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COMMON FIXED POINTS FOR NONEXPANSIVE AND ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract.

The main aim of the present paper is to prove the existence of common fixed points for nonexpansive and asymptotically nonexpansive mappings in convex metric spaces. Such spaces, introduced by Takahashi, include Banach spaces; and our results generalize those of Bahtin, De Marr, Goebel and Kirk, Hu, Kirk, and others.

Key Words and Phrases.
Convex metric space, nonexpansive and asymptotically nonexpansive mappings, fixed points.

Classification. 47H10, 52H25.

§0. Introduction.

In 1970, Takahashi [14] introduced a notion of convexity in metric spaces (see Definition 0.1) and generalized some fixed-point theorems in Banach spaces.

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Subsequently, Machado [8], Tallman [15], and Naimpally and Singh [9], among others, have studied convex metric spaces and fixed-point theorems. This paper is a continuation of these investigations.

In Section 1, we prove common fixed-point theorems for nonexpansive mappings. Section 2 deals with the existence of common fixed points for asymptotically nonexpansive mappings. All of these are in the setting of convex metric spaces and generalize or extend results in the Banach-space setting.

We begin with some definitions.

Definition 0.1. Let X be a metric space and I be the closed-unit interval. A continuous mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x,y \in X$ and $\lambda \in I$, $d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1 - \lambda)d(u,y)$ for all $u \in X$. X together with a convex structure is called a *convex metric space*.

Clearly, a Banach space, or any convex subset of it, is a convex metrix space with $W(x,y,\lambda) = \lambda x + (1 - \lambda)y$. More generally, if X is a linear space with a translation-invariant metric satisfying $d(\lambda x + (1 - \lambda)y,0) \leq \lambda d(x,0) + (1 - \lambda)d(y,0)$, then X is a convex metric space. There are many other examples, but we consider these as paradigmatic.

Definition 0.2. Let X be a convex metric space. A nonempty subset $K \subseteq X$ is *convex* if $W(x,y,\lambda) \in K$ whenever $x,y \in K$ and $\lambda \in I$.

Takahashi [14] has shown that the open spheres, $B(x,r) = \{y \in X : d(x,y) < r\}$, and closed spheres, $B[x,r] = \{y \in X : d(x,y) \leq r\}$, are convex. Also, if $\{K_\alpha : \alpha \in A\}$ is a family of convex subsets of X, then $\bigcap\{K_\alpha : \alpha \in A\}$ is convex.

Definition 0.3. Let X be a convex metric space. The *convex hull* of $A \subsetneq X$, $co(A)$, is the intersection of all convex sets containing A. The *closed convex hull*, $\overline{co}(A)$, is the intersection of all closed convex sets containing A.

The following result of Takahashi [14, Proposition 5] will be used frequently.

Lemma 0.4. Let M be a nonempty compact subset of a convex metric space X, and let $K = \overline{co}M$. If the diameter of M, $\delta(M)$, is positive, then there exists $u \in K$ such that $\sup\{d(x,u) : x \in M\} < \delta(M)$.

§1. Families of Nonexpansive Mappings.

A common fixed point for a commutative family of nonexpansive mappings is found. Results of Bahtin [1] are generalized.

Definition 1.1. Let X be a metric space. A mapping $T : X \to X$ is said to be *demicompact* if every bounded sequence $\{x_n\}$, such that $d(x_n,Tx_n) \to 0$ as $n \to \infty$, contains a convergent subsequence.

Lemma 1.2. Let X be a metric space, and let D be a closed, bounded subset of X. If $T : D \to D$ is a continuous demicompact mapping, then $F(T) = \{x \in D : Tx = x\}$ is compact.

Proof. Let $\{x_n\}$ be a sequence in $F(T)$. Then, $d(x_n,Tx_n) = 0$ for each n. Since T is demicompact and D is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x$ for some $x \in D$. By the continuity of T, $Tx = x$.

Theorem 1.3. Let X be a convex metric space, and let C be a nonempty,

closed, bounded, and convex subset of X. Let $G : C \to C$ be a family of commuting nonexpansive mappings with nonempty fixed-point set. Suppose there is at least one mapping in G, which is demicompact; then, the family G has a common fixed point.

