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A LIOUVILLE THEOREM FOR NONLINEAR ELLIPTIC SYSTEMS WITH ISOTROPIC NONLINEARITIES

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Abstract: We show that if $u=(u_1,\dots,u_m)$ is a solution with bounded gradient in $\mathbb{R}^n$ of an elliptic system of the form:

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(|\nabla u|^2)\frac{\partial u_\alpha}{\partial x_j}\right)=0,\ 1\le\alpha\le m,$$

then each u_α is an affine function on $\mathbb{R}^n$.

Key words: elliptic systems, Liouville theorem, regularity, Harnack inequality.

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I Introduction:

We consider here a nonlinear second-order elliptic system of the following form:

$$(1)\qquad -\frac{\partial}{\partial x_i}\left(a_{ij}(|\nabla u|^2)\frac{\partial u_\alpha}{\partial x_j}\right)=0 \text{ in } \mathbb{R}^n,\ u=(u_1,\dots,u_m),\quad 1\le\alpha\le m.$$

Throughout all the paper we will assume that $a_{ij}\in C^1(\mathbb{R})$ (for $1\le i,j\le n$) and that (1) is very strongly elliptic in the sense that for every η and $\xi\ne 0$

$$(2)\qquad a_{ij}(|\eta|^2)\,\xi_i^\alpha\xi_j^\alpha+2\,a'_{ik}(|\eta|^2)\,\eta_i^\alpha\eta_j^\beta\xi_i^\alpha\xi_j^\beta>0$$

We prove below that if u has a bounded gradient on $\mathbb{R}^n$, then each component u_α of u is affine on $\mathbb{R}^n$.

This result is clearly a Liouville type theorem. Let us explain now how this result is related to various facts from nonlinear second-order elliptic systems theory. To this end, let us consider a general second order elliptic system:

$$(3) \qquad -\frac{\partial}{\partial x_i}\left(a_i^\alpha(x,u,\nabla u)\right) + a^\alpha(x,u,\nabla u) = f^\alpha(x) \text{ in } \Omega$$

where $1 \leq \alpha \leq m$, $u = (u_1,\dots,u_m)$ and Ω is a bounded domain in $\mathbb{R}^n$. The very strong ellipticity of the system (3) is expressed by the following condition:

$$(4) \qquad \frac{\partial a_i^\alpha}{\partial \gamma_j^\beta}(x,\xi,\gamma)\ \xi_i^\alpha \xi_j^\beta > 0, \qquad \xi \neq 0.$$

Of course, when (3) reduces to (1), (4) is nothing else than (2). Assuming that u is a Lipschitz solution of (3), one may ask the following natural (and fundamental) question: is u of class C^1 or even $C^{1,\mu}$ (for some $\mu \in (0,1)$) ?

As shown by M.Giaquinta and J.Nečas [2] , this regularity question turns out to be, in some sense, equivalent to the following Liouville type condition: (3) is said to satisfy <u>the Liouville condition</u> (in short $L(\mathbb{R}^n)$) provided the following implication holds: for all $x^0 \in \Omega$, $\xi \in \mathbb{R}^m$, if $v = (v_1,\dots,v_m)$ is a solution with bounded gradient of

$$(3') \qquad -\frac{\partial}{\partial x_i}\left(a_i^\alpha(x^0,\xi,\nabla v)\right) = 0 \quad \text{in } \mathbb{R}^n,$$

then each v is affine on $\mathbb{R}^n$. More precisely, in [2] it is proved that if the system (3) (where we assume (4) with a_i^α, $a^\alpha \in C^1(\bar{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{mn})$ satisfies $L(\mathbb{R}^n)$ and $p > n$, then for every $\nu > 0$ and every compact set $K \subset \Omega$ there is $c(\nu, K) < \infty$ such that

$$(5) \qquad \| u_\alpha \|_{C^{1,\mu}(K)} \leq c(\nu, K), \qquad 1 \leq \alpha \leq m,$$

with $\mu = 1 - (n/p)$, whenever $f^\alpha \in L^p(\Omega)$ and u is a Lipschitz solution of (3) such that

$$\| u \|_{[W^{1,\infty}(\Omega)]^m} + \| f \|_{[L^p(\Omega)]^m} \leq \nu .$$

Conversely, in some sense, $L(\mathbb{R}^n)$ is a consequence of regularity results of the form (5) - see J.Nečas [6],[7] or M.Giaquinta [1].

