Commentationes Mathematicae Universitatis Carolinae

David M. Berman

A note on choosability in planar graphs

Commentationes Mathematicae Universitatis Carolinae, Vol. 23 (1982), No. 3, 537--540

Persistent URL: `http://dml.cz/dmlcz/106174`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

23,3 (1982)

A NOTE ON CHOOSABILITY IN PLANAR GRAPHS

David M. BERMAN

Abstract: We call a graph k-choosable if for every assignment of a list of k colors to each vertex, the graph can be properly colored so that each vertex is colored with one of the colors on its list. Erdös, Rubin and Taylor have conjectured that every planar graph is 5-choosable. In this note we show that in a minimal counter-example to this conjecture every vertex of degree five must have a neighbor of degree at least seven.

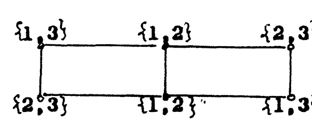
Key words: Choosability, coloring, planar graph.

Classification: 05C15

In [1] Erdös, Rubin and Taylor developed the idea of choosability in graphs. Suppose each vertex of graph G has assigned to it a list of k colors. We say that G is k-choosable or can be k-list-colored if for every assignment of lists, G has a proper coloring with each vertex assigned a color on its list.

We call the minimum k for which G is k-choosable the list-chromatic number of G. It is immediate that the list-chromatic number of G is at least as great as the chromatic number.

That this inequality may be strict is shown by the following example:



The graph is 2-chromatic but the assignment of lists shown in the diagram shows that it is not 2-choosable. It might be noted that to show that a particular graph is k-colorable one needs only to exhibit a k-coloring; to show that it is k-choosable one must show that it can be properly colored from any assignment of lists to the vertices.

It is clear that any planar graph is 6-choosable. The proof is by induction. Delete from the graph a vertex x of degree at most five, and 6-list-color the remaining graph.

When x is restored, it has at most five neighbors already colored, so there must be at least one of the six colors on the list for x that is not used for any of the neighbors. Therefore x can be colored with that sixth color.

In [1] the conjecture is made that every planar graph is 5-choosable.

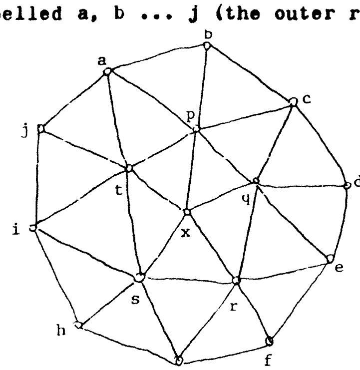
It appears that the techniques used to attack the four color problem cannot be applied to the conjective of five-choosability. Kempe chains for instance cannot be used because the step of recoloring a vertex, say from red to blue, requires that blue be on the list for that vertex.

It is clear that a minimal counter-example to his conjecture can have no vertex of degree four or less. The proof mimics that for the 6-choosability.

In this note we show that in a minimal counter-example to the conjecture every vertex of degree five must have a neighbor of degree at least seven.

The proof is by contradiction. Suppose in a minimal counter-example G we have a vertex x of degree five, all of whose neighbors are of degree six or less. It clearly suffices to consider only the case that all neighbors of x have degree six.

We then have the configuration shown with the five neighbors of x labelled p, q, r, s, t (the inner ring) and their neighbors labelled a, b ... j (the outer ring)



As a matter of notation: let $L(v)$ denote the set of colors on the list of v; let $k(v)$ be the color assigned to v in a coloring of G. Say $L(v) = \{1,2,3,4,5\}$.

Delete x and 5-list-color the remaining graph. When x is restored it will be possible to color x unless the five vertices in the inner ring have been colored using all five of the colors in $L(x)$.

Without loss of generality, say p, q, r, s, t are colored 1, 2, 3, 4, 5 respectively. If p can be re-colored with a color other than 1, then x can be colored 1. We cannot re-color p with any color used for a, b, or c but there must be at least one color on $L(p)$ other than 1, $k(a)$, $k(b)$, $k(c)$.

If this color is neither 2 nor 5, then re-color p with this color. Then x can be colored 1.

Say therefore without loss of generality $L(p) = \{1,2, k(a), k(b),k(c)\}$. Recolor p with color 2. Now q must be re-colored. $L(q)$ must have a fifth color other than 2, $k(c)$, $k(d)$, $k(e)$. If this color is other than 3, then re-color q with this color and color x with 2.

Say therefore that $L(q) = \{2,3, k(c),k(d),k(e)\}$. Repeating the above procedure with vertex r, then s, then t we see that the only obstruction to coloring x occurs if $L(r) = \{3,4, k(e), k(f),k(g)\}$; $L(s) = \{4,5,k(g),k(h),k(i)\}$, $L(t) = \{5,1,k(i),k(j), k(a)\}$.

To look at this another way, we see that if x and the entire ring p , q, r, s, t are all deleted and the remaining graph is 5-list colored, then the only obstruction to restoring x, p, q, r, s, t and extending the coloring to them is if the lists for the inner ring and the coloring of the outer ring are as described above.

We now propose two distinct methods of reduction, which one to be used depending on the lists for the inner ring. In each case the reduction will make impossible the coloring of the outer ring in a manner such that the coloring cannot be extended to a coloring of G.

Case i: The lists for the vertice in the inner ring have in their union at least ten colors, including therefore at least five colors other than those in $L(x)$; say they include 6, 7, 8, 9, 10.

Define graph G' by deleting vertices x, p, q, r, s, t and replacing them with a new vertex x' that is joined to each vertex a, b, c ... j of the outer ring. Assign to x' the list $L(x') = \{6,7,8,9,10\}$. By induction G' can be 5-list colored; say with x' colored 6. But since x' is adjacent in G' to each vertex of the outer ring, this gives a coloring with none of the vertices of the outer ring colored 6. Since 6 is on the list of one of the vertices of the inner ring, this list coloring can be extended in to x, p, q, r, s, t.

Case ii: The lists for the vertices in the inner ring have in their union at most nine colors. Then we must have some two of the ten colors $k(a), k(b) \ldots k(j)$ the same. Say $k(v) = k(w)$ for some $v, w \in \{a, b, \ldots, j\}$.

Define graph G' by deleting x, p, q, r, s, t and joining vertex v to each other vertex of the outer ring. By induction G' can be 5-list-colored. In this coloring no other vertex of the outer ring can be colored the same as v. Therefore this coloring can be extended in to color x, p, q, r, s, t.

Reference

[1] P. ERDÖS, A.L. RUBIN, H. TAYLOR: Choosability in Graphs, Proc. West Coast Conference on Combinatorics, Graph Theory, and Computing, Arcala, Calif., 1979 (Utilitas Math. Publ. Inc., Winnipeg 1980), 125-157.

University of New Orleans, New Orleans, LA 70148, U.S.A.

(Oblatum 30.4. 1982)