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REMARKS ON SUBSETS OF CARTESIAN PRODUCTS OF METRIC SPACES

Boris S. KLEBANOV

Abstract: In the paper, results on the structure of certain subsets of Cartesian products of metric spaces are presented. We give an affirmative answer to a question posed in [2]. The problem of extending the Sierpiński-Stone theorem is considered, too.

Key words: Metric space, Cartesian product, G_δ-set, retraction.

Classification: 54B10, 54E35, 54C15

I. Problems which are considered in the paper are close to those treated in sections 1, 2 of our note [1]. Here we generalize some results of [1]. Let us note that the main construction used in the present paper is essentially the same as in [1]. In this paper, we give a positive answer to a question posed by R. Pol and E. Puzio-Pol [2]. We also examine the question of extending the Sierpiński-Stone theorem concerning retractions of zero-dimensional x) metric spaces over Cartesian products of such spaces. An example presented in the final section shows that a closed G_δ-subset of a Cartesian product of zero-dimensional metric spaces need not be its retract.

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x) In the paper zero-dimensionality is understood in the sense of covering dimension dim.

Let X be the Cartesian product of sets X_α, $\alpha \in A$ and $Y_\alpha \subset X_\alpha$ for $\alpha \in A$. Following [3, 2], the subset $Y = \prod\{Y_\alpha : \alpha \in A\}$ of X will be called a cube. By D(Y) we shall denote the set $\{\alpha \in A : Y_\alpha \neq X_\alpha\}$. If $|D(Y)| \leq \tau$ for some $\tau \geq \aleph_0$ (resp. D(Y) is finite), then Y is called a τ-cube (resp. an f-cube). (In this paper we shall deal only with $\aleph_0$-cubes and f-cubes.) A cube Y is called elementary if $|Y_\alpha| = 1$ for all $\alpha \in D(Y)$.

In [3, 2] the sets which are the closures of unions of τ-cubes in X were examined in the cases when X is a Cartesian product of spaces the weight or the character of which does not exceed some cardinal number. We impose on the factors a restriction of other kind: metrizability; the object of our study are the sets which are the closures of unions of $\aleph_0$-cubes in Cartesian products of metric spaces.

Let $X = \prod\{X_\alpha : \alpha \in A\}$ and $B \subset A$. By X_B we denote $\prod\{X_\alpha : \alpha \in B\}$; $p_\alpha : X \longrightarrow X_\alpha$ and $p_B : X \longrightarrow X_B$ are projections. A set $U \subset X$ is called B-distinguished if $U = p_B^{-1}(U)$. Following [4], we say that a set $U \subset X$ has a countable type if U is B-distinguished for some countable B.

The union of a family γ of sets is denoted by $\bigcup \gamma$; Int S denotes the interior of a set S. Since there is a difference in the terminology, let us note that we rank finite sets among the countable ones.

Let us proceed to formulating theorems (the proofs will be presented in section II).

<u>Theorem 1</u>. Let $X = \prod\{X_\alpha : \alpha \in A\}$, where every X_α is a metric space, and let $F \subset X$ be the closure of a union of $\aleph_0$-

cubes. Then

(i) there exists a σ-discrete family λ of open f-cubes such that $\bigcup\lambda = X \setminus F$,

(ii) there exists a σ-discrete family μ of open f-cubes such that $\bigcup\mu = \mathrm{Int}\, F$,

(iii) if all the spaces X_α are zero-dimensional, then λ and μ consist of closed-and-open sets.

In connection with this theorem let us make the following remark. Let X be a Cartesian product of topological spaces. Clearly, a G_δ-subset of X is a union of $\aleph_0$-cubes. On the other hand, if all factors of X are spaces of a countable pseudocharacter, then an $\aleph_0$-cube in X is a union of G_δ-sets. Hence, the set F above can be defined equivalently as the closure of a union of G_δ-sets in X.

Using properties of λ stated in Theorem 1(i), one can obtain

<u>Corollary</u>. Let X be a Cartesian product of metric spaces and let $F \subset X$ be the closure of a union of $\aleph_0$-cubes. Then F is a functionally closed subset of X.

To prove this corollary, note first that an open f-cube in a Cartesian product of metric spaces is functionally open. Thus, the family λ consists of functionally open sets. Since functional openess is preserved by the operations of taking the union of a discrete family and the countable union, $\bigcup\lambda$ is a functionally open set. Hence $F = X \setminus \bigcup\lambda$ is functionally closed in X.

