Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 24 (1983), No. 1, 63--87

Persistent URL: http://dml.cz/dmlcz/106205

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24,1 (1983)

LOCALLY NICE SPACES UNDER MARTIN'S AXIOM

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Abstract. The starting point of the paper is a "σ-discrete extension" of Szentmiklóssy's theorem that under Ma+ + ¬CH, a countably tight compact T_2 space has no hereditarily separable, non-Lindelöf subspaces. Then a parallelism (under MA+ ¬CH, again) is established between the theory of trees of height ω_1 and cardinality $< 2^{\omega}$ and the theory of locally compact, locally countable spaces of cardinality $< 2^{\omega}$. As applications in infinite combinatorics, Baumgartner's theorem on Aronszajn trees and a result of Wage on almost disjoint countable sets are deduced. It is proved under MA+ ¬CH that in a "locally nice" space hereditarily collectionwise T_2 implies paracompact iff the space does not contain a perfect preimage of the ordinal space ω_1. Moreover, conditions are given under which "hereditarily" can be omitted. These results improve a set of results of M.E. Rudin, D. Lane and G. Gruenhage among which the first was an affirmative answer under MA+ ¬CH, to the Alexandroff's old conjecture that a perfectly normal manifold is metrizable.

Key words and phrases: Countably tight spaces, tree, collectionwise T_2, ordinal space ω_1, nonmetrizable manifold.

Classification: Primary 54A35, 54D30
Secondary 54D45, 54E35

Introduction. The aim of the present paper is perfectly expressed by the title: we are going to give a structural analysis of some locally nice (locally countable, locally hereditarily Lindelöf, locally compact etc.) spaces assuming Martin s Axiom plus the negation of the Continuum Hypothesis (abbreviated, as usual, Ma + ¬CH).

The content of the paper is arranged in four sections.

In the first section we prove an extension of Szentmiklossy's theorem that countably tight compact T_2 spaces contain no S subspaces. This extension (Theorem 1.1) will then be a starting point in our further investigations. (For the history, see the beginning of the section and the Acknowledgement.)

In the second section locally compact, locally countable spaces are dealt with. Theorem 2.2 is a strengthening of a result (Corollary 2.4) of Gruenhage [9]. It shows that once "the tree has no ω_1-branch" is substituted by "the space contains no perfect preimage of the ordinal space ω_1" there is a surprising parallelism (under MA + $\neg$CH) between the theory of trees of height ω_1 and cardinality $< 2^{\omega}$ and the more general theory of locally compact, locally countable spaces of cardinality $< 2^{\omega}$. This parallelism is, in one direction, explained by the (known) fact that certain combinatorial structures admit a natural locally compact topology. We shall illustrate this point by deducing a couple of familiar theorems from infinite combinatorics: Baumgartner's theorem on Aronszajn trees and a result of Wage on almost disjoint sets.

The third section mainly concerns hereditarily collectionwise T_2, locally hereditarily Lindelöf, locally compact spaces. The reader should recall at this point that the long line, the most common example of a nonparacompact manifold and the (nonparacompact) ordinal space ω_1 are such spaces. Now one of the main results of this section says that if we exclude from the subspaces the perfect preimages of the ordinal space ω_1 then, under MA + $\neg$CH, such spaces are paracompact. Further results show that although "hereditarily collectionwise T_2" cannot be weakened to just "collectionwise T_2" in general, it can be weakened so if the space is either of Lindelöf degree $< 2^{\omega}$ or connected and hereditarily normal. A set of results of Rudin, Lane and Gruenhage concerning perfectly normal, locally compact spaces follows. Another (new) consequence concerns locally compact spaces with a G_δ-diagonal. At the end of this section we explain, with the aid of known examples, why there is little room to improve our results.

The fourth (and last) section is simply a specialization of the results of the third section to manifolds (more generally, to locally compact, locally connected spaces). At the end of this section we point out how there might be much room to improve our results if we restrict ourselves to manifolds.

Throughout the paper we use the terminology and notation of the current set theory and set-theoretic topology (as used in Kunen [13] and Engelking [5], for example). If $\varkappa$ is a cardinal, A is a set, then by definition $[A]^{\varkappa} = \{A' \subset A : |A'| = \varkappa\}$, $[A]^{<\varkappa} = \{A' \subset A : |A'| < \varkappa\}$. All spaces are meant to be topological and regular T_1. Some deviation from the standard usage is that like Fremlin [6], we say "X is countably tight" instead of "X has countable tightness". $\bar{A}$ always denotes the closure of A in the space X, whatever space the letter X denotes in that context.

1. Locally countable spaces in countably tight compact spaces

In 1981 the author observed that the proof of Szentmiklóssy's famous result [21] on the non-existence, under MA + $\neg$CH, of S subspaces of countably tight compact spaces also applies to prove, more generally,

Theorem $\Sigma^-(MA(\omega_1))$. Every locally countable subspace of cardinality ω_1 in a countably tight compact space is the union of countably many discrete subspaces.

He then proved a number of consequences which form part of this paper.

The proof of Theorem Σ^- relied on the order type of ω_1, as did Szentmiklóssy's proof. However, having been informed on the proof of Theorem Σ^-, D. Fremlin [6] was able to get rid of this restriction and proved

Theorem Σ (MA + $\neg$CH). Every locally countable subspace of cardinality $< 2^{\omega}$ in a countably tight compact space is the union of countably many discrete subspaces.