<u>Proof.</u> Let $D = \{B \subseteq C : B$ is nonempty, closed, and convex, and $T(B) \subseteq B$ for each $T \in G\}$. Clearly, D is nonempty, since C belongs to D. Partially order D by set inclusion, and let $\{B_\alpha : \alpha \in \Delta\}$ be a chain in D. $A = \bigcap\{B_\alpha : \alpha \in \Delta\}$ is closed, convex, and invariant under each $T \in G$. Let $T_o \in G$ be a demicompact mapping, and let $F_\alpha = \{x \in B_\alpha : T_o x = x\}$. Each F_α is nonempty and compact; therefore, $F = \bigcap\{F_\alpha : \alpha \in \Delta\}$ is a nonempty subset of A. Thus, $A \in D$; and, by Zorn's Lemma, there exists a minimal nonempty, closed convex set $B_o \subseteq C$ such that $T(B_o) \subseteq B_o$ for each $T \in G$.

Let $H = \{x \in B_o : T_o x = x\}$. Then, H is a nonempty, compact subset of B_o. From $Tx = TT_o x = T_o Tx$ for any $x \in H$ and $T \in G$, it follows that $T(H) \subseteq H$ for each $T \in G$. Again, by Zorn's Lemma, there is a minimal. nonempty, compact set $M \subseteq C$ such that $T(M) \subseteq M$ for each $T \in G$. Clearly, $M \subseteq B_o$, and the minimality of M in H implies $T(M) = M$ for each $T \in G$.

Suppose M consists of more than one element. Then, by Lemma 0.4, we conclude that there is $u \in \overline{co}(M)$ such that $\rho = \sup\{d(x,u) : x \in M\} < \delta(M)$. Since B_o is convex and $M \subseteq B_o$, it follows that u belongs to B_o. For each $x \in M$, $u \in B[x,\rho]$. Let $N = \bigcap_{x \in M} B[x,\rho]$ and $P = N \cap B_o$. Then, P closed and convex. Also, $T(P) \subseteq P$ for each $T \in G$. To see this, since $T(B_o) \subseteq B_o$ for each $T \in G$, it suffices to show that $T(N) \subseteq N$ for each $T \in G$. Let $z \in N$ and $T \in G$. Then, $d(z,x) \leq \rho$ for each $x \in M$. Since

$T(M) = M$, there is a $y \in M$ such that $Ty = x$; hence, $d(Tz,x) = d(Tz,Ty) \leq d(z,y) \leq \rho$ for each $x \in M$. Thus, $Tz \in B[x,\rho]$ for each $x \in M$; that is, $T(N) \subseteq N$ for each $T \in G$. Thus, P belongs to D; and by the minimality of B_o in D, we have $P = B_o$

Since T is continuous and M is compact, there are elements x,y in M such that $d(x,y) = \delta(M)$. The element y does not belong to $B[x,\rho]$, and, consequently, y does not belong to B_o, which is a contradiction. Thus, $M = \{x\}$ for some x in C, and $Tx = x$ for each T in G.

Before stating our next result, we need to recall the following.

Definition 1.4 [7]. Let X be a metric space and D be a bounded subset of X. The *measure of noncompactness* of D, denoted by $\gamma(D)$, is defined as follows: $\gamma(D) = \inf\{\varepsilon > 0 : D$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$.

$\gamma(D)$ has the following properties:

(1) $0 \leq \gamma(D) \leq \delta(D)$.

(2) $\gamma(D) = 0$ if and only if D is precompact (i.e., $\overline{D}$ is compact).

(3) $\gamma(D) = \gamma(\overline{D})$.

(4) $\gamma(C \cup D) = \max\{\gamma(C), \gamma(D)\}$.

(5) $C \subset D$ implies $\gamma(C) \leq \gamma(D)$.

(6) $\gamma(S(B,r)) \leq \gamma(B) + 2r$, where $S(B,r) = \{x \in X : d(x,B) < r\}$.

If X is a Banach space, then in addition we have:

(7) $\gamma(C + D) \leq \gamma(C) + \gamma(D)$, where $C + D = \{c + d : c \in C, \ d \in D\}$.

(8) $\gamma(\alpha(D)) = |\alpha|\gamma(D)$, where α is any real number.

Closely related to the notion of measure of noncompactness is the concept of k-set contraction.

Definition 1.5. Let X be a metric space. A continuous mapping $T : X \to X$ is said to be a *k-set contraction* if for any bounded subset D of X we have $\gamma(T(D)) \leq k\gamma(D)$. T is said to be *densifying* if for any bounded subset D of X such that $\gamma(D) \neq 0$, $\gamma(T(D)) < \gamma(D)$.

An elegant discussion of measure of noncompactness and densifying mappings may be found in [10] and [12].

Theorem A [10, 11, 13]. Let X be a Banach space and C be a closed, bounded, convex subset of X. Let $T : C \to C$ be a densifying mapping. Then, T has at least one fixed point in C.

Lemma 1.6. Let X be a metric space and D be a nonempty, closed, bounded subset of X. Let $T : D \to D$ be a densifying mapping. Then, T is demicompact.