This corollary gives a positive answer to a question formulated in [2]. Let us note also that it generalizes Theorem 1 of [5].

A.H. Stone [6], having strengthened a result of W. Sierpiński [7], proved that if X is a metric space, $\dim X = 0$, and F is a closed subset x) of X, then there exists a continuous mapping $r: X \to F$ such that the restriction $r|F$ is the identity mapping (i.e., r is a retraction). This retraction has the following property: the set $r(X \setminus F)$ is σ-discrete in X.

We became interested in the question whether a similar statement holds for certain closed subsets of Cartesian products of zero-dimensional metric spaces. First of all, it should be found out of what sort these subsets may be. It is clear that not any suits (otherwise each zero-dimensional compact Hausdorff space, being homeomorphic to a subspace of a certain Cantor cube D^{τ}, would be dyadic, which is wrong). On the other hand, if a closed subset of such a product has a countable type, then it is a retract of the product. Indeed, let $X = \prod \{X_{\alpha} : \alpha \in A\}$, where all X_{α}'s are zero-dimensional metric spaces, F is closed in X and $F = p_C^{-1} p_C(F)$ for some countable $C \subset A$. Since $p_C(F)$ is closed in the zero-dimensional metric space X_C, by the Sierpiński-Stone theorem there exists a retraction r of X_C onto $p_C(F)$. Obviously, the mapping $r \times id_Z$, where $Z = X_{A \setminus C}$, is a retraction of X onto F.

By virtue of a theorem of R. Engelking [3] (a less general formulation of it was given by B.A. Efimov [8]), if X is a Cartesian product of spaces of a countable weight and $F \subset X$ is the closure of a union of $\aleph_0$-cubes, then F has a countable type in X. Therefore, if $X = \prod \{X_{\alpha} : \alpha \in A\}$, where every

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x) All subsets up to the end of the section are assumed to be non-empty.

X_α is a metric space of a countable weight with $\dim X_\alpha = 0$, and $F \subset X$ is the closure of a union of $\aleph_0$-cubes, then F is a retract of X. In view of the last assertion it is natural to put the question: is the restriction imposed in it on the weight of the factors essential? Below, in section III, an example is presented which shows that the answer is positive. Moreover, we establish that in the case when not all factors have a countable weight even a sequentially continuous mapping $r: X \longrightarrow F$ with $r|F = id_F$ may not exist.

Still, for X being a Cartesian product of zero-dimensional metric spaces and $F \subset X$ being the closure of a union of $\aleph_0$-cubes, the statement that generalizes the Sierpiński-Stone theorem is succeeded in proving. For convenience s sake of its formulation let us introduce first the notion of a c-mapping.

Let $X = \Pi\{X_\alpha : \alpha \in A\}$ and $Y \subset X$. By $\mathcal{S}(Y)$ we shall denote the set of all convergent sequences of points of the space Y. A mapping $f: Y \longrightarrow Z$, where $Z \subset X$, will be called a c-mapping if for each sequence $S = \{y_n\} \in \mathcal{S}(Y)$ there exists a set $A_f(S) \subset A$ such that $A \setminus A_f(S)$ is countable and $p_\alpha(\lim_{n\to\infty} f(y_n)) = \lim_{n\to\infty} p_\alpha f(y_n)$ for $\alpha \in A_f(S)$.

Clearly, every sequentially continuous mapping of Y to Z is a c-mapping.

Theorem 2. Let $X = \Pi\{X_\alpha : \alpha \in A\}$, where every X_α is a zero-dimensional metric space, and let $F \subset X$ be the closure of a union of $\aleph_0$-cubes. Then for each countable $\tilde{A} \subset A$ there exists a c-mapping $r: X \longrightarrow F$ such that

(a) $\tilde{A} \subset A_r(S)$ for all $S \in \mathcal{S}(X)$,

(b) $r|F = id_F$,

(c) $r(X\setminus F)$ is the union of a σ-discrete family of elementary $\aleph_0$-cubes.

Since the sequential continuity of a mapping is equivalent to the continuity when the domain is a metric space, the Sierpiński-Stone theorem follows from Theorem 2 if one takes one-element sets as A and $\widetilde{A}$. Note that, by virtue of the corollary stated above, the set F indicated in Theorem 2 is a G_δ-subset of X.

II. In the proof of Theorems 1 and 2 one common construction is used. This construction is similar to (and was suggested by) that due to S.P. Gul'ko (see the proof of Theorem 1 from [9]).