In some of the results of this paper the following more general version of Theorem Σ will be extremely useful:

Theorem 1.1 (MA + $\neg$CH). Let X be a countably tight compact space, Z be a locally countable subspace of X with $|Z| < 2^{\omega}$, and $\mathcal{V}$ be a family of $< 2^{\omega}$ open subsets of X such that

(a) $Z \subset \bigcup \mathcal{V}$

(b) For every $V \in \mathcal{V}$ there is an open subset U_V of X such that $\overline{V} \subset U_V$ and $|U_V \cap Z| \leq \omega$.

Then $Z = \bigcup_{n \in \omega} A_n$ such that each A_n is a closed discrete subset of the subspace $Y = \bigcup \mathcal{V}$.

To prove Theorem 1.1 we need the following two results.

Lemma 1.2 (Szentmiklóssy [21], in essence). Suppose that $\{K_\xi : \xi \in \omega_1\}$ is a family of pairwise disjoint finite sets and $\mathcal{B}$ is a family of sets such that

(a) $\mathcal{B}$ is closed under finite unions

(b) For every $B \in \mathcal{B}$, $|B \cap (\bigcup_{\xi \in \omega_1} K_\xi)| \leq \omega$;

(c) There is a sequence $\{B_\xi : \xi \in \omega_1\} \subset \mathcal{B}$ such that for every $\eta < \xi < \omega_1$ we have $B_\xi \cap K_\eta \neq \emptyset$.

Then there is a set $D \in [\bigcup_{\xi \in \omega_1} K_\xi]^{\omega_1}$, and a sequence $\{B_\xi : \xi \in \omega_1\}$ such that for every $C \in [D]^{\omega_1}$ there is an $\alpha \in \omega_1$ with $\{C\} \cup \{B_\xi : \xi \in \omega_1 - \alpha\}$ centered.

Lemma 1.3 (Galvin and Hajnal). Suppose MA holds, P is a c.c.c. poset of cardinality $< 2^\omega$ and $\mathcal{D}$ is a family of $< 2^\omega$ dense subsets of P. Then $P = \bigcup_{n \in \omega} G_n$ such that each G_n is P-generic over $\mathcal{D}$.

The heart of the proof of Theorem 2.1 (viz. that the poset we set up is c.c.c.) will proceed parallel to the proof of Lemma 44 B in Fremlin [6]. However, we have to set up a different poset from that of in [6] and, therefore, part of the notation of [6] is not applicable here. Thus we think to make our paper more readable (and, perhaps, the statement of Theorem 1.1 more convincing) by giving the details here.

Proof of Theorem 1.1. Let $P = [Z]^{<\omega} \times [\mathcal{V}]^{<\omega}$ with the following partial order:

$$\langle K, \mathcal{H} \rangle \geq \langle K', \mathcal{H}' \rangle \text{ iff } K \subset K',\ \mathcal{H} \subset \mathcal{H}' \text{ and } (K' - K) \cap (\bigcup \mathcal{H}) = \emptyset.$$

Suppose P is not c.c.c. Then there is a family $p_\xi = \langle K_\xi, \mathcal{H}_\xi \rangle$ $(\xi < \omega_1)$ of pairwise incompatible members of P. We may suppose that

(i) $\{K_\xi : \xi \in \omega_1\}$ forms a Δ-system with root K;

(ii) For every $\xi \in \omega_1$, $K^*_\xi \cap \bigcup_{\eta < \xi} (\bigcup \mathcal{H}_\eta) = \emptyset$, where $K^*_\xi = K_\xi - K$ $(\xi \in \omega_1)$.

To see that we can make sure (ii) note that $|\bigcup_{\eta < \xi} (\bigcup \mathcal{H}_\eta) \cap Z| \leq \omega$ for every $\xi \in \omega_1$, and the K^*_ξ's are pairwise disjoint.

Since the p_ξ's are pairwise incompatible by (ii) we have

$\eta < \xi < \omega_1$ implies $K^*_\eta \cap (\bigcup \mathcal{H}_\xi) \neq \emptyset$.

Thus the family $\mathcal{B} = \{(\bigcup \mathcal{H}) : \mathcal{H} \in [\mathcal{V}]^{<\omega}\}$ and the sequence $\{K^*_\xi : \xi \in \omega_1\}$ satisfy the conditions of Lemma 2.2. Thus there is a set $D \in [Z]^{\omega_1}$ and a sequence $\{B_\xi : \xi \in \omega_1\} \subset \mathcal{B}$ such that

($*$) for every $C \in [D]^{\omega_1}$ there is an $\alpha \in \omega_1$ with $\{C\} \cup \{B_\xi : \xi \in \omega_1 - \alpha\}$ centered.

Note that each B_ξ has the form $B_\xi = \bigcup \mathcal{H}'_\xi$ for some $\mathcal{H}'_\xi \in [\mathcal{V}]^{<\omega}$ and so

($**$) $\overline{B}_\xi \subset \bigcup_{V \in \mathcal{H}'_\xi} U_V$.

Let us choose inductively a couple of sequences $\alpha(\xi) \in \omega_1$, $G_\xi \subset X (\xi \in \omega_1)$ in the following way:

(1) $G_\xi = \bigcup \{ U_V : V \in \bigcup_{\eta < \xi} \mathcal{H}'_\eta \}$;

(2) $(D - G_\xi) \cap \bigcap_{\xi \in c} B_\xi \neq \emptyset$ for every $c \in [\omega_1 - \alpha(\xi)]^{<\omega}$

(3) $\eta < \xi$ implies $\alpha(\eta) < \alpha(\xi)$.

(2) is possible by ($*$). (Remember that by condition (b) of Theorem 1.1 each G_ξ is countable!)

Now let, for every $\xi \in \omega_1$,

$$F_\xi = \bigcap_{\rho \geq \alpha(\xi)} \overline{B}_\rho .$$

By ($**$), $F_\xi \subset \overline{B}_{\alpha(\xi)} \subset G_{\xi+1}$. On the other hand (2) implies

($***$) $F_\xi \cap (X - G_\xi) \neq \emptyset$ for every $\xi \in \omega_1$.

