the number of elements of $F^{-1}(f)$ is finite, odd and locally constant.

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