Roman Frič; Martin Gavalec

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Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 2, 341--347

Persistent URL: http://dml.cz/dmlcz/106232

COMMENTATIONES MATHEMATICAE UNIVERSITATIC CAROLINAE
24,2(1983)

ON THE NOVÁK COMPLETION OF CONVERGENCE GROUPS

ROMAN FRIČ, MARTIN GAVALEC

Abstract: Some properties of a convergence commutative group G are not inherited by its finest completion G_1 (constructed by J. Novák). We study two such properties (G is Fréchet or torsion-free, respectively). The results shed more light on the interplay between algebraic and closure properties of group completions.

Key words and phrases: Convergence commutative group, Novák completion, Fréchet space, divisible group, torsion-free group.

Classification: Primary 54H13, 54C20
Secondary 54D55, 54B05

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1. Introduction. In terminology and notation on convergence spaces and groups we follow [4] and [5]. Some facts, however, are recollected below.

A convergence commutative group, abbreviated to cc-group, is a quadruple $(G, \mathcal{L}, \wp, +)$ such that $(G, +)$ is a commutative group, $(G, \mathcal{L}, \wp)$ is a convergence space (i.e., $\mathcal{L} \subset G^N \times G$ defines a sequential convergence satisfying axioms $(\mathcal{L}_0)$, $(\mathcal{L}_1)$, $(\mathcal{L}_2)$, and $\wp : 2^G \longrightarrow 2^G$ is the induced convergence closure operator - it need not be idempotent), and the algebraic and closure structures are compatible (i.e., $\mathcal{L}$ satisfies: (SG) If $x = \mathcal{L}\text{-}\lim x_n$ and $y = \mathcal{L}\text{-}\lim y_n$, then there is a subsequence $\langle i_n \rangle$ of $\langle n \rangle$ such that $x - y = \mathcal{L}\text{-}\lim x_{i_n} - y_{i_n}$. As a rule, $\mathcal{L}^*$ denotes the largest convergence inducing the same closure operator $\wp$. We say that $\langle x_n \rangle$ is a Cauchy sequence if for every subsequence $\langle x_{i_n} \rangle$ of $\langle x_n \rangle$ the sequence

$\langle x_n - x_{i_n} \rangle$ $\mathcal{Y}^*$- converges to the neutral element 0 of G, and G is complete if every Cauchy sequence $\mathcal{Y}^*$- converges in G. A complete co-group $(\bar{G}, \bar{\mathcal{Y}}, \bar{\gamma}, +)$ is a completion of $(G, \mathcal{Y}, \gamma, +)$ if G is a $\bar{\gamma}^{\Omega}$-dense subspace of $(\bar{G}, \bar{\mathcal{Y}}, \gamma)$ and a subgroup of $(G, +)$.

For every co-group $(G, \mathcal{Y}^*, \gamma, +)$ J.Novák has constructed, in [5], a completion $(G_1, \mathcal{Y}_1, \gamma_1, +)$. It was shown in [1] that the completion has nice categorical properties (it yields an epi-reflector into complete co-groups); $(G_1, \mathcal{Y}_1, \gamma_1, +)$ will be called the Novák completion of $(G, \mathcal{Y}, \gamma, +)$. Note that (unlike in the case of a topological group) a co-group can have more nonequivalent completions. In [2], V.Koutník pointed out that if G is a Fréchet space (unique sequential limits), then G_1 need not be a Fréchet space. He also proved that if G is Fréchet, then G_1 is Fréchet iff the quotient group G_1/G is finite.

Example 1. Consider the group Q of all rational numbers equipped with the usual convergence of sequences. It is a Fréchet co-group. The Novák completion of Q yields the group of all real numbers equipped with a rather strange convergence and closure. In view of Koutník's result, it is not a Fréchet co-group.

Some features of Example 1 are further developed in the next section. In the last section we show that the Novák completion of a torsion-free co-group need not be torsion-free. We also mention some related problems.

2. Closure order. Recall that if $(L, \mathcal{L}, \lambda)$ is a convergence space, then for each ordinal number α a closure operator λ^{α} is defined inductively: for $A \subset L$ put $\lambda^0 A = A$ and $\lambda^{\alpha} A = \bigcup_{\beta < \alpha} \lambda(\lambda^{\beta} A)$ for $\alpha > 0$. If Ω is the first uncountable ordinal, then λ^{Ω} is idempotent, hence a topology. The smallest ordinal α for which

λ^{α} is idempotent is said to be the t o p o l o g i c a l o r d e r of λ; it will be denoted by $t\sigma(\lambda)$. Fréchet spaces (unique sequential limits) are precisely those convergence spaces $(L, \mathcal{L}, \lambda)$ for which $t\sigma(\lambda) = 1$.

In [3], L.Mišík has constructed a co-group the topological order of which is greater than 1 but it has a dense subgroup the topological order of which equals 1. Our first result shows that such groups are not rare.

<u>Theorem</u>. Let $(G, \mathcal{Y}^{*}, \gamma, +)$ be an incomplete co-group such that $t\sigma(\gamma) = 1$ and let $(G_1, \mathcal{Y}_1^{*}, \gamma_1, +)$ be its Novák completion. If $(G_1, +)$ is a divisible group, then $t\sigma(\gamma_1) > 1$.

<u>Proof</u>. If G_1 is divisible, then the quotient group G_1/G is also divisible. Since $G \subsetneqq G_1$ and since the only finite divisible group is trivial, the group G_1/G is infinite. The assertion now follows from the before mentioned result of Koutník (cf. [2]).