Thus $F_\xi \subsetneq F_{\xi+1}$, i.e. $\{F_\xi : \xi \in \omega_1\}$ is a monotone increasing family of compact subsets. Since X is countably tight, this implies that $F = \bigcup_{\xi \in \omega_1} F_\xi$ is compact. Since $F \subset \bigcup_{\xi \in \omega_1} G_\xi$ and the G_ξ's are increasing, there is a $\xi \in \omega_1$ with $F \subset G_\xi$, in contradiction with ($***$).

Thus P is a c.c.c. poset of cardinality $< 2^\omega$. For every $V \in \mathcal{V}$ consider

$$E_V = \{\langle K, \mathcal{H}\rangle \in P : V \in \mathcal{H}\}.$$

E_V is P-dense, since any $p = \langle K_p, \mathcal{H}_p\rangle \in P$ can be extended by $p' = \langle K_p, \mathcal{H}_p \cup \{V\}\rangle \in E_V$. Let $\mathcal{D} = \{E_V : V \in \mathcal{V}\}$, and consider a $G \subset P$ such that G is P-generic over $\mathcal{D}$. Let us define

$$A = \bigcup\{K : (\exists \mathcal{H})\langle K, \mathcal{H}\rangle \in G\}.$$

We claim that A is closed discrete in $Y = \bigcup \mathcal{V}$. Once we prove that claim, Theorem 1.1 follows from Lemma 1.3.

To see that our claim is true, it is enough to verify that $|V \cap A| < \omega$ for every $V \in \mathcal{V}$. To see this, let $p = \langle K_p, \mathcal{H}_p\rangle \in E_V \cap G$. Now, if we had $V \cap A \not\subset K_p$ then there would be a $z \in V \cap A - K_p$. Then, by $z \in A$ we could find a $p' = \langle K_{p'}, \mathcal{H}_{p'}\rangle \in G$ with $z \in K_{p'}$. We may assume $p' \leq p$. Then

$$z \in V \cap (K_{p'} - K_p) \subset (\bigcup \mathcal{H}_p) \cap (K_{p'} - K_p).$$

But $p' \leq p$ implies $(\bigcup \mathcal{H}_p) \cap (K_{p'} - K_p) = \emptyset$, a contradiction. Thus $V \cap A \subset K_p$, i.e., $|V \cap A| < \omega$, q.e.d.

Remarks. There are lots of strengthenings of Theorem 1.1. For example, we have the following combinatorial strengthening which then enables us, under some additional conditions, to prescribe some points to be accumulation points of the A_n's in X.

Strengthening 1. Suppose that the conditions of Theorem 1.1 are satisfied, and $\mathcal{B}$ is a family of $< 2^{\omega}$ subsets of Z such that no member of $\mathcal{B}$ can be covered by a finite subfamily of $\mathcal{V}$. Then, in addition to the conclusion of Theorem 1.1. we can make $Z = \bigcup_{n\in\omega} A_n$ so that $|B \cap A_n| \geq \omega$ holds for every $n \in \omega$ and $B \in \mathcal{B}$.

To see that this can be done. consider the poset P of the proof of Theorem 1.1 and let, for every $m \in \omega$ and $B \in \mathcal{B}$,

$D_m(B) = \{\langle K, \mathcal{H}\rangle \in P : |K \cap B| \geq m\}$.

$D_m(B)$ is dense in P, since for any $p = \langle K_p, \mathcal{H}_p\rangle \in P$ there is a $K \in [B - \bigcup \mathcal{H}_p]^m$, and then p can be extended by $p' = \langle K_p \cup K, \mathcal{H}_p\rangle \in D_m(B)$. Now add to the family $\mathcal{D}$ at the end of the proof of Theorem 1.1 all of the $D_m(B)$'s ($m \in \omega$, $B \in \mathcal{B}$). Let $G \subset P$ be P-generic over this $\mathcal{D}$ and $A = \bigcup\{K : (\exists \mathcal{H}) \langle K, \mathcal{H}\rangle \in G\}$. Then it can be easily verified that $B \in \mathcal{B}$ implies $|B \cap A| \geq m$ for every $m \in \omega$.

Strengthening 2. Suppose that the conditions of Theorem 1.1 are satisfied. Further suppose that $Y = \bigcup\{\overline{V} : V \in \mathcal{V}\}$ and $F \subset \overline{Z} - Y$ is a closed subspace of X such that F has an outer base $\mathcal{B}^*$ of cardinality $< 2^\omega$ in $\overline{Z}$. Then, in addition to the conclusion of Theorem 1.1 we can make $Z = \bigcup_{n \in \omega} A_n$ so that $\overline{A}_n \supset F$ holds for every $n \in \omega$.

To see that this strengthening is possible, let $\mathcal{B} = \{B \cap Z : B \in \mathcal{B}^*, B \cap F \neq \emptyset\} \subset P(Z)$. Suppose indirectly that $B \cap Z \subset \bigcup\{\overline{H} : H \in \mathcal{H}\}$ for some $B \cap Z \in \mathcal{B}$ and $\mathcal{H} \in [\mathcal{V}]^{<\omega}$. Then $\overline{B} = \overline{B \cap Z} \subset \bigcup\{\overline{H} : H \in \mathcal{H}\} \subset Y$ in contradiction with $F \cap B \neq \emptyset$. Thus we can apply Strengthening 1 to conclude that for every $B \cap Z \in \mathcal{B}$ and $n \in \omega$ $|B \cap Z \cap A_n| \geq \omega$ holds. Since $\mathcal{B}^*$ is an outer base for F in $\overline{Z}$ this means that $\overline{A}_n \supset F$.

