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ON THE SYMMETRY OF APPROXIMATE DINI DERIVATES OF ARBITRARY FUNCTIONS

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Abstract: In the article the strongest relation connecting the approximate Dini derivates of arbitrary functions is found.

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In [2] (see also [1]) the strongest relation connecting the Dini derivates of an arbitrary real function which holds except on a first category set is found. The corresponding problem for approximate Dini derivates was partially solved in [3] by the following theorem.

Theorem A. Let f be an arbitrary function on R. Then there exists a σ-porous set P such that for any $t \in R - P$ or

(i) $\bar{f}^+_{ap}(t) = \bar{f}^-_{ap}(t)$, $\underline{f}^+_{ap}(t) = \underline{f}^-_{ap}(t)$ or

(ii) $\max(|\bar{f}^+_{ap}(t)|, |\underline{f}^+_{ap}(t)|) = \max(|\bar{f}^-_{ap}(t)|, |\underline{f}^-_{ap}(t)|) = +\infty$.

The second author proved Theorem 1 which is more general than Theorem A and, after reading his preprint, the first author added an example showing that this result gives actually the desired strongest relation.

In the following the symbol μ stands for the outer Lebesgue measure in R. The right upper density of $M \subset R$ at $x \in R$ is denoted by $d^+(M,x)$. For definitions of porous and σ-porous sets see e.g. [2].

We shall use the following simple lemma.

Lemma. Let $M \subset R$, $x \in R$ and $0 < c < 1$ be given. Then
$\limsup_{h \to 0+} \mu(M \cap (x+ch, x+h)) / (1-c)h \geqq d^+(M,x)$.

Proof. Let $h^{-1}\mu(M \cap (x,x+h)) \geqq a$. Since $(x,x+h) = \bigcup_{n=0}^{\infty} \langle x + c^{n+1}h,\ x + c^n h)$ it is easy to see that $\mu(x + c^{n+1}h,\ x + c^n h)\ (c^n h - c^{n+1}h)^{-1} \geqq a$ for an index n. From this the conclusion of our lemma easily follows.

Proposition. Let f be an arbitrary function on R. Then the set $M(f) := \{x;\ \bar{f}^-_{ap}(x) < \bar{f}^+_{ap}(x) < +\infty\}$ is σ-porous.

Proof. Define $g(y,x) = (f(y) - f(x))(y - x)^{-1}$. For rational numbers $R < s < S$ put

$$M(R,s,S) = \{x;\ \bar{f}^-_{ap}(x) < R < s < \bar{f}^+_{ap}(x) < S\}.$$

Obviously $M(f) = \bigcup M(R,s,S)$. Let rationals $R < s < S$ be fixed. For positive integers n, k denote by $M(n,k)$ the set of all points x for which

(1) $\qquad d^+(\{z;\ g(z,x) > s\}, x) > 1/n$,

(2) $\mu(\{y;\ g(y,x) > S,\ x < y < x+h\}) \cdot h^{-1} < C$ for $h < 1/k$, and

(3) $\mu(\{y;\ g(y,x) > R,\ x-h < y < x\}) \cdot h^{-1} < C$ for $h < 1/k$,

where $C = \min\ (1/4(s-R)(S-R)^{-1},\ 1/2n)$. Obviously $M(R,s,S) \subset \subset \bigcup M(n,k)$ and therefore it is sufficient to prove that $M(n,k)$ is a porous set for fixed positive integers n, k. Let $x \in M(n,k)$ be given. Choose a number $0 < p < 1/2$ such that

(4) $\quad 2p(1-p)^{-1}\ (S - R - (s-R)/2) < (s-R)/2$.