Let X be the Cartesian product of metric spaces X_α, $\alpha\in A$, $Q\subset X$ be the union of $\aleph_0$-cubes, and $F = cl\, Q$. We shall assume that $F\neq X$, $F\neq\emptyset$. For every countable $B\subset A$ fix a metric ρ_B on the metrizable space X_B and define the pseudometric d_B on X by the formula $d_B(x,y) = \rho_B(p_B(x),p_B(y))$.

<u>The Main Construction</u>. For an integer $n = 0,1,\dots$ let us construct by induction the families λ_n, μ_n and ν_n of subsets of X such that the following conditions C1 - C8 hold:

C1. $\varphi_n = \lambda_n \cup \mu_n \cup \nu_n$ is a family of open sets, both locally finite and σ-discrete;

C2. members of φ_n have a countable type;

C3. let $U\in\varphi_n$; then $U\in\lambda_n$ if $U\cap F = \emptyset$, $U\in\mu_n$ if $U\subset F$, $U\in\nu_n$ if $U\cap F\neq\emptyset$ and $U\setminus F\neq\emptyset$;

C4. $\nu_0 = \{X\}$; for $n\geq 1$ φ_n is a cover of $\bigcup\nu_{n-1}$ which refines ν_{n-1}.

C5. for each $U \in \nu_n$, $n \geq 1$, the family $\xi(U) = \{V \in \nu_{n-1} : U \cap V \neq \emptyset\}$ is finite;

C6. to each $U \in \nu_n$ the points $a(U)$, $a'(U) \in U$ and a countable set $B(U) \subset A$ are assigned such that $p^{-1}_{B(U)} p_{B(U)}(a(U)) \subset U \cap F$ and $p^{-1}_{B(U)} p_{B(U)}(a'(U)) \subset U \setminus F$;

C7. to each $U \in \nu_n$, a countable set $R(U) \subset A$ and a pseudometric d_U on X are assigned such that

(a) for $n \geq 1$, $R(U) = \bigcup\{R(V) : V \in \xi(U)\} \cup B(U)$ and $d_U = \max\{d_{R(U)}, \max\{d_V : V \in \xi(U)\}\}$,

(b) $d_U(x,y) = 0$ iff $p_{R(U)}(x) = p_{R(U)}(y)$, and the metric on $X_{R(U)}$ that naturally corresponds to d_U induces on $X_{R(U)}$ the existing topology,

(c) if $U' \in \nu_n$, then $U \cap U'$ is $R(U)$-distinguished;

C8. to each $U \in \lambda_n \cup \nu_n$, $n \geq 1$, the set $k(U) \in \nu_{n-1}$ is assigned such that

(a) $U \subset k(U)$,

(b) if $k(U) = V$, $U \in \nu_n$, then U is $R(V)$-distinguished and the d_V-diameter of U is less than $1/n$.

The initial step of induction. Put $\lambda_o = \mu_o = \{\emptyset\}$, $\nu_o = \{X\}$. Choose an arbitrary point $a(X) \in Q$. From the definition of Q it follows that there is a countable $C(X) \subset A$ such that $p^{-1}_{C(X)} p_{C(X)}(a(X)) \subset Q$. Taking some $a'(X) \in X \setminus F$, we find then, using the openess of $X \setminus F$, a finite $D(X) \subset A$ such that $p^{-1}_{D(X)} p_{D(X)}(a'(X)) \subset X \setminus F$. Define $B(X)$ as $C(X) \cup D(X)$. Let $\widetilde{A}$ be some countable subset of A. Put $R(X) = B(X) \cup \widetilde{A}$ (the inclusion $\widetilde{A} \subset R(X)$ will be needed for the proof of Theorem 2) and $d_X = d_{R(X)}$.

Assume that for all $n \leq m$ the construction has been carried out so that conditions C1 - C8 are satisfied. Let us fix a $U \in \nu_m$ and put $\eta(U) = \{U \cap U' : U' \in \nu_m\}$. The family $\eta(U)$ is locally finite and consists of R(U)-distinguished sets (see C7(c)). Let $\sigma(U)$ be an open cover of X by R(U)-distinguished sets each of which intersects at most a finite number of elements of $\eta(U)$. It is known that every open cover of a metric space has an open (covering) refinement which is both locally finite and σ-discrete. Using this fact, one can readily find an open refinement $\sigma_1(U)$ of $\sigma(U)$ such that: 1) $\sigma_1(U)$ consists of R(U)-distinguished sets, is locally finite and σ-discrete, 2) d_U-diameter of every member of $\sigma_1(U)$ is less than $1/(m+1)$. The family $\gamma(U) = \{U \cap U' \neq \emptyset : U' \in \sigma_1(U)\}$ is locally finite, σ-discrete, consists of open R(U)-distinguished sets, and $\bigcup \gamma(U) = U$. One may assume that all members of ν_m are well-ordered somehow by a relation $<$. If $V \in \gamma(U)$, $V \setminus F \neq \emptyset$ and there is no $\tilde{U} \nleq U$ such that $V \in \gamma(\tilde{U})$, then we put $k(V) = U$.