2. Locally countable, locally compact spaces and infinite combinatorics

Lemma 2.1. The following are equivalent for a countably tight locally compact space X:

(a) The one-point compactification $X^* = X \cup \{x^*\}$ of X is countably tight;

(b) X does not contain a perfect preimage of the ordinal space ω_1.

Proof. To show (a) $\Longrightarrow$ (b), it is enough to verify that if a space X has a perfect map f onto the ordinal space ω_1, then X cannot be embedded into a compact space $\tilde{X}$ of countable tightness. Suppose indirectly that it can be embedded. Then, since $X = f^{\leftarrow}(\omega_1)$ is not compact, there is a point $x \in cl_{\tilde{X}} X - X$. By $t(\tilde{X}) = \omega$ there is a countable $A \subset X$ with $x \in cl_{\tilde{X}} A - X$. Let $\alpha \in \omega_1$ be such big that $A \subset f^{\leftarrow}(\alpha \cup \{\alpha\}) = X_\alpha$. By the perfectness of f, X_α is a compact subset of X so that $x \notin cl_{\tilde{X}} X_\alpha$, in contradiction with $A \subset X_\alpha$.

A less trivial task is to prove (b) $\Longrightarrow$ (a), but this essentially follows from the following result of Gruenhage and Burke (see [8],[3]):

(GB) If Y is a noncompact space and every separable closed subspace of Y is compact, then Y contains a perfect preimage of the ordinal space ω_1.

Now assume that X^* does not have countable tightness. Then there is an $A \subset X$, $|A| > \omega$ such that $Y = cl_X A$ is not compact but $cl_X A'$ is compact for every $A' \in [A]^\omega$. Then, since Y is countably tight, every separable closed subspace of Y is contained in $cl_Y A' = cl_X A'$ for some $A' \in [A]^\omega$, and so is compact. Thus (GB) is applicable.

Theorem 2.2 (MA $+ \neg$CH). Suppose that Y is a locally compact, locally countable space of cardinality $< 2^\omega$. Then the following conditions are equivalent:

(a) The one-point compactification X of Y is countably tight;

(b) Y does not contain a perfect preimage of ω_1;

(c) Y is the union of countably many closed discrete subspaces;

(d) Y is a Moore space.

Proof. The equivalence of (a) and (b) follows from Lemma 2.1. Therefore it is enough to show (a) $\Longrightarrow$ (c), (c) $\Longrightarrow$ (d) and (d) $\Longrightarrow$ (b).

(a) $\Longrightarrow$ (c). Let $\mathcal{V}$ be a base of cardinality $< 2^{\omega}$ for Y consisting of open sets with compact closures. Since Y is an open, locally compact, locally countable subspace of X, it is easy to find, for each $V \in \mathcal{V}$, an open subset U_V of X such as required in the conditions of Theorem 1.1. Then applying Theorem 1.1 with Z = Y finishes the proof.

(c) $\Longrightarrow$ (d). It is wellknown, more generally, that a first countable space which is the union of countably many closed discrete subspaces, is developable.

(d) $\Longrightarrow$ (b). Since a countably compact Moore space is compact, and subspaces of Moore spaces are Moore, we infer that a perfect preimage of ω_1, being a countably compact noncompact space, cannot be embedded in any Moore space.

Remark. As one can easily prove, (c) and (d) are equivalent in ZFC even if Y is only supposed to be locally countable and first countable. We, however, will not need this fact in the present paper.

Corollary 2.3 (MA + $\neg$CH). A locally compact space Y of cardinality $< 2^{\omega}$ is a Moore space if and only if it has a G_δ-diagonal.

Proof. It is enough to note that by a result of J. Chaber [4] a countably compact noncompact space (in particular, a perfect preimage of ω_1) cannot have a G_δ-diagonal.

Remark. There are examples (Gerlitz [7], Burke [2], W. Weiss [22]) of locally compact nondevelopable spaces with G_δ-diagonals. Those spaces are locally countable and have cardinality 2^ω. It is an interesting consequence of Corollary 2.3 that there are no such spaces of cardinality ω_1 in ZFC (cf. Corollary 3.14 and Remark 3 at the end of the third section).

Corollary 2.4 (G. Gruenhage [9], MA + $\neg$CH). A locally compact space of cardinality $< 2^\omega$ is a Moore space if and only if it is perfect.

Proof. No perfect preimage of ω_1 is a perfect space. (Otherwise its perfect image, ω_1 would be a perfect space which it is not.)

The proof of the following folklore result is omitted.

Proposition 2.5. Let T be a tree of height ω_1. Equip T with the tree topology. Then the one-point compactification of T is countably tight if and only if T has no ω_1-branch.

Corollary 2.6 (Baumgartner, MA + $\neg$CH). Suppose T is a tree of cardinality $< 2^\omega$ and T has no ω_1-branch. Then T is the union of countably many antichains.

Proof. Note first that if $A \subset T$ is closed discrete in the tree topology then A is the union of countably many antichains.

(Indeed, if A is closed discrete then for every $x \in A$, $\hat{x} \cap A = \{y \in A : y \leq_T x\}$ is finite. Now consider, for each $n \in \omega$ the antichain $A_n = \{x \in A : \hat{x} \cap A = n+1\}$. Then $A = \bigcup_{n \in \omega} A_n$.)
By Proposition 2.5, the one-point compactification of T is countably tight so that Theorem 2.2 implies that T is the

union of countably many closed discrete subspaces, and thus, of countably many antichains.

The last corollary is essentially due to Wage (see [18], p. 500).