<u>Corollary</u>. Let G be a subgroup of the co-group R of all real numbers such that $Q \subset G \subsetneqq R$ and let G_1 be its Novák completion. Then G_1 is not a Fréchet space.

<u>Proof</u>. It follows from the construction of the Novák completion that $(G_1, +)$ is the group of all real numbers. It is divisible and hence G_1 is not a Fréchet space.

However, the divisibility is not a necessary condition for the Novák completion to be Fréchet. We present an example of a Fréchet co-group G such that its Novák completion G_1 is not Fréchet and G_1 is not a divisible group.

<u>Example 2</u>. Let G be the ring of all finite subsets of a countable infinite set X. Then G equipped with the symmetric difference as a group operation and with the usual convergence of subsets of X is a co-group and the Novák completion G_1 of G is the

set of all subsets of X equipped with the symmetric difference and a convergence different from the usual convergence of subsets (cf. [5]). It is easy to see that G is a Fréchet space, G_1 fails to be divisible (each $A \in G_1$, $A \neq \emptyset$, has order 2), and G_1 fails to be a Fréchet space (G_1/G is infinite).

<u>Problem 1</u>. Does there exist an incomplete Fréchet cc-group G such that the Novák completion G_1 of G is also a Fréchet space?

<u>Problem 2</u>. Let G be an incomplete cc-group and G_1 its Novák completion. Describe the relationship between $t\sigma(g)$ and $t\sigma(g_1)$. Is it true that if $t\sigma(g)=1$, then $t\sigma(g_1) \leqq 2$?

<u>3. Algebraic order</u>. In this section we consider the relationship between the (algebraic) order of elements of a cc-group and the order of elements of its completion.

It is known that the completion operator for topological groups does not preserve torsion-type properties. E.g., the complete topological group T of all complex numbers having absolute value 1 has two dense subgroups, one of which is a torsion-free group (an infinite cyclic group) and the other one is a torsion group (the subgroup of all elements of finite order). These groups are first countable. Hence, they can be considered as Fréchet cc-groups. Then T equipped with the corresponding convergences and closures yields the Novák completions of the two cc-groups. Consequently, torsion-type properties are not preserved by the Novák completion of cc-groups.

We present here an example of a countable incomplete Abelian torsion-free cc-group G the Novák completion G_1 of which is not torsion-free and all nonzero elements have either infinite order or order 2.

Note that G is a first countable Hausdorff topological group and the topological completion $\bar{G}$ of G has the same algebraic properties as G_1 (see Remark).

<u>Example 3</u>. Let G be the weak direct product of countably many copies of the group Z of all integers. The group G can be visualized as the group of all mappings of N into Z having finite support (for each $g \in G$, $g(n) = 0$ for all but finitely many $n \in N$) eqipped with the usual pointwise addition. For $k \in N$, let H_k be the set of all $g \in G$ for which $g(1) = g(2) = \cdots = g(k-1) = 0$ and $\sum g(n)$ is an even integer. Then $\langle H_k \rangle$ is a decreasing sequence of subgroups of G the intersection of which contains only the neutral element 0 of G. Consequently, H_k's can be taken as a clopen basis at 0 and G becomes a first countable Hausdorff topological group. It follows from Corollary 4 in [4] that G is also a Fréchet cc-group in which a sequence $\langle g_n \rangle$ converges to 0 iff for each $k \in N$ the sequence $\langle g_n \rangle$ is in H_k for all but finitely many $n \in N$. Denote the resulting cc-group by $(G, \mathcal{G}^*, \mathfrak{g}, +)$. Let $(G_1, \mathcal{G}_1^*, \mathfrak{g}_1, +)$ be its Novák completion. We show that G_1 has the desired properties.

Recall that two Cauchy sequences $\langle g_n \rangle$, $\langle h_n \rangle$ are equivalent if $0 = \lim (g_n - h_n)$. The group G_1 consists of the set of all equivalence classes of Cauchy sequences (each point of G is identified with the class containing the corresponding constant sequence) equipped with the natural group structure and a certain convergence of sequences. Each divergent Cauchy sequence $\langle h_n \rangle$ in G converges in G_1 to the equivalence class $[\langle h_n \rangle]$ it belongs to.

Let $\langle h_m \rangle$ be a divergent Cauchy sequence in G. Then there are two possibilities.

1. For each $k \in N$ there exists $m(k) \in N$ such that $h_n(k) = 0$ whenever $m > m(k)$. Then $\langle 2h_m \rangle$ converges in G to 0 and the ideal point $[\langle h_m \rangle] \in G_1$ has order 2.

2. There exists $k \in N$ such that $h_m(k) \neq 0$ for infinitely many $m \in N$. Then for each $m \in N$ the sequence $\langle m h_m \rangle$ cannot converge in G to 0. Hence the ideal point $[\langle h_m \rangle] \in G_1$ has infinite order.

It can be easily verified that G_1 is not a Fréchet space.

Remark. Since G is a first countable topological group, the topological completion $\overline{G}$ of G is the group G_1 (consisting of equivalence classes of Cauchy sequences in G) equipped with the corresponding topological and uniform structures. Thus each nonzero element of $\overline{G}$ has either infinite order or order 2.

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Matematický ústav SAV
Karpatská 5
040 01 Košice
Československo

Prírodovedecká fakulta
UPJŠ v Košiciach
nám. Febr. víť. 9
041 54 Košice
Československo

(Oblatum 29.11. 1982)