Let $\varphi_{m+1} = \bigcup \{\gamma(U) : U \in \nu_m\}$. Clearly, we have $\bigcup \varphi_{m+1} = \bigcup \nu_m$. Since ν_m and each of the families $\gamma(U)$ is locally finite and σ-discrete, so is φ_{m+1}. Define λ_{m+1}, μ_{m+1} and ν_{m+1} in accordance with C3. Let us observe that if ν_{m+1} turned out to be empty, there is no need in continuing induction, and the construction should be ended. For every $U \in \nu_{m+1}$ we have $U \cap F \neq \emptyset$, therefore $U \cap Q \neq \emptyset$. Take an $a(U) \in U \cap Q$. As U is open and Q is a union of $\aleph_0$-cubes, there exist a finite $L(U) \subset A$ and a countable $M(U) \subset A$ such that $p^{-1}_{L(U)} p_{L(U)}(a(U)) \subset U$ and $p^{-1}_{M(U)} p_{M(U)}(a(U)) \subset Q$. Then $p^{-1}_{C(U)} p_{C(U)}(a(U)) \subset U \cap Q$, where $C(U) = L(U) \cup M(U)$. Since $U \setminus F$ is non-empty and open, there are a point $a'(U)\in U\setminus F$ and a finite set $D(U)\subset A$ such that $p^{-1}_{D(U)}p_{D(U)}(a'(U))\subset U\setminus F$. Evidently, the set $B(U) = C(U)\cup D(U)$ satisfies C6.

From the construction of ν_{m+1} it follows that the family $\xi(U)$ is finite, and hence, taking into account the corresponding inductive assumptions, $R(U)$ is countable (the definitions of $\xi(U)$ and $R(U)$ see in C5 and C7(a) resp.). Finally, let us define the pseudometric d_U as indicated in C7(a). This completes the construction for $n = m+1$. It is not hard to see that C1 - C8 are satisfied for this n. Thus, the induction is carried out.

One easily verifies that for $n\geq 1$

(1) if $U\in\nu_n$ and $V\in\xi(U)$, then $k(V)\in\xi(k(U))$.

Indeed, $V\in\xi(U)$ means that $U\cap V\neq\emptyset$, and since $V\subset k(V)$ and $U\subset k(U)$ (see C8(a)), then $k(V)\cap k(U)\neq\emptyset$, i.e., $k(V)\in\xi(k(U))$.

Proof of Theorem 1. We shall begin with the proof of (i). Put $\lambda^* = \bigcup\{\lambda_n : n\geq 1\}$. By virtue of C3, $\bigcup\lambda^*\subset X\setminus F$. We shall show that $\bigcup\lambda^* = X\setminus F$. Let, on the contrary, a point $x\in X\setminus F$ be not covered by λ^*. Then, by C4 and C3, for each n there exists a set $U_n\in\nu_n$ such that $x\in U_n$. Let $R_n = R(k(U_n))$ and $d_n = d_{k(U_n)}$, $(n\geq 1)$. Clearly, $U_n\in\xi(U_{n+1})$, whence, by virtue of (1) and C7(a) we have