Corollary 2.7 (MA + $\neg$CH). Let $\varkappa < 2^{\omega}$ be an uncountable cardinal and $\{H_\ell : \ell \in L\}$ be an almost disjoint family of countable subsets of $\varkappa$ with $|L| < 2^{\omega}$. Then $\varkappa = \bigcup_{n\in\omega} A_n$ such that for each $n \in \omega$, $\{H_\ell : \ell \in L\} \cup \{A_n\}$ is almost disjoint.

Proof. Define a topological space X in the following way.

The underlying set of X is the disjoint union of L and $\varkappa$. Define the topology of X by the following two conditions:

(1) $\varkappa$ is an open discrete subspace;

(2) $\{\{\ell\} \cup (H_\ell - F) : F \in [\varkappa]^{<\omega}\}$ is a neighbourhood base for ℓ in X.

It is easily verified that (1) and (2) define a locally compact T_2 topology. Suppose indirectly that X contains a perfect preimage $P = f^{\leftarrow}(\omega_1)$ of the ordinal space ω_1. Then, since P is countably compact and L is closed discrete in X, it follows that $|P \cap L| < \omega$. Thus there is an $\alpha \in \omega_1$ with $f^{\leftarrow}(\omega_1 - \alpha) \subset \varkappa$. Since $\varkappa$ is a discrete subspace and $f^{\leftarrow}(\omega_1 - \alpha)$ is countably compact, $|f^{\leftarrow}(\omega_1 - \alpha)| < \omega$, a contradiction.

Now, by Theorem 2.2, $\varkappa$ (more generally, the whole space X) is the union of countably many closed discrete subspaces of X. Then we can finish the proof by observing that each H_ℓ, being a compact set in X, has a finite intersection with any closed discrete subspace.

3. On locally nice spaces

Definition 3.1. We shall say that a space X is σ-collectionwise T_2 if for every closed discrete subset A of X we have $A = \bigcup_{n\in\omega} A_n$ such that for each $n \in \omega$, the points of A_n can be simultaneously separated by open subsets of X.

Remark. Collectionwise T_2 spaces are σ-collectionwise T_2. Normal σ-collectionwise T_2 spaces are collectionwise T_2.

Lemma 3.2. Let X be a locally hereditarily Lindelöf, locally hereditarily separable, hereditarily σ-collectionwise T_2 space. Then X is the topological sum of clopen subspaces each having the Lindelöf degree $\leq \omega_1$.

Proof. First of all we claim that if $F \subset X$ is any subspace of X then there is a σ-disjoint collection $\mathcal{G}$ of hereditarily Lindelöf, open subsets of X such that $(\bigcup\mathcal{G}) \cap F$ is dense in F.

To prove this, let $\mathcal{U}$ be a maximal family of pairwise disjoint, relatively open, separable subsets of F, and let, for every $U \in \mathcal{U}$, $S(U) = \{x_n(U) : n \in \omega\}$ be a dense subset of U. Further, for every $U \in \mathcal{U}$ let us choose an open subset $\tilde{U}$ of X such that $\tilde{U} \cap F = U$. Now, for every $n \in \omega$, $S_n = \{x_n(U) : U \in \mathcal{U}\}$ is a closed discrete subset of the open subspace $\bigcup\{\tilde{U} : U \in \mathcal{U}\}$. Since $\bigcup\{\tilde{U} : U \in \mathcal{U}\}$ is σ-collectionwise T_2, there is a σ-disjoint collection $\{G_n(U) : U \in \mathcal{U}\} = \mathcal{G}_n$ of open hereditarily Lindelöf subspaces of X such that $x_n(U) \in G_n(U)$ for every $U \in \mathcal{U}$. Then $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ is as required.

Now define by induction a sequence $\{\langle F_\alpha, \mathcal{G}_\alpha\rangle : \alpha \in \omega_1\}$

in the following way. $F_0 = X$ and $\mathcal{G}_0$ is a collection of pairwise disjoint, open, hereditarily Lindelöf subspaces of X such that $\bigcup \mathcal{G}_0$ is dense in X. If $\beta \in \omega_1 - \{0\}$ and $\{\langle F_\alpha, \mathcal{G}_\alpha \rangle : \alpha \in \beta\}$ is already defined, then let $F_\beta = X - \bigcup_{\alpha \in \beta} (\bigcup \mathcal{G}_\alpha)$ and let $\mathcal{G}_\beta$ be a σ-disjoint collection of hereditarily Lindelöf, open subspaces of X such that $(\bigcup \mathcal{G}_\beta) \cap F_\beta$ is dense in F_β.

Then $X = \bigcup_{\alpha \in \omega_1} (\bigcup \mathcal{G}_\alpha)$. Indeed, if there was an $x \in X - \bigcup_{\alpha \in \omega_1} (\bigcup \mathcal{G}_\alpha) = \bigcap_{\alpha \in \omega_1} F_\alpha$, then we could take a <u>hereditarily</u> Lindelöf neighbourhood V of x and, since $(\bigcup \mathcal{G}_\alpha) \cap F_\alpha = F_\alpha$, we could take a point $x_\alpha \in V \cap (\bigcup \mathcal{G}_\alpha) \cap F_\alpha$ for each $\alpha \in \omega_1$. Then $\{x_\alpha : \alpha \in \omega_1\}$ would be an uncountable scattered subspace of V, in contradiction with the assumption that V is hereditarily Lindelöf.