Let a $\delta > 0$ be given. By (1) and Lemma there exists $h < \min(\delta, 1/k)$ such that

(5) $\mu(\{z;\ g(z,x) > s\} \cap (x + (1-p)h,\ x+h))\ (ph)^{-1} > 1/n.$

We shall prove that

(6) $(x + (1-2p)h,\ x + (1-p)h) \cap M(n,k) = \emptyset.$

Suppose on the contrary that there exists $y \subset (x + (1-2p)h,$ $x + (1-p)h) \cap M(n,k)$. Then, of course, $g(y,x) \geqq R + (s-R)/2$ or $g(y,x) < R + (s-R)/2$. We shall show that the both possibilities yield a contradiction.

a) The case $g(y,x) \geqq R + (s-R)/2$.

In this case

(7) $(x, x + \omega(y-x)) \subset \{z;\ g(x,z) > S\} \cup \{z;\ g(z,y) > R\},$

where $\omega = 1/2\ (s-R)(S-R)^{-1}$.

In fact, suppose that (7) does not hold. Then there exists $z \in (x, x + \omega(y-x))$ such that $g(z,x) \leqq S$ and $g(z,y) \leqq R$. Consequently we have

$$g(y,x) = \frac{(f(y) - f(z)) + (f(z) - f(x))}{y - x} \leqq \frac{(y-z)R + (z-x)\ S}{y - x} =$$

$$= R + (z-x)\ (S-R)/(y-x) < R + \omega\ (S-R) = R + (s-R)/2$$

and this is a contradiction. Since $y - x < 1/k$ we obtain by (7), (2) and (3) $\omega(y-x) < C(y-x) + C(y-x)$ which contradicts to the definitions of the numbers C, ω .

b) The case $g(y,x) < R + (s-R)/2$.

In this case

(8) $(x + (1-p)h, x+h) \cap \{z;\ g(z,x) \geqq x\} \subset \{z;\ g(z,y) > S\}.$

In fact, suppose that (8) does not hold. Then there exists $z \in (x + (1-p)h,\ x+h)$ such that $g(z,x) \geqq s$ and $g(z,y) \leqq S$. Using (4), we consequently obtain

$$s \leqq g(z,x) = \frac{(f(z) - f(y)) + (f(y) - f(x))}{z - x} \leqq$$

$$\leqq \frac{(z-y)S + (y-x)(R + (s-R)/2)}{z - x} = R + (s-R)/2 +$$

$+ (z - y)(z - x)^{-1}(S - R - (s-R)/2) \leq R + (s-R)/2 +$

$+ 2p\,(1-p)^{-1}\,(S - R - (s-R)/2) < s$

and this is a contradiction. Since $h < 1/k$ we obtain by (8),(5) and (2) $ph/n < 2phC$ which contradicts the definition of C.

Since δ is an arbitrary positive number (6) yields that $M(n,k)$ is porous at x. Therefore $M(n,k)$ is a porous set and the proof of Proposition is complete.

Theorem 1. Let f be an arbitrary function on R. Then there exists a σ-porous set P such that for any $x \in R-P$ at least one from the following relations holds:

(i) $\bar{f}^+_{ap}(x) = \bar{f}^-_{ap}(x)$ and $\underline{f}^+_{ap}(x) = \underline{f}^-_{ap}(x)$

(ii) $\bar{f}^+_{ap}(x) = +\infty$ and $\underline{f}^-_{ap}(x) = -\infty$

(iii) $\underline{f}^+_{ap}(x) = -\infty$ and $\bar{f}^-_{ap}(x) = +\infty$.

Proof. Suppose that for an $x \in R$ no from the relations (i), (ii),(iii) holds and $\max(|\bar{f}^+_{ap}(x)|, |\underline{f}^+_{ap}(x)|) = \max(|\bar{f}^-_{ap}(x)|, |\underline{f}^-_{ap}(x)|) = +\infty$. Then it is easy to see that $x \in M(f(x)) \cup \cup M(-f(x)) \cup -M(f(-x)) \cup -M(-f(-x))$, where $M(g)$ has the same sense as in Proposition. From this observation, Theorem A and Proposition, our theorem easily follows.