Proof. Let $\{x_n\}$ be a bounded sequence in D such that $d(x_n, Tx_n) \to 0$. We need to show that $\{x_n\}$ has a convergent subsequence, or, equivalently, it is enough to show that $\gamma\{x_n\} = 0$. Let $M = \{x_n\}$, so $T(M) = \{T(x_n)\}$. Since $d(x_n, Tx_n) \to 0$, it follows that for any $\varepsilon > 0$, $B(T(M), \varepsilon) = \bigcup\{B(y, \varepsilon) : y \in T(M)\}$ contains all but a finite number of elements of M. Thus, $\gamma(M) \leq \gamma(B(T(M), \varepsilon)) \leq \gamma(T(M)) + 2\varepsilon$. Hence, $\gamma(M) \leq \gamma(T(M))$; and, since T is densifying, $\gamma(M) = 0$.

The following result of Bahtin [1] follows as a special case of Theorem 1.3.

Corollary 1.7 [1, Theorem 1]. Let X be a real Banach space. Let C be a nonempty, bounded, closed convex subset of X. Let F be a commutative family of nonexpansive mappings of C into itself. Let there be at least one densifyin mapping in F. Then, the operators T in F have a common fixed point.

Proof. It follows from Theorem A that $F(T)$, the fixed-points set of T in C, is nonempty for each T in F. An appeal of Lemma 1.6 establishes the demicompactness of T.

The converse of Lemma 1.6 is not true, as can be seen from the following examples.

Example 1.7. Let $X = [0,1]$ with the usual metric. Define $T : X \to X$, as $Tx = x/2$ if $x \neq 0$ and $T(0) = 1$. Then T is not densifying, due to the lack of continuity of T. However, T is demicompact. Indeed, if $\{x_n\}$ is any bounded sequence in X such that $x_n - Tx_n \to 0$ as $n \to \infty$; then, from the Bolzano-Weierstrass theorem, it follows that $\{x_n\}$ has a convergent subsequence.

Example 1.8. Let $B = \{e_1, e_2, \ldots, e_n, \ldots\}$ be the usual orthonormal basis for ℓ^2. Define $T : B \to B$ by $T(e_i) = e_{i+1}$. Then, T is continuous (in fact, an isometry), but not densifying. However, T is demicompact. Indeed, if $\{e_i\}_{i=1}^{\infty}$ is a bounded sequence in B such that $e_i - Te_i$ converges, then $\{e_i\}_{i=1}^{\infty}$ must be finite.

Definition 1.9. A convex metric space X is said to be *strictly convex* if for any $x, y \in X$ and $\lambda (0 \leq \lambda \leq 1)$, there exists a unique element $z \in X$ such that $\lambda d(x,y) = d(z,y)$ and $(1 - \lambda) d(x,y) = d(x,z)$.

Lemma 1.10. Let X be a strictly convex metric space and K be a nonempty convex subset of X. If $T : K \to K$ is nonexpansive, then $F(T) = \{x \in K : Tx = x\}$ is convex.

Proof. Let $x, y \in F(T)$. Since K is convex, it follows that $W(x,y,\lambda) \in K$. We need to show that $T(W(x,y,\lambda)) = W(x,y,\lambda)$. Now,

$d(x,T(W(x,y,\lambda))) = d(Tx,T(W(x,y,\lambda))) \leq d(x,W(x,y,\lambda)) \leq (1 - \lambda)d(x,y)$ and

$d(y,T(W(x,y,\lambda))) \leq \lambda d(x,y)$.

Thus,

$$d(x,y) = d(Tx,Ty) \leq d(Tx,T(W(x,y,\lambda))) + d(T(W(x,y,\lambda)),Ty) \leq (1 - \lambda)d(x,y) + \lambda d(x,y)$$
$$= d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y) = d(x,y).$$

By the strict convexity of the space, it follows that $W(x,y,\lambda) = T(W(x,y,\lambda))$.

Theorem 1.11. Let X be a strictly convex metric space. Let $F : X \to X$ be a family of commuting nonexpansive mappings with the properties: (a) at least one of the mappings $T_o \in F$ is demicompact, (b) mapping T_o has at least one fixed point, and (c) there are no fixed points of the mapping T_o outside

$S = \{y \in X : d(x,y) \leq r\}$. Then, the family F has a common fixed point.

Proof. It follows from the continuity of T_o and Lemma 1.10 that $D = \{x \in X : T_o x = x\}$ is bounded, closed, and convex. Also, for any $T \in F$, $T(D) \subseteq D$. Indeed, $Tx = TT_o x = T_o Tx$ for any $x \in D$; that is, $Tx \in D$. Therefore, $T(D) \subset D$. The existence of a common fixed point now follows from the properties of the set D and Theorem 1.3.