Thus $\mathcal{G}^* = \bigcup_{\alpha \in \omega_1} \mathcal{G}_\alpha$ is an open cover of X and $\mathcal{G}^*$ is the union of $\leq \omega_1$ disjoint collections. Since every member of $\mathcal{G}^*$ is c.c.c., it follows that every $G \in \mathcal{G}^*$ intersects only $\leq \omega_1$ other members of $\mathcal{G}^*$. Thus, by an easy induction argument, X can be decomposed into disjoint open (and, thus, closed) subspaces each of which is the union of $\leq \omega_1$ members of $\mathcal{G}^*$ and has, therefore, Lindelöf degree $\leq \omega_1$.

<u>Theorem 3.3</u> (MA + ¬CH). Let X be a locally hereditarily Lindelöf hereditarily σ-collectionwise T_2 space. Suppose that X can be embedded into a countably tight compact space. Then X is paracompact.

<u>Proof</u>. Note first that since MA + ¬CH implies a countably tight compact space contains no L subspaces [21] X is also locally hereditarily separable. Further, a locally Lindelöf space is paracompact iff it is the topological sum of

its clopen Lindelöf subspaces. If the space is, in addition, locally hereditarily Lindelöf and locally hereditarily separable, then these clopen subspaces are automatically hereditarily Lindelöf and hereditarily separable.

By Lemma 3.2 we only have to consider the case when the Lindelöf degree of X is ω_1. Let $\mathcal{U} = \{U_\alpha : \alpha \in \omega_1\}$ be an open cover of X by open, hereditarily Lindelöf subspaces such that $U_\alpha - \bigcup_{\beta \in \alpha} U_\beta \neq \emptyset$ for every $\alpha \in \omega_1$. Let

$$A = \{\alpha \in \omega_1 : \overline{\bigcup_{\beta \in \alpha} U_\beta} \neq \bigcup_{\beta \in \alpha} U_\beta\}.$$

To prove that X has a topological decomposition such as described above it is enough to show that A is non-stationary in ω_1.

Suppose indirectly that A is stationary in ω_1. Then choose, for every $\alpha \in A$, a point $x_\alpha \in \overline{\bigcup_{\beta \in \alpha} U_\beta} - \bigcup_{\beta \in \alpha} U_\beta$. Since $\alpha \geq \beta$ implies $x_\alpha \notin U_\beta$, the subspace $\{x_\alpha : \alpha \in A\}$ is locally countable. Thus we can apply Theorem Σ to get $A = \bigcup_{n \in \omega} A_n$ such that each $\{x_\alpha : \alpha \in A_n\}$ is a discrete subspace. Let A_{n_0} be non-stationary in ω_1 and let, for every $\alpha \in A_{n_0}$, V_α be an open subset of X such that $V_\alpha \cap \{x_\alpha : \alpha \in A_{n_0}\} = \{x_\alpha\}$. Then $D = \{x_\alpha : \alpha \in A_{n_0}\}$ is a closed discrete subspace of the open subspace $S = \bigcup\{V_\alpha : \alpha \in A_{n_0}\}$. Since X is σ-collectionwise T_2, there is a stationary set $A'_{n_0} \subset A_{n_0}$ such that we can find a family $\{G_\alpha : \alpha \in A'_{n_0}\}$ of disjoint open subsets of S (and thus, of X) such that $x_\alpha \in G_\alpha$ for every $\alpha \in A'_{n_0}$. Since for every $\alpha \in A'_{n_0}$ $x_\alpha \in \overline{\bigcup_{\beta \in \alpha} U_\beta}$ it follows that for every $\alpha \in A'_{n_0}$ there is an $f(\alpha) \in \alpha$ with $G_\alpha \cap U_{f(\alpha)} \neq \emptyset$. Since A'_{n_0} is stationary, the Pressing Down Lemma implies that there is a $\beta \in \omega_1$ with $|f^{-1}(\beta)| = \omega_1$. Thus U_β intersects uncountably many G_α's in contradiction with our assumption that U_β is a hereditarily Lindelöf (and thus, c.c.c.) subspace.

Theorem 3.4 (MA + ¬CH). Let X be a locally compact, locally hereditarily Lindelöf, hereditarily σ-collectionwise T_2 space. Then X is paracompact if and only if X does not contain a perfect preimage of the ordinal space ω_1.

Proof. Only the "if" part needs proof. So let X not contain a perfect preimage of ω_1. Then by Lemma 2.1 X can be embedded into a compact space of countable tightness. Thus Theorem 3.3 is applicable.

Theorem 3.5 (MA + ¬CH). Let W be a locally hereditarily Lindelöf, σ-collectionwise T_2 space with Lindelöf degree $< 2^\omega$. Suppose that W can be embedded into a countably tight compact space X. Then W is hereditarily σ-collectionwise T_2.

Proof. Let Z be an uncountable discrete subspace of W. Let $\mathcal{U}$ be a cover of W by open subsets of X such that $U \cap W$ is hereditarily Lindelöf, and thus, $U \cap Z$ is countable, for every $U \in \mathcal{U}$. By regularity of X there is a cover $\mathcal{V}$ of W by open subsets of X such that for every $V \in \mathcal{V}$ there is a $U_V \in \mathcal{U}$ with $\overline{V} \subset U_V$. Since $L(W) < 2^\omega$ we may assume $|\mathcal{V}| < 2^\omega$. By Theorem 1.1 $Z = \bigcup_{n \in \omega} A_n$ such that each A_n is closed discrete in $Y = \bigcup \mathcal{V}$, and thus, in W. Since W is σ-collectionwise T_2, it follows that Z is the union of countably many collections $\{A_{nk} : n,k \in \omega\}$ such that the points of each A_{nk} can be simultaneously separated by open subsets in W.

Combining Theorems 3.3 and 3.5 together, we get

Theorem 3.6 (MA + ¬CH). Let X be a locally hereditarily Lindelöf, σ-collectionwise T_2 space with Lindelöf degree

$< 2^{\omega}$. Suppose that X can be embedded into a countably tight compact space. Then X is paracompact.