(2) $$R_n\subset R_{n+1} \text{ and } d_n\leq d_{n+1}.$$

Since $U_n\cap U_{n+2}\neq\emptyset$ and $U_{n+2}\subset k(U_{n+2})$, $U_n\in\xi(k(U_{n+2}))$. Hence, by C7(a), $R(U_n)\subset R_{n+2}$. Condition C7(a) yields also that $R_n\subset R(U_n)$. The last two inclusions, along with the inclusion in (2), imply that $\bigcup\{R(U_n) : n\geq 1\} = \bigcup\{R_n : n\geq 1\}$. Denote this union by R. For each $n\geq 1$ let us define the point $x_n\in X$ from the conditions: $p_\alpha(x_n) = p_\alpha(a(U_n))$ if $\alpha \in R(U_n)$, $p_\alpha(x_n) = p_\alpha(x)$ if $\alpha \in A \setminus R(U_n)$. As $B(U_n) \subset R(U_n)$, so $x_n \in p^{-1}_{B(U_n)} p_{B(U_n)}(a(U_n))$, and hence $x_n \in U_n \cap F$ (see C6). Then from C8(b) and (2) we infer that $\lim_{n \to \infty} d_n(p_{R_m}(x), p_{R_m}(x_n)) = 0$ for each $m \geq 1$. Therefore, by C7(b), $p_{R_m}(x) = \lim_{n \to \infty} p_{R_m}(x_n)$, and thus $p_R(x) = \lim_{n \to \infty} p_R(x_n)$. Since $p_\alpha(x_n) = p_\alpha(x)$ for $\alpha \in A \setminus R$, we have $x = \lim_{n \to \infty} x_n$. As all x_n s belong to F and F is closed, $x \in F$. This contradicts the assumption $x \in X \setminus F$.

Each space X_α being metric, there is a σ-discrete base in X_α. Applying this fact, one readily shows that an open subset of X having a countable type is a union of a σ-discrete family of open f-cubes. By virtue of C2, every $U \in \lambda^*$ has a countable type, so that for it there exists a σ-discrete family $\psi(U)$ of open f-cubes such that $\bigcup \psi(U) = U$. Condition C1 implies that the family λ^* is open and σ-discrete. Since $\bigcup \lambda^* = X \setminus F$, we conclude that $\lambda = \bigcup \{\psi(U) : U \in \lambda^*\}$ is the family sought for.

Let us prove (ii). Put $\mu^* = \bigcup \{\mu_n : n \geq 1\}$. Let us check that $\bigcup \mu^* = \operatorname{Int} F$. The inclusion $\bigcup \mu^* \subset \operatorname{Int} F$ follows from C3. Take an $x \in F \setminus \bigcup \mu^*$. To prove the desired equality, we must show that x is an accumulation point of $X \setminus F$. It is readily seen that for each n there exists a set $U_n \in \nu_n$ which contains x. Conducting further reasonings similar to those in the proof of (i), one can find a sequence $\{x_n\} \subset X \setminus F$ converging to x (the only difference from the former reasonings is that now the points $a'(U)$ are used in place of $a(U)$). Arguing as in the proof of (i) (see the transition from λ^* to λ), one can construct a σ-discrete family μ of open f-cubes such that $\cup\mu = \cup\mu^*$. This ends the proof of (ii).

Now let all X_α's be zero-dimensional. Then (see, e.g., [10]), 1) every open cover of X has a clopen (= closed-and-open) discrete refinement, 2) if $C \subset A$ is countable, then $\dim X_C = 0$. Making use of these properties, the construction in the case of zero-dimensional X_α's can be carried out in such a way that the following strengthening of condition C1 holds:

$C1_o$. $\varphi_n = \lambda_n \cup \mu_n \cup \nu_n$ is a discrete family of clopen sets.

Therefore in the present case one can consider all members of λ_n (and thus, of λ^*) to be clopen. Since a zero-dimensional metric space has a σ-discrete base consisting of clopen sets, the members of $\psi(U)$, $U \in \lambda^*$, can be assumed to be clopen. Then λ also consists of clopen sets. The analogous statement concerning μ is proved similarly. Thus, (iii) is established.

<u>Proof of Theorem 2</u>. We use the main construction again. One can suppose that it has been carried out so that conditions $C1_o$ and C2 - C8 are satisfied.

Fix an $n \geq 1$ and a set $U_n \in \lambda_n$. Adopt the notation $V_n = k(U_n)$, $Q(V_n) = p^{-1}_{B(V_n)} p_{B(V_n)}(a(V_n))$. For every point $x \in U_n$ let us define the point $y_x \in X$ from the conditions: $p_\alpha(y_x) = p_\alpha(a(V_n))$ if $\alpha \in R(V_n)$, $p_\alpha(y_x) = p_\alpha(x)$ if $\alpha \in A \setminus R(V_n)$. By virtue of C7(a) and C6, $R(V_n) \supset B(V_n)$ and $Q(V_n) \subset V_n \cap F$, so that

(3) $$y_x \in V_n \cap F.$$

Put $r(x) = x$ if $x \in F$ and $r(x) = y_x$ if $x \in X \setminus F$. Let us show that the mapping r is defined on $X \setminus F$ correctly. Note that $\bigcup \lambda_m$ is disjoint from $\bigcup \lambda_n$ for $m \neq n$. Indeed, if, for instance, $m > n$, by C4 we have $\bigcup \lambda_m \subset \bigcup \nu_n$, whereas $\bigcup \nu_n$ is disjoint from $\bigcup \lambda_n$. In addition, the families λ_n consist of disjoint sets and - as it was established when proving Theorem 1(i) - in total cover $X \setminus F$.