As a corollary of Theorem 1.11, we have the following.

Corollary 1.12 [1, Theorem 2]. If X is a strictly convex Banach space and if in the commutative set F of nonexpansive operators on X there exists at least one densifying operator A_o with the properties (a) operator A_o has at least one fixed point and (b) outside a ball $\|x\| \leq r$, there are no fixed points of operator A_o; then, the operators $A \in F$ have a common fixed point.

§2. Asymptotically Nonexpansive Mappings.

In this section, we find a common fixed point for a commuting family of asymptotically nonexpansive mappings on certain subsets of a convex metric space. These results extend those of Goebel, Kirk and Thele [4], Hu [5], Kirk [6], and DeMarr [2].

Definition 1.2. Let X be a metric space, and let $T : X \to X$ be a mapping. A subset K of X is said to be *sequentially closed under* T if for each $x \in K$, every subsequential limit point of $\{T^n x\}$ lies in K.

Theorem 2.2. Let K be a compact convex subset of a convex metric space X. Let F be a commutative family of continuous mappings of K into itself, satisfying the following condition: for all $x \in K$ and $T \in F$,

(K) $\lim_{i\to\infty} \sup\{\sup_{y \in K} d(T^i x, T^i y) - d(x,y)\} \leq 0$. Then, F has a common fixed point.

Proof. Applying Zorn's Lemma, we obtain a set $L \subseteq K$, which is minimal with respect to being nonempty, compact, convex, and sequentially closed under each $T \in F$. Again, by Zorn's Lemma, there is a set $M \subseteq L$ such that M is minimal with respect to being nonempty, compact, and sequentially closed under each $T \in F$. Fix $S \in F$, and let $N = M \cap S(M)$.

By the continuity of S, N is compact. To show that N is nonempty, let $x \in M$. Then, $\{S^n x\} \subseteq K$ and there exists a subsequence $\{S^{n_i} x\}$ and $w \in K$ such that $S^{n_i} x \to w$. Since $x \in M$ and M is sequentially closed under S, $w \in M$. Consequently, $Sw \in S(M)$. Also, continuity of S implies that $S(w) = S(\lim_{i\to\infty} S^{n_i} x) = \lim_{i\to\infty} S^{n_i+1} x$; and, since M is sequentially closed under S, $S(w) \in M$. Hence, $S(w) \in M \cap S(M) = N$.

Finally, N is sequentially closed under each $T \in F$. Let $x \in N$ and $z = \lim_{i\to\infty} T^{n_i} x$. Since $x \in M$ and M is sequentially closed under T, $z \in M$. Also, $x \in S(M)$ implies $x = Sy$ for some $y \in M$. Hence, $S(T^{n_i} y) = T^{n_i}(Sy) = T^{n_i} x \to z$ as $i \to \infty$. Since L is compact $\{T^{n_i} y\}$ has a convergent subsequence $\{T^{n_{i_j}} y\}$, and there is $v \in L$ such that $T^{n_{i_j}} y \to v$ as $j \to \infty$. From the fact that $y \in M$ and M is sequentially closed under T, $v \in M$. The continuity of S implies that $S(T^{n_{i_j}} y) \to S(v)$ as $j \to \infty$; consequently, $S(v) = z$; i.e., $z \in S(M)$. Hence, $z \in N$ and N is sequentially closed under each $T \in F$.

Thus, we see that N is a nonempty, compact subset of M, which is sequentially closed under each $T \in F$. Hence, by the minimality of M, $N = M$; and, thus, $M \subseteq S(M)$. Since $S \in F$ was arbitrary, $M \subseteq S(M)$ for all $S \in F$.

If $\delta(M) = 0$, we are finished. Assume that $\delta(M) > 0$. Then, by Lemma 0.4, there exists $x_o \in L$ such that $0 < r = \sup\{d(x_o,y) : y \in M\} < \delta(M)$. Define $D = \{\bigcap_{y\in M} B[y,r]\} \cap L$. Clearly, D is convex and compact. Also, $D \neq \phi$, since $x_o \in D$. Since M is compact, there exist points $x_1, x_2 \in M$ with $d(x_1,x_2) = \delta(M) > r$. Hence, $x_2 \notin B[x_1,r]$ and $x_2 \notin D$. On the other hand, $x_2 \in M \subseteq L$; and, thus, D is properly contained in L.