Therefore the Liouville result we prove in this paper immediately yields the $C^{1,\mu}$ regularity for special systems of form:

$$(6) \qquad - \frac{\partial}{\partial x_i} (a_{ij}(x,u,|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_j}) + a^\alpha (x,u,\nabla u) = f^\alpha(x) \text{ in } \Omega$$

(for $1 \leq \alpha \leq m$). At this point, we want to point out that this regularity result (a consequence of our result and an equivalent when a_{ij} depend on $|\nabla u|^2$ only) was established by P.A.Ivert [4] in a generalization of deep results due to K.Uhlenbeck [8]. Thus, in some sense, the result we present here is not new and could be derived from Uhlenbeck - Ivert results. On the other hand, our method of proof is quite different from those of [4], [8] and, we believe, much simpler. Let us also mention that it is straightforward to adapt our method of proof to show directly the $C^{1,\mu}$ regularity result (looking, roughly speaking, at little balls instead of large balls).

Let us conclude this introduction by a few words on our method of proof. In section II below, we present a general result on nonlinear elliptic systems which implies in particular that, if we denote by $\omega = \nabla u$, we have: there is $\epsilon_0 > 0$ such that if $\Phi_\omega(R) < \epsilon_0$, then for every $\rho \in (0,R)$

$$(7) \qquad \Phi_\omega(\rho) \leq c_0 \, \Phi_\omega(R)$$

where C_0 depends only on $\|\omega\|_{L^\infty(B_R)}$ and where for a vector valued function g we denote:

$$\Phi_g(\rho) = \frac{1}{\rho^n} \int_{B_\rho} |g(x) - (g)^\rho|^2 \, dx,$$

$$(g)^\rho = (1/|B_\rho|) \int_{B_\rho} g(x)dx.$$

By an easy use of Poincaré inequality, we see that in order to conclude (using(7)) we just need to show that $\omega = \nabla u$ has the so-called Saint-Venant property:

$$\text{(8)} \qquad \lim_{R\to\infty} R^{-n+2} \int_{B_R} |\nabla \omega(x)|^2 \, dx = 0 \quad .$$

The main idea used to prove (7) goes back to a fundamental lemma of E.Giusti - see e.g. [2] .

Next, in section III, we state and prove a Liouville type theorem. This is done by remarking - following [4],[8] - that $|\nabla u|^2 = w$ satisfies:

$$\text{(9)} \qquad -\frac{\partial}{\partial x_i}\left(A_{ij} \frac{\partial w}{\partial x_j}\right) + \alpha |D^2u|^2 \leq 0 \quad \text{in } \mathbb{R}^n$$

for some $\alpha > 0$, and for some uniformly elliptic coefficients A_{ij}. Using this inequality and a Harnack type inequality proved in D.Gilbarg and N.S.Trudinger [3] (for example), we show that (8) holds and thus ω is constant.

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II A general result on quasilinear elliptic systems:

In this section we consider a solution $\omega = (\omega_1, \ldots, \omega_N)$ of

$$(10) \qquad -\frac{\partial}{\partial x_i}\left[A_{ij}^{\alpha\beta}(\omega)\,\frac{\partial \omega_\beta}{\partial x_j}\right] = 0 \quad \text{in } \mathbb{R}^n, \quad \alpha = 1,\dots,N,$$

where $A_{ij}^{\alpha\beta}$ are continuous on $\mathbb{R}^m$ and where the ellipticity condition

$$(11) \qquad A_{ij}^{\alpha\beta}(\xi)\,\xi_i^\alpha\,\xi_j^\beta > 0 \quad \text{for } \xi \neq 0$$

holds.

Theorem II. 1: Let $R > 0$, let ω be a bounded solution of (10) in $(H^1(B_R))^N$ and let us assume that (11) holds. We denote $\mu = \|\omega\|_{L^\infty(B_R)}$. Then there exist $\varepsilon_0 > 0$, $C_0 > 0$ such that the following statement holds:

if $\qquad \Phi_\omega(R) \leq \varepsilon_0^2$,

then $\qquad \Phi_\omega(\rho) \leq C_0\, \Phi_\omega(R)$

whenever $\rho \in (0,R)$. In addition ε_0, C_0 depend only on μ and on the ellipticity constants in (11).

Before giving the proof of Theorem II.1, let us mention the

Corollary II.1: Let ω be a bounded solution of (10) in $(H^1_{loc}(\mathbb{R}^n))^N$ satisfying the Saint-Venant property

$$\lim_{R\to\infty} R^{-n+2} \int_{B_R} |\nabla \omega(x)|^2\, dx = 0,$$

and let us assume that (11) holds. Then ω is a constant vector.