Clearly, $r(U_n)$ is the elementary $\aleph_0$-cube $Q(V_n)$. As $Q(V_n) \subset \subset V_n \in \nu_{n-1}$ and ν_{n-1} is σ-discrete, we get that the family $\Theta = \{Q(V_n) : V_n \in \nu_{n-1}, n = 1,2,\dots\}$ is σ-discrete as well. It is readily seen that $r(X \setminus F) = \bigcup \Theta$, which proves (c).

Let us verify that r is a c-mapping. Let $S = \{x_n\} \in \mathcal{S}(X)$ and $x = \lim_{n\to\infty} x_n$. We shall consider the case when $x \in F$, $\{x_n\} \subset \subset X \setminus F$ (other cases either are trivial or come to this one).

For every point x_n there exists a (unique) member of λ^* which contains it. Let it be a set $U_{i_n} \in \lambda_{i_n}$. Put $W_n = k(U_{i_n})$. The points x_n and $r(x_n)$ being contained in W_n (see C8(a) and (3)), it follows from C8(b) that $d_{k(W_n)}(x_n, r(x_n)) < 1/(i_n - 1)$ for $i_n > 1$. According to the construction, for every $U \in \nu_m$, $m \geq 0$, we have $R(X) \subset R(U)$ and $d_X \leq d_U$ ($R(X)$ and d_X were defined at the initial step of induction). Taking $k(W_n)$ as U, we conclude that $d_X(x_n, r(x_n)) < 1/(i_n - 1)$. The condition $x = \lim_{n\to\infty} x_n$ implies that the sequence $\{i_n\}$ increases unboundedly, whence $\lim_{n\to\infty} d_X(x_n, r(x_n)) = 0$. Therefore, by C7(b), $p_{R(X)}(x) = \lim_{n\to\infty} p_{R(X)}(x_n)$. Since $\widetilde{A} \subset R(X)$, we infer that $p_{\widetilde{A}}(x) = \lim_{n\to\infty} p_{\widetilde{A}}(x_n)$. Let R be the union of all the sets $R(W_n)$. Since each $R(W_n)$ is countable, then so is R. From the definition of r it follows that $p_{A \setminus R}(x_n) = p_{A \setminus R}(x)$ for each n. Put $A_r(S) =$

$= \tilde{A} \cup (A \setminus R)$. It is clear then that $p_\alpha r(x) = p_\alpha(x) = \lim_{n\to\infty} p_\alpha r(x_n)$ for $\alpha \in A_r(S)$. We also have: $|A \setminus A_r(S)| = |R \setminus \tilde{A}| \leq |R| \leq \aleph_0$. The theorem is proved.

__Remarks__. 1. The main construction described above is applicable not only to the whole Cartesian product but also to its appropriate subsets. This enables us to obtain a generalisation of Theorem 2. To state it, we shall need the following

__Definition__. Let $X = \prod\{X_\alpha : \alpha \in A\}$. A set $Y \subset X$ is said to be $\aleph_0$-convex, provided that for any points $y_1, y_2 \in Y$ and any $B \subset A$ such that $|B| \leq \aleph_0$ the point $x \in X$ which is defined from the conditions: $p_B(x) = p_B(y_1)$, $p_{A\setminus B}(x) = p_{A\setminus B}(y_2)$ belongs to Y.

Examples of $\aleph_0$-convex subsets of a Cartesian product are Σ_τ-products for all $\tau \geq \aleph_0$, a σ-product, and defined in the case of metric factors a Σ_*-product (the definitions see, e.g., in [9, 11]).

Let X and F be the same as in Theorem 2. The statement of this theorem will remain true if one replaces in it X by any its $\aleph_0$-convex subset Y, F - by $F_Y = F \cap Y \neq \emptyset$, and item (c) - by the following: $r(Y \setminus F_Y)$ is the union of a σ-discrete family each member of which is the intersection of an elementary $\aleph_0$-cube with Y.