Theorem 2. Whenever $\bar{D}^+, \underline{D}^+, \bar{D}^-, \underline{D}^- \in \bar{R}$ are such that at least one from the following relations holds

(i) $\bar{D}^+ = \bar{D}^-$ and $\underline{D}^+ = \underline{D}^-$,

(ii) $\bar{D}^+ = +\infty$ and $\underline{D}^- = -\infty$,

(iii) $\underline{D}^+ = -\infty$ and $\bar{D}^- = +\infty$,

then there is a function f such that $\bar{D}^+_{ap}f(x) = \bar{D}^+$, $\underline{D}^+_{ap}f(x) = \underline{D}^+$, $\bar{D}^-_{ap}f(x) = \bar{D}^-$ and $\underline{D}^-_{ap}f(x) = \underline{D}^-$ holds for every x belonging to some residual subset of R.

Proof. If (i) holds, then the desired functions are constructed in Examples 2, 3 in [3].

If (ii) holds, let $d_n^+, d_n^- \in R$ be such that $d_n^+ \longrightarrow \underline{D}^+$, $d_n^- \longrightarrow \overline{D}^-$ and $|d_n^+| + |d_n^-| \leq 2^n$ and let $A, B \subset R$ be disjoint measurable sets such that $\mu(I \cap A) > 0$ and $\mu(I \cap B) > 0$ for every interval I. Let I_n be a sequence of all rational intervals. By induction we shall construct sequences g_n, h_n of functions ($n = 0,1,\dots$) and sequences $T_n \subset I_n$ of open intervals and F_n of disjoint compact nowhere dense subsets of A ($n = 1,2,\dots$) such that

(a) $0 \leq g_n \leq g_{n+1} \leq h_{n+1} \leq h_n \leq 1$,

(b) for every interval I there is an interval $J \subset I$ such that $\sup g_n|J < \inf h_n|J$,

(c) $\mu T_n \leq 2^{-n}$, $\mu T_n \leq 2^{-2n}$ dist (T_n, F_n) and $|h_{n+1}(u) - g_{n+1}(v)| \leq 2^{-n-1}$ dist (T_n, F_n) for all $u, v \in T_n$,

(d) for every $t \in T_n$ there is $s \in (0, 2^{-n})$ such that $\mu((t-s, t+s) - F_n) \leq 2^{-n}s$, and

(e) for every $t \notin T_n$ either $\mu F_n \leq 2^{-n}$ dist$^2(t, F_n)$ or $h_{n+1}(t) \leq \inf \{h_{n+1}(u) - 2^n(|x-u| + |x-t|);\ u \in T_n,\ x \in F_n\}$.

We put $g_1 = 0$, $h_1 = 1$ and, whenever g_n and h_n have been defined, we find an interval $J \subset I_n$ and $c, d \in (0,1)$ such that $g_n \leq c < d \leq h_n$ on J. Next we find an interval $K \subset J$ such that $\mu K \leq 2^{-3n-4}(d-c)^2$, $\mu K \leq 2^{-n}$ dist$^2(R-J, K)$ and $\mu(K-A) < 2^{-n-2}\mu K$. Let $F_n \subset K \cap A$ be a compact nowhere dense set which does not contain the center of K such that $\mu(K - F_n) \leq 2^{-n-2}\mu K$. Finally we find an open interval T_n containing the center of K such that $\mu T_n \leq 2^{-2n}$ dist (T_n, F_n) and $\mu T_n \leq 2^{-n-2}\mu K$ and we put
$g_{n+1}(t) = g_n(t)$ and $h_{n+1}(t) = h_n(t)$ for $t \notin J$,
$g_{n+1}(t) = \max(c, d - 2^{-n-1}$ dist $(T_n, F_n))$ and $h_{n+1}(t) = d$ for $t \in T_n$,

and

$g_{n+1}(t) = c$ and $h_{n+1}(t) = (c+d)/2$ for $t \in J-T_n$.