Proof: Observe that we have by Poincaré inequality:

$$(12) \qquad R^{-n} \int_{B_R} |\omega(x) - (\omega)^R|^2\, dx \leq c_1 R^{-n+2} \int_{B_R} |\nabla \omega(x)|^2\, dx.$$

(Here and below $c_1, c_2, \dots$ denote various positive contents independent of R, ω, u.) Thus we see that (8) implies: $\lim_{R\to\infty} \Phi_\omega(R) = 0$. Therefore by Theorem II.1, $\Phi_\omega(\rho) = 0$ for all $\rho > 0$ and the proof is complete.

<u>Proof of Theorem II.1:</u> First of all, in view of (11), there exists $\nu > 0$ such that for every ξ and $|\xi| \leq \mu$ we have

$$[A_{ij}^{\alpha\beta}(\xi)\, A_{ij}^{\alpha\beta}(\xi)]^{1/2} \leq \frac{1}{\nu}\,,\ A_{ij}^{\alpha\beta}(\xi)\, \xi_i^{\alpha}\, \xi_j^{\beta} \geq \nu\, |\xi|^2.$$

Let us also recall that it is known (see e.g.[2]) that there exists c_2 ($= c_2(\mu, \nu)$) such that we have:

$$\Phi_\omega(\tau) \leq c_2 \tau^2.\ \Phi_\omega(1), \qquad 0 < \tau \leq 1, \tag{13}$$

if ω is a solution of the system:

$$-\frac{\partial}{\partial y_i}\left(A_{ij}^{\alpha\beta}(\xi)\frac{\partial \omega_\beta}{\partial y_j}\right) = 0 \ \text{ in } B_1$$

where $|\xi| \leq \mu$.

Next, let $\tau \in (0,1)$. We are first going to prove that there exist $\varepsilon_0 = \varepsilon_0(\mu, \tau, \nu) > 0$ such that

$$\Phi_\omega(\tau) \leq 2\, c_2\, \tau^2\, \Phi_\omega(1) \tag{14}$$

where ω solves (10) and satisfies: $\|\omega\|_{L^\infty(B_1)} \leq \mu$, $\Phi_\omega(1) \leq \varepsilon_0^2$.

Let us argue by contradiction and let us thus assume that there exists a sequence $(\omega^n)_{n\geq 1}$ of solutions of (10) satisfying:

$$\|\omega^n\|_{L^\infty(B_1)} \leq \mu,\ \{\Phi_{\omega^n}(1)\}^{1/2} = \varepsilon_n \longrightarrow 0, \tag{15}$$
$$\Phi_{\omega^n}(\tau) > 2c_2 \tau^2 \varepsilon_n^2.$$

To simplify notations, we will use indifferently the notations $\Phi_{\omega^n}(\tau)$ or $\Phi(\omega^n, \tau)$. We then set: $\sigma^n = \frac{1}{\varepsilon_n}[\omega^n - (\omega^n)^1]$.

Obviously we have:

$$\int_{B_1} |\sigma^n(x)|^2 dx = 1;\ \Phi(\sigma^n, \tau) > 2c_2 \tau^2; \tag{16}$$

$$(17) \qquad - \frac{\partial}{\partial x_i} \left(A_{ij}^{\alpha\beta}(\omega^n) \frac{\partial \sigma_\beta^n}{\partial x_j} \right) = 0 \,.$$

Without loss of generality we may assume that:

$\sigma^n \longrightarrow \sigma$ weakly in $(L^2(B_1))^N$, $\varepsilon_n \sigma^n \longrightarrow 0$ in $(L^2(B_1))^N$ and a.e.,

for some $\sigma \in (L^2(B_1))^N$. In addition, in wiew of (16): $\Phi_\sigma(1) \le 1$. Furthermore, recalling that we have:

$$\omega^n = \varepsilon_n \sigma^n + (\omega^n)^1, \quad \| \omega^n \|_{L^\infty(B_1)} \le \mu,$$

we see that $|(\omega^n)^1| \le \mu$ and $\omega^n - (\omega^n)^1 \longrightarrow 0$ a.e. Since we may assume without loss of generality that $(\omega^n)^1 \longrightarrow \xi$ $(|\xi| \le \mu)$, we finally deduce: $\omega^n \longrightarrow \xi$. a.e. .