The proof of this result is actually the same as of Theorem 2. Let us show where in the proof the $\aleph_0$-convexity of Y is employed. First, when proving Theorem 2, we used the equality $\cup \lambda^* = X \setminus F$, which ensures that the retraction is defined on the whole X. It was established in the proof of Theorem 1(i). Proving the analogous equality for $Y \setminus F_Y$, we need the $\aleph_0$-convexity of Y to obtain that the points x_n (introduced in the course of proving Theorem 1(i)) belong to Y. Besides, defining the mapping r on Y, we need all the points y_x (i.e., the range of r) to be contained in Y. This is also guaranteed by the $\aleph_0$-convexity of Y.

2. The family μ indicated in Theorem 1(ii) gives an inner approximation of the set F. Another approximation of that kind is expressed by

Theorem 3. Let X be a Cartesian product of metric spaces and $F = cl \bigcup \gamma$, where γ is a family of $\aleph_0$-cubes in X. Then there exists a σ-discrete subfamily δ of γ such that $F = cl \bigcup \delta$.

Theorem 3 can be established by applying a known construction due to A.M. Gleason (cf. the proof of Theorem 1 of [2]). In the proof only the existence of a σ-discrete network in factors of X is used.

3. Suppose that in Theorems 1 and 3 all factors of X have the weight $\leq \tau$ $(\tau \geq \aleph_0)$. Then the families λ, μ and γ have cardinality $\leq \tau$. This follows immediately from the fact that every discrete family of non-empty τ-cubes in a Cartesian product of spaces of the weight $\leq \tau$ has cardinality $\leq \tau$ (see [3], Th. 3).

III. Let X be a Cartesian product of zero-dimensional metric spaces and F a closed G_δ-subset of X. We shall present now an example which shows that there may not exist a sequentially continuous mapping $r: X \to F$ such that $r|F = id_F$.

For every integer $n \geq 1$ fix a certain set J^n of real numbers such that $|J^n| = \aleph_1$ and $1/(n+1) < x < 1/n$ for each $x \in J^n$.

On the set $J = \{0\} \cup \bigcup \{J^n : n \geq 1\}$, introduce the metric $\hat{\rho}$ by letting $\hat{\rho}(x,y) = x + y$ if $x \neq y$ and $\hat{\rho}(x,x) = 0$. The metric space $(J, \hat{\rho})$ denote by $\hat{J}$. Clearly, all points of $\hat{J}$ except zero are isolated, and the neighbourhoods of zero in $\hat{J}$ are the same as in the Euclidean topology.

Let $D^{\aleph_1} = \prod \{D_\alpha : \alpha < \omega_1\}$, where each D_α is the two-point discrete space $\{0_\alpha, 1_\alpha\}$. Put $X = \hat{J} \times D^{\aleph_1}$. Obviously, all the factors of X are zero-dimensional metric spaces. Let $\pi_\alpha : D^{\aleph_1} \to D_\alpha$, $p_1 : X \to \hat{J}$ and $p_2 : X \to D^{\aleph_1}$ be projections. Let us number the points of each J^n by countable ordinals: $J^n = \{j^n_\beta : \beta < \omega_1\}$. For $\beta < \omega_1$ set $S_\beta = \{a \in D^{\aleph_1} : \pi_\alpha(a) = 0_\alpha$ for all $\alpha < \beta\}$ and $F^n_\beta = \{x \in X : p_1(x) = j^n_\beta,\ p_2(x) \in S_\beta\}$. Let $F = p_1^{-1}(0) \cup \bigcup \{F^n_\beta : n \geq 1, \beta < \omega_1\}$. It is not hard to see that F is a closed G_δ-subset of X.

Let us prove that there is no sequentially continuous mapping $r : X \to F$ such that $r|F = \mathrm{id}_F$. Let, on the contrary, such an r exist. Put $Y = \{x \in X : |\{\alpha \in A : \pi_\alpha p_2(x) \neq 0_\alpha\}| \leq 1\}$. Note that Y is closed in X. Since Y is contained in the Σ-product of metric spaces $\hat{J}$ and D_α, $\alpha < \omega_1$, which is a Fréchet space (see [10], Ex. 3.10.D), Y is also a Fréchet space. Hence, $r|Y$ – being a sequentially continuous mapping of a Fréchet space – is continuous. Let $\Phi^n_\beta = F^n_\beta \cap Y$. The set Φ^n_β is open in $F \cap Y$, because j^n_β is an isolated point of $\hat{J}$, and $r(\Phi^n_\beta) = \Phi^n_\beta$. Hence, by the continuity of $r|Y$, each point $y \in \Phi^n_\beta$ has a neighbourhood Oy in Y such that $p_1(Oy) = \{j^n_\beta\}$ and $r(Oy) \subset \Phi^n_\beta$. One can suppose that Oy has the form $O^*y \cap Y$, where O^*y is open in X, $p_1(O^*y) = \{j^n_\beta\}$ and the set $K(y) = \{\alpha \in A : \pi_\alpha p_2(O^*y) \neq D_\alpha\}$ is finite. The set $p_1^{-1}(j^n_\beta)$, homeomorphic to S_β, is compact, and so is its closed subspace Φ^n_β.