Finally, to show that D is sequentially closed under each $T \in F$; let $z \in D$, suppose $\lim_{j\to\infty} T^{n_j}(z) = w$, and show that $w \in D$. Let $\varepsilon > 0$ be given. Since $\lim_{i\to\infty} \sup\{\sup_{x\in K} [d(T^i z,T^i x) - d(z,x)]\} \leq 0$, there is an integer N such that if $i \geq N$, $\sup_{x\in K}\{d(T^i z,T^i x) - d(z,x)\} < \varepsilon$. Also, since $T^{n_i} z \to w$ as $i \to \infty$, there is $n_j \geq N$ such that $d(w,T^{n_j} z) < \varepsilon$. Since $T(M) \subseteq M$ for each $T \in F$, $M \subseteq T^{n_j}(M)$. Let $y \in M \subseteq T^{n_j}(M)$. Then, there exists a $u \in M$ such that $T^{n_j}(u) = y$ and $d(w,y) \leq d(w,T^{n_j} z) + d(T^{n_j} z,y) \leq d(w,T^{n_j} z) + d(T^{n_j} z,T^{n_j} u)$ $< \varepsilon + d(z,u) + \varepsilon = 2\varepsilon + d(z,u) \leq 2\varepsilon + r$. Since ε was arbitrary, $d(w,y) \leq r, w \in B[y,r]$. Thus, $w \in \bigcap_{y\in M} B[y,r]$. On the other hand, since $z \in D \subseteq L$ and L is sequentially closed under F, $w = \lim_{i\to\infty} T^{n_i} z \in L$. Thus, $w \in (\bigcap_{y\in M} B[y,r]) \cap L = D$, showing D is sequentially closed under each $T \in F$. Thus, we see that D is a nonempty, compact, convex proper subset of L and that it is sequentially closed under each $T \in F$. This contradicts the minimality of L. Thus, M consists of a single point, which must be a fixed point of F.

Definition 2.3. Let X be a metric space and K be a nonempty subset of X. A mapping $T: K \to K$ is called *asymptotically nonexpansive* if for each $x,y \in K$, $d(T^i x, T^i y) \leq k_i d(x,y)$; $i = 1,2,\ldots$, where $\{k_i\}$ is a fixed sequence of positive real numbers such that $k_i \to 1$ as $i \to \infty$.

Corollary 2.4. Let K be a nonempty, compact, convex subset of a convex metric space X. If $F: K \to K$ is a commutative family of continuous, asymptotically nonexpansive mappings, then there is $x \in K$ such that $Tx = x$ for all $T \in F$.

Proof. It follows analogously to Kirk's result [6] for Banach spaces that if T is asymptotically nonexpansive, then T satisfies condition (K) in Theorem 2.2. The result now follows.

Example 2.5 [3]. Let B denote the unit ball in the Hilbert space ℓ^2, and let T be defined as follows: $T(x_1,x_2,x_3,\ldots) \to (0,x_1^2,A_2x_2,A_3x_3,\ldots)$, where A_i is a sequence of numbers such that $0 < A_i < 1$ and $\prod_{i=2}^{\infty} A_i = 1/2$ (for example, A_i may be taken as $1 - \frac{1}{i^2}$). Then, T is asymptotically nonexpansive; however, T is not nonexpansive. Thus, the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings.

Example 2.6. Let $X = [0,1]$, with the usual metric. Define $T: X \to X$ by $Tx = \frac{1}{2}\sqrt{x}$. Then, $T^n x \to 0$ uniformly as $n \to \infty$; so, T satisfies condition (K), but T is not asymptotically nonexpansive. In fact, T does not satisfy a Lipschitz condition.

The following results of Goebel, Kirk, and Thele [4], Hu [5], and DeMarr [2] are special cases of Theorem 2.2 and Corollary 2.4.

Corollary 2.7 [4, Theorem 3.1]. Let K be a nonempty, compact, convex subset of a Banach space X, and let F be a commutative semigroup of asymptotically nonexpansive mappings of K into K. Then, there exists a point $x \in K$ such that $Tx = x$ for each $T \in F$.

Corollary 2.8 [5]. Let X be a Banach space and M be a compact, convex subset of X. If F is a nonempty commuting family of continuous mappings of X into itself, satisfying the condition that for each $x \in X$ and $f \in F$, $\lim_{i\to\infty} \sup\{ \sup_{y \in X} [\|f^i x - f^i y\| - \|x - y\|]\} \leq 0$; then, F has a common fixed point in X.

Corollary 2.9 [2]. Let X be a Banach space, and let M be a nonempty, compact, convex subset of X. If F is a nonempty commuting family of nonexpansive mappings of X into itself, the family F has a common fixed point in X.

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