The following result of Nyikos [15] is an application of Jones' Lemma.

Lemma 3.7 (Nyikos [15]). Suppose that $cf(2^{\omega}) > \omega_1$. Then every separable, locally hereditarily Lindelöf, hereditarily normal space has the Lindelöf degree $< 2^{\omega}$.

Remark. In [15] Nyikos made the assumption "2^{ω} is a successor" but his proof only requires "$cf(2^{\omega}) > \omega_1$".

One easily checks that Nyikos' argument is valid to prove the following more general version:

Lemma 3.7'. Suppose that $2^{\omega} > \omega_1$ and for every $\lambda < 2^{\omega}$, $2^{\lambda} = 2^{\omega}$ holds. Then every locally hereditarily Lindelöf, hereditarily normal space of density $< 2^{\omega}$ has the Lindelöf degree $< 2^{\omega}$.

(Note that the second cardinal assumption in Lemma 3.7' implies that 2^{ω} is regular.)

Iterating Lemma 3.7', we get

Lemma 3.8. Suppose that $2^{\omega} > \omega_1$ and for every $\lambda < 2^{\omega}$, $2^{\lambda} = 2^{\omega}$ holds. Let X be a locally hereditarily Lindelöf, locally hereditarily separable, hereditarily normal space and x be an arbitrary point of X. Then x is contained in a clopen set of the Lindelöf degree $< 2^{\omega}$.

Proof. Note first that since X is locally hereditarily separable, every subspace $Y \subset X$ with $L(Y) < 2^{\omega}$ also satisfies $d(Y) < 2^{\omega}$ and thus, by Lemma 3.7', $L(\overline{Y}) < 2^{\omega}$. Therefore we may construct an increasing sequence $\{X_{\alpha} : \alpha \in \omega_1\}$ of open subsets of X such that

(a) $x \in X_0$;

(b) $L(X_\alpha) < 2^\omega$ for every $\alpha \in \omega_1$,

(c) $\overline{X}_\alpha \subset X_{\alpha+1}$ for every $\alpha \in \omega_1$.

Then $Y = \bigcup_{\alpha \in \omega_1} X_\alpha$ is an open subset containing x. By (c), $Y = \bigcup_{\alpha \in \omega_1} \overline{X}_\alpha$, which implies $Y = \overline{Y}$, since X is countably tight. Finally $L(Y) < 2^\omega$ follows from (b) and from $cf(2^\omega) > \omega_1$.

Remark. There are other cardinal assumptions which make Lemma 3.8 true. (One such is, for example, $2^\omega = \omega_n (n \geq 2)$.) However, the assumptions of Lemma 3.8 suit us best, since we are going to assume MA + ¬CH which implies $(\forall \lambda < 2^\omega)(2^\lambda = 2^\omega)$.

Putting Lemma 3.8 and Theorem 3.6 together yields

Theorem 3.9 (MA + ¬CH). Let X be a connected, locally hereditarily Lindelöf, hereditarily normal, σ-collectionwise T_2 space. Suppose that X can be embedded into a countably tight compact space. Then X is hereditarily Lindelöf.

Proof. Since a countably tight compact space contains no L subspaces, X is locally hereditarily separable. Lemma 3.8 and connectedness together then imply $L(X) < 2^\omega$ so that Theorem 3.6 is applicable to get paracompactness of X. A paracompact locally hereditarily Lindelöf space, however, is the free sum of clopen hereditarily Lindelöf subspaces, so X is hereditarily Lindelöf again by connectedness.

Remark. To get paracompactedness of X we made use of connectedness only through Lemma 3.8. Therefore, if we could prove some analogue of Lemma 3.2 with "hereditarily normal" in place of "hereditarily σ-collectionwise T_2" then we could quote Theorem 3.6 to prove

Conjecture 3.10. Suppose that all the conditions of Theorem 3.9 except connectedness of X are satisfied. Then X is paracompact.

Looking back to Lemma 2.1 gives us the following "locally compact versions" of Theorems 3.6 and 3.9, respectively.

Theorem 3.11 (MA $+\neg$CH). Let X be locally compact, locally hereditarily Lindelöf, σ-collectionwise T_2 space with the Lindelöf degree $< 2^{\omega}$. Then X is paracompact iff X does not contain a perfect preimage of ω_1.

Theorem 3.12 (MA $+\neg$CH). Let X be a connected, locally compact, locally hereditarily Lindelöf, hereditarily normal, σ-collectionwise T_2 space. Then X is paracompact iff X does not contain a perfect preimage of ω_1.

In [9] Gruenhage proved that under MA $+\neg$CH, every perfectly normal, locally compact space is paracompact provided it is collectionwise normal with respect to compact sets. A stronger form of Gruenhage's result is a corollary to our Theorem 3.4:

Corollary 3.13 (MA $+\neg$CH). If X is a locally compact, perfect, σ-collectionwise T_2 space, then X is paracompact.

Proof. It can be easily seen that a perfect, σ-collectionwise T_2 space is hereditarily σ-collectionwise T_2. Thus, making use of the fact that perfect compact spaces are hereditarily Lindelöf and that a perfect preimage of the ordinal space ω_1 is not a perfect space (cf. the proof of Corollary 2.4), we can apply Theorem 3.4.

Corollary 3.14 (MA $+\neg$CH). If X is a locally compact, σ-collectionwise T_2 space with a G_δ-diagonal, then X is paracompact provided one of the following conditions holds:

(a) X is hereditarily σ-collectionwise T_2;

(b) X has the Lindelöf degree $< 2^{\omega}$;

(c) X is T_5 and connected.