Choose from the open cover $\{Oy : y \in \Phi_\beta^n\}$ of Φ_β^n a finite subcover ψ_β^n and put $O_\beta^n = \bigcup \psi_\beta^n$, $K_\beta^n = \bigcup\{K(y) : Oy \in \psi_\beta^n\}$. The set K_β^n is finite. We have: 1) $p_1(O_\beta^n) = \{j_\beta^n\}$, 2) $p_2(O_\beta^n)$ is the intersection of a certain K_β^n-distinguished set with Y, 3) $r(O_\beta^n) = \Phi_\beta^n$. This implies that for $y \in Y$ we have

($*$) if $p_1(y) = j_\beta^n$ and $\pi_\alpha p_2(y) = 0_\alpha$ for all $\alpha \in K_\beta^n$, then $r(y) \in \Phi_\beta^n$.

For $\beta < \omega_1$ pur $R_\beta^n = \{\alpha : \alpha < \beta\} \setminus K_\beta^n$.

<u>Lemma</u>. There exist sequences $n_1 < n_2 < \dots$ of natural numbers and $\beta_1, \beta_2, \dots$ of countable ordinals such that $\bigcap\{R_{\beta_i}^{n_i} : i \geq 1\} \neq \emptyset$.

Suppose that the lemma has been already proved. Let $\gamma \in R_{\beta_i}^{n_i}$ for all $i \geq 1$. Define the point $t \in D^{\aleph_1}$ from the conditions: $\pi_\alpha(t) = 0_\alpha$ if $\alpha \neq \gamma$, $\pi_\gamma(t) = 1_\gamma$. Let us consider the following points of Y: $z = (0, t)$ and $z_i = (j_{\beta_i}^{n_i}, t)$, $i \geq 1$. Obviously, $z = \lim_{i \to \infty} z_i$. Since $\gamma \notin K_{\beta_i}^{n_i}$, the definition of t implies that $\pi_\alpha p_2(z_i) = \pi_\alpha(t) = 0_\alpha$ for each $\alpha \in K_{\beta_i}^{n_i}$. Then, according to ($*$), $r(z_i) \in \Phi_{\beta_i}^{n_i}$. As $\gamma < \beta_i$ and $p_2 r(z_i) \in S_{\beta_i}$, by the definition of S_{β_i} we have $\pi_\gamma p_2 r(z_i) = 0_\gamma$. On the other hand, $\pi_\gamma p_2 r(z) = \pi_\gamma p_2(z) = \pi_\gamma(t) = 1_\gamma$. Since the sequence $\{z_i\}$ converges to z, we infer that $\pi_\gamma p_2 r|Y$, and hence $r|Y$ too, cannot be continuous at the point z. The contradiction obtained completes the proof.

It remains to prove the lemma. Suppose that it does not hold. Then for every $\gamma < \omega_1$ there exists a natural number

$n(\gamma)$ such that $\gamma \notin R^n_\beta$ for all $n \geq n(\gamma)$ and $\beta < \omega_1$. This implies that $\gamma \in K^n_\beta$ whenever $n \geq n(\gamma)$ and $\gamma < \beta$. As $\{\gamma : \gamma < \omega_1\}$ is uncountable, there exists an m such that the set $M = \{\gamma : \gamma < \omega_1, n(\gamma) = m\}$ is uncountable. Take an $L \subset M$ with $|L| = \aleph_0$ and a $\delta \in M$ such that $\gamma < \delta$ for all $\gamma \in L$. Clearly, $\gamma \in K^m_\delta$ for each $\gamma \in L$. This contradicts the finiteness of K^m_δ. The lemma is proved.

In conclusion we take the opportunity of indicating that the mapping r in Theorem 1 of [1] is a c-mapping - not sequentially continuous as it was stated. Theorem 2 of [1], which bases itself on that theorem, does not hold. To exclude these incorrect statements, changes were made at our request in English translation of [1] (Soviet Math. Dokl. 21(1980), 303-306).

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