Then (a),(b) and (c) are obvious and (d) follows from
$\mu((t-\mu K/2, t+\mu K/2)-F_n) \leq 2^{-n-2}\mu K + \mu T_n \leq 2^{-n-1}\mu K$.
To prove (e) assume that $t \notin T_n$, $u \in T_n$ and $x \in F_n$ are such that
$h_{n+1}(t) \geq h_{n+1}(u) - 2^n (|x-u| + |x-t|)$. If $t \notin J$ then
$\mu F_n \leq \mu K \leq 2^{-n}$ dist^2 $(R-J, K) \leq 2^{-n}$ dist$^2(t, F_n)$. If $t \in J$ then, according to the definition of the function h_{n+1}, we get
$2^n(|x-u| + |x-t|) \geq (d-c)/2$, hence μK+dist $(t, F_n) + \mu K \geq 2^{-n-1}(d-c)$
and dist $(t, F_n) \geq 2^{-n-1}(d-c) - 2\mu K \geq 2^{-n-2}(d-c)$. Hence
$\mu F_n \leq \mu K \leq 2^{-3n-4}(d-c)^2 \leq 2^{-n}$ dist$^2(t, F_n)$.
Let $G = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T_n - \bigcup_{n=1}^{\infty} F_n$ and choose $s_n \in T_n$. Put

$f(x) = \inf_n h_n(x)$ for $x \in G$,

$f(x) = h_{n+1}(s_n) + d_n^+(x-s_n)$ for $x \in F_n$, $x > s_n$,

$f(x) = h_{n+1}(s_n) + d_n^-(x-s_n)$ for $x \in F_n$, $x < s_n$, and

$f(x) = 2$ in all other cases.

Then G is a residual subset of R of measure zero, $f \leq 1$ on G and the set $\{x;\ f(x)=2\}$ is of positive measure in every interval, therefore (see, e.g.,[4], Proposition 5)
$\bar{d}(\{t;\ f(t) = 2\}, x) = 1$ for all x from a residual subset of R and hence $\bar{D}^+_{ap} f(x) = +\infty$ and $\underline{D}^-_{ap} f(x) = -\infty$ holds in a residual subset of G. If $t \in G$, let $C = \bigcup \{F_n;\ t \in T_n\}$ and
$D = \bigcup \{F_n;\ \mu F_n \leq 2^{-n}$dist$^2(t, F_n)\}$. From (d) we see that $\bar{d}(C,t) = 1$ and obviously $d(D,t) = 0$. If $x \in R-(C \cup D \cup G)$ and $x \notin \bigcup_{k<m} F_k$, then either $f(x) = 2$ and hence $f(x) - f(t) \geq 1$ or x belongs to some F_n with $n \geq m$. In the latter case $f(x) \geq h_{n+1}(s_n) - 2^n|x-s_n|$, hence $f(x) - f(t) \geq 2^n|x-t|$ according to (e). If $x \in C$, $x \in F_n$ and $x > t$, then $|f(x) - f(t) - d_n^+(x-t)| \leq |f(t) - h_{n+1}(s_n)| + 2^n|t-s_n| \leq$
$\leq 2^{-n}$ dist $(T_n, F_n) + 2^n \mu T_n \leq 2^{-n+1}$dist $(T_n, F_n) \leq 2^{-n+1}|x-t|$.

Similarly we get $|f(x) - f(t) - d_n^-(x-t)| \le 2^{-n+1}|x-t|$ for $x \in C$, $x \in F_n$, $x < t$. Hence $\underline{D}_{ap}^+ f(t) = \underline{D}^+$ and $\overline{D}_{ap}^- f(t) = \overline{D}^-$ for every $t \in G$, which finishes the proof.

Finally we note that the case (iii) follows from (ii) by symmetry.

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