Next, we obtain from (16) and (17):

$$(18) \qquad \int_{B_k} |\nabla \sigma^n(y)|^2 \, dy \le C(k) \quad \text{for} \quad k \in (0,1),$$

thus we may suppose that $\sigma^n \longrightarrow \sigma$ weakly in $(H^1(B_k))^N$ (for all $k < 1$). Thus, passing to the limit in (17), we get:

$$- \frac{\partial}{\partial x_i} \left(A_{ij}^{\alpha\beta}(\xi) \frac{\partial \sigma_\beta}{\partial x_j} \right) = 0 \quad \text{in} \quad B_1 .$$

In addition, since $\sigma_n \longrightarrow \sigma$ in $(L^2(B_k))^N$ (for all $k < 1$), we deduce from (16): $\Phi(\sigma, \tau) \ge 2 c_2 \tau^2 \ge 2c_2 \tau^2 \Phi(\sigma,1)$. This contradicts (13) and the contradiction shows our claim.

Let us choose now $\tau \in (0,1)$ satisfying: $2c_2 \tau^2 \le 1$. Given $\rho \in (0,1)$, let $k \ge 0$ be the integer such that: $\tau^{k+1} \le \rho < \tau^k$. Now, if ω solves (10) and satisfies: $\| \omega \|_{L^\infty(B_1)} \le \mu$, $\Phi_\omega(1) \le \varepsilon_0^2$, we have in view of (14):

$$\tau^n \rho^{-n} \int_{B_\rho} |\omega - (\omega)^\rho|^2 \, dx \le (\rho / \tau^k)^n \rho^{-n} \int_{B_\rho} |\omega - (\omega)^\rho|^2 \, dx \le$$

$$\leq (\tau^k)^{-n} \int_{B_\rho} |\omega - (\omega)^{\tau^k}|^2 \, dx \leq (\tau^k)^{-n} \int_{B_{\tau^k}} |\omega - (\omega)^{\tau^k}|^2 \, dx \leq$$

$$\leq \int_{B_1} |\omega - (\omega)^1|^2 \, dx;$$

that is, we proved: $\Phi_\omega(\rho) \leq \tau^{-n} \Phi_\omega(1)$.

The proof of Theorem II.1 is easily completed by considering the function $\tilde{\omega}(x) = \omega(x/R)$.

Remark II.1: We now show how the preceding results are related to the system (1): indeed, if $u \in (H^2_{loc}(\mathbb{R}^n))^m$ is a solution of (1) then, for $1 \leq k \leq n$, $\frac{\partial u_\alpha}{\partial x_k}$ satisfies:

$$- \frac{\partial}{\partial x_i} \left[A^{\alpha\beta}_{ij}(\nabla u) \frac{\partial}{\partial x_j} \left(\frac{\partial u_\beta}{\partial x_k} \right) \right] = 0 \quad \text{in } \mathbb{R}^n, \quad 1 \leq \alpha \leq m,$$

where $A^{\alpha\beta}_{ij}(\nabla u) = a_{ij}(|\nabla u|^2)\, \delta_{\alpha\beta} + 2\, a'(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_\ell} \frac{\partial u_\beta}{\partial x_j}$.

Thus $\omega = \nabla u$ satisfies a system of the form (10) and (11) is a consequence of (2).

III The main result:

Let $u = (u_1, \dots, u_m)$ be a solution of (1):

$$- \frac{\partial}{\partial x_i} \left(a_{ij}(|\nabla u|^2) \frac{\partial u_\alpha}{\partial x_j} \right) = 0 \quad \text{in } \mathbb{R}^n, \quad 1 \leq \alpha \leq m.$$

Theorem III.1: We assume the ellipticity condition (2) and $\nabla u \in (L^\infty(\mathbb{R}^n))^{nm}$. Then each component u_α of u is affine on $\mathbb{R}^n$.

Proof: Standard arguments yield $u \in W^{2,2}_{loc}(\mathbb{R}^n)$; cf. [7] or [1] .
In view of the results of the preceding section and of Remark II.1, it is enough to show:

$$\lim_{R \to \infty} R^{-n+2} \int_{B_R} |D^2 u|^2 \, dx = 0. \tag{19}$$

In order to prove (19), we first observe that an easy computation yields:

$$(20)\qquad -\frac{\partial}{\partial x_i}\left[A_{ij}(\nabla u)\,\frac{\partial}{\partial x_j}(|\nabla u|^2)\right] + a_{ij}\,\frac{\partial^2 u_\alpha}{\partial x_i\partial x_k}\,\frac{\partial^2 u_\alpha}{\partial x_j\partial x_k} + 2\,a'_{ik}\,\frac{\partial u_\alpha}{\partial x_k}\,\frac{\partial u_\beta}{\partial x_j}\,\frac{\partial^2 u_\alpha}{\partial x_i\partial x_s}\,\frac{\partial^2 u_\beta}{\partial x_j\partial x_s} = 0 \quad ,$$

where $A_{ij}(\nabla u) = \frac{1}{2}\;a_{ij}(|\nabla u|^2) + a'_{ik}(|\nabla u|^2)\frac{\partial u_\alpha}{\partial x_k}\;\frac{\partial u_\alpha}{\partial x_j}$