Proof. In order to apply the corresponding theorems proved above it is enough to note that a compact space with a G_δ-diagonal is second countable and a perfect preimage of ω_1 cannot have a G_δ-diagonal (cf. the proof of Corollary 2.3).

Remarks. 1. Some corollaries of the theorems of this section which concern manifolds (more generally, locally compact, locally connected spaces) will be included in the fourth (and last) section.

2. The Kunen line [11] is an example, under CH, of a locally compact, locally countable, submetrizable, perfectly normal, hereditarily collectionwise normal space of Lindelöf degree ω_1 which is hereditarily separable, does not contain a perfect preimage of ω_1, but still fails to be paracompact. (Note that "separable + paracompact" implies "Lindelöf".) This shows that MA + $\neg$CH is an essential assumption in Theorems 3.3, 3.4, 3.6, 3.11 and Corollaries 3.13, 3.14(a),(b). (One, however, has to write $L(X) = \omega_1$ instead of $L(X) < 2^{\omega}$ in Theorem 3.11 and Corollary 3.14 (b).) Note that MA + $\neg$CH is also essential in some other results of this section, where it was assumed (cf. Remark 1 in the fourth section).

3. W. Weiss [22] constructed a naive example of a normal, collectionwise T_2, locally compact, separable, submetrizable space which is not paracompact. This example shows that "hereditarily σ-collectionwise T_2" in Theorems 3.3, 3.4 and Corollary 3.14 (a) cannot be weakened to "σ-collectionwise T_2".

Further, it shows that $L(X) < 2^{\omega}$ cannot be omitted in Theorem 3.6, Corollary 3.11 and Corollary 3.14 (b).

4. If MA + $\neg$CH holds, then there is an example ([17], p. 47) of a locally compact, normal, nonmetrizable Moore space. Thus "σ-collectionwise T_2" is essential in Corollary 3.13.

4. On locally compact, locally connected spaces

In this final (and really very short) section we reformulate some of the results of the third section for locally compact, locally connected spaces (in particular, for manifolds). Our aim in doing so is, on the one hand, to get some further corollaries, on the other hand, to point out some possible improvements of our results if only manifolds (or more generally, locally compact, locally connected spaces) are considered.

Theorem 4.1 (MA + $\neg$CH). Let X be a locally compact, locally hereditarily Lindelöf, locally connected, conecocted space (in particular, let X be a manifold). Suppose that X contains no perfect preimage of ω_1. Then X is Lindelöf if one of the following conditions holds:

(a) X is hereditarily σ-collectionwise T_2;

(b) X is σ-collectionwise T_2 and $L(X) < 2^{\omega}$;

(c) X is σ-collectionwise T_2 and hereditarily normal.

Corollary 4.2 (Lane [14], Rudin [20], MA + $\neg$CH). Let X be a perfectly normal, locally compact, locally connected, connected space (in particular, let X be a perfectly normal manifold). Then X is Lindelöf.

Proof. A perfectly normal, locally compact, locally connected space is, by a result of Alster and Zenor [1], hereditarily collectionwise T_2. Since a perfect preimage of ω_1 is not a perfect space (cf. the proof of Corollary 2.4), and a perfect locally compact space is locally hereditarily Lindelöf, we can apply Theorem 4.1 (a).

Corollary 4.3 (MA + ¬CH). Let X be a locally compact, locally connected, connected space (in particular, a manifold) with a G_δ-diagonal. Then X is second countable, provided one of the conditions (a),(b),(c) of Theorem 4.1 holds.

The proof goes parallel to the proof of Corollary 3.14.

Remarks. 1. M.E. Rudin and Zenor [19] constructed, under CH a perfectly normal, hereditarily separable, hereditarily collectionwise T_2, non-metrizable manifold of weight ω_1. This shows that MA + ¬CH is a necessary assumption in Theorem 4.1 and Corollary 4.2 (and, a fortiori, in Theorem 3.12). We do not know whether it can be omitted from Corollary 4.3.

2. The Prüfer manifold is a separable, nonmetrizable Moore manifold. Since it is Moore, it is perfect and does not contain a perfect preimage of ω_1. Thus the additional hypotheses (a),(b),(c) in Theorem 4.1 and Corollary 4.3 and normality in Corollary 4.2 cannot be omitted.

3. P. Nyikos [16] conjectures that, under MA + ¬CH, every T_5 manifold of dim>1 is metrizable. (In dimension 1, the long line is counterexample.) The best results we are able to prove in connection with this conjecture are Theorem 4.1 (a),(b),(c). These results give us raise the following questions:

Question 1. Suppose MA + ¬CH (or PFA). Does every nonmetrizable T_5 manifold contain a perfect preimage of ω_1?

Question 2. Suppose MA $+\neg$CH (or PFA). Does every normal, nonmetrizable manifold of weight $< 2^{\omega}$ contain a perfect preimage of ω_1?

By Theorem 4.1 (b),(c) we could give affirmative answers to Questions 1 and 2 if we had an affirmative answer (under MA $+\neg$CH, or PFA) to the following question of Alster and Zenor [1]:

Question 3 (Alster and Zenor [1]). Is every normal manifold collectionwise T_2?

Acknowledgement. Having been informed of Theorem Σ^- in 1981 in Prague, Heikki Junnila observed that he had a similar result. Returning to Helsinki, he checked his 1979 notes and he kindly informed the author [12] that he actually proved Theorem Σ, two years before the author and Fremlin (re-)discovered it. He, however, was reluctant to publish his result, because he could see no applications. Thus Theorem Σ should also be attributed to Heikki Junnila.

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(Oblatum 30.9. 1981, revisum 3.8. 1982)