In view of (2), we see that (for more details, see [4])

$$(21)\quad \exists\,\nu > 0,\quad \forall\,\xi \in \mathbb{R}^n,\quad A_{ij}(\nabla u(x))\,\xi_i\,\xi_j \geq \nu\,|\xi|^2\;,$$

$$\left\{A_{ij}(\nabla u(x))\;A_{ij}(\nabla u(x))\right\}^{1/2} \leq \frac{1}{\nu}\quad \text{a.e. in } \mathbb{R}^n$$

and (20) implies:

$$(22)\qquad -\frac{\partial}{\partial x_i}\,(A_{ij}(\nabla u)\;\frac{\partial}{\partial x_j}\;(|\nabla u|^2)) + \alpha\,|D^2u|^2 \leq 0 \quad \text{in } \mathbb{R}^n,$$

for some $\alpha > 0$. We denote $\quad M = \|\,|\nabla u|^2\,\|_{L^\infty(\mathbb{R}^n)}$.

We are now going to prove:

$$(23)\qquad R^{-n+2}\int_{B_{R/2}} |D^2u|^2\,dx \leq c_3 R^{-n}\int_{B_{2R}} (M - |\nabla u|^2)\,dx.$$

To this end we introduce $\eta \in H^1_0\,(B_{2R})$, the solution of:

$$(24)\qquad -\frac{\partial}{\partial x_i}\,(A_{ji}\;\frac{\partial \eta}{\partial x_j}\;) = \frac{1}{R^2}\quad \text{in } B_{2R}\quad .$$

Standard results yield: $\eta \geq 0$ in B_{2R} and

$$(25)\qquad \|\eta\|_{L^\infty(B_{2R})} \leq c \qquad\qquad \operatorname*{infess}_{B_{R/2}}\;\eta \geq c_5 > 0.$$

Then multiplying (22) by η^2 and using (24), (25), we deduce:

$$c_6 \int_{B_{R/2}} |D^2u(x)|^2\,dx \leq \int_{B_{2R}} A_{ij}\,\frac{\partial \eta^2}{\partial x_i}\,\frac{\partial}{\partial x_j}\,(M-|\nabla u|^2)\,dx \leq$$

$$\leq 2\int_{B_{2R}} \frac{1}{R^2}\,\eta\,(M-|\nabla u|^2)\,dx - 2\int_{B_{2R}} A_{ij}\,\frac{\partial \eta}{\partial x_i}\,\frac{\partial \eta}{\partial x_j}\,(M-|\nabla u|^2)dx$$

and this yields:

$$\int_{B_{R/2}} |D^2u(x)|^2\,dx \leq \frac{c_7}{R^2}\int_{B_{2R}} (M-|\nabla u|^2)dx$$

and (23) is proved.

To conclude, we see that (19) follows from (23), applying the following lemma to $w = |\nabla u|^2$, $\alpha_{ij}(x) = A_{ij}(\nabla u(x))$.

<u>Lemma III.1</u>: Let $w \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy:

$$-\frac{\partial}{\partial x_i}\left(\alpha_{ij}(x)\,\frac{\partial w}{\partial x_j}\right) \leq 0 \quad \text{in } \mathbb{R}^n$$

where $\alpha_{ij} \in L^\infty(\mathbb{R}^n)$ satisfy:

$$\{\alpha_{ij}(x)\;\alpha_{ij}(x)\}^{1/2} \leq \frac{1}{\nu}\,,\quad \alpha_{ij}(x)\,\xi_i\,\xi_j \geq \nu|\xi|^2 \quad \forall\,\xi \in \mathbb{R}^n,$$

$$\text{a.e. in } \mathbb{R}^n$$

for some $\nu > 0$. If $M = \sup\operatorname{ess}_{\mathbb{R}^n} w$, then we have:

$$(26)\qquad \lim_{R\to\infty}(1/|B_R|)\int_{B_R} w(x)dx = M.$$

<u>Proof</u>: This lemma is proved by the use of a weak Harnack inequality (cf.[3], for example) which implies:

$$(27)\qquad R^{-n}\int_{B_{2R}} z(x)\,dx \leq c_8 \inf_{B_R}\operatorname{ess}\, z$$

with $z = M - w$. Now if we let $R \to \infty$, we obtain (26) since $\inf\operatorname{ess}_{B_R} z \to \inf\operatorname{ess}_{\mathbb{R}^n} z = 0$; and $z \geq 0$ a.e. in R^n.

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