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ISOMORPHISMS OF LIE ALGEBRAS OF VECTOR FIELDS

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Abstract : Many examples of Lie subalgebras of the Lie algebra of vector fields with compact supports of a manifold are known to characterize the manifold. Introducing the localizable subalgebras, we get rid of the restriction on the supports extending and generalizing thus these results to a large class of subalgebras.

Key words : Smooth manifold - Infinite dimensional Lie algebra - Lie algebra of vector fields - Isomorphism of Lie algebras.

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1. Introduction

Many examples of subalgebras of the Lie algebra $\mathcal{X}(M)$ of smooth vector fields of a smooth manifold M are known to characterize the differentiable structure of M. Most of of them are more precisely subalgebras of vector fields with compact supports [4],[5],[8],[9] (exceptions being the so-called quasi-foliations of Amemiya [1] and the Lie algebra of infinitesimal automorphisms of a vector bundle [6]).

Our aim is to try to get rid of the restriction on the supports, which seems to rely on the techniques of proofs, rather than on geometric features.

In this paper, for each manifold M, we introduce a class A_M of Lie subalgebras of $\mathcal{X}(M)$. the so-called *localizable* **subalgebras**, for the isomorphisms of which we obtain the following description : given a transitive $A \in \mathcal{A}_M$, a transitive $A' \in \mathcal{A}_{M'}$, and an isomorphism of Lie algebras $\phi : A \to A'$, there exist an open dense subset Ω of M, an open dense subset Ω' of M' and a diffeomorphism $f : \Omega \to \Omega'$ such that $\phi = f_*$ on Ω. We moreover give various general conditions on A and on A' which guarantee that $\Omega = M$ and that $\Omega' = M'$. This allows us to obtain the expected extensions and generalizations mentioned above.

2. Localizable subalgebras

The manifolds considered in this note are connected, Hausdorff, second countable and of class C_∞.

We denote by $\mathcal{X}(M)$ the Lie algebra of smooth vector fields of a manifold M and by $C_\infty(M)$ the space of real valued smooth functions on M.

Let $A \subset \mathcal{X}(M)$ be a subalgebra. For each $x \in M$, set $A_x = \{X_x | X \in A\}$ and for each open subset $\omega \subset M$, $A_\omega = \{X \in A : \text{supp } X \subset \omega\}$ and $N_\omega(A) = \{X \in A : X|\omega = 0\}$.

Definition 2.1. A Lie subalgebra $A \subset \mathcal{X}(M)$ is called *localizable* if

(I) for each finite open cover $\mathcal{O}$ of M, including a neighborhood of ∞, $A = \sum_{\omega \in \mathcal{O}} A_\omega$,

(II) for each $x \in M$ and each $X \in A$ such that $X_x \neq 0$, there exists an open neighborhood ω of x such that $A = L_X A + N_\omega(A)$.

(III) for each $x \in M$, $A_x \neq 0$.

Finite open covers including a neighborhood of ∞ will be called *proper covers*.

We denote by $\mathcal{A}_M$ the family of all localizable Lie subalgebras of $\mathcal{X}(M)$. Let us now give some examples of localizable subalgebras. We denote by $\mathcal{X}(M)_c$ the Lie algebra of smooth vector fields with compact support in M.

Examples 2.2. a) *If* $A \in \mathcal{A}_M$, *then* $A_c = A \cap \mathcal{X}(M)_c \in \mathcal{A}_M$.

This is obvious

b) *The Lie algebra* $\mathcal{X}(M)$ *and, for each closed regular submanifold* N *of* M, *the Lie algebra* A(N) *of all vector fields on* M *tangent to* N *belong to* $\mathcal{A}_M$.

This is easily verified.

Interesting structures on M provide further examples of localizable subalgebras.

c) *If* η *is a contact form on* M, *the Lie algebra of infinitesimal conformal contact transformations of* (M,η) *is localizable.*

See [7] and [8].

If η is a symplectic form (resp. a volume element) of M, set $A_\eta = \{X \in \mathcal{X}(M) : L_X\eta = 0\}$ and $A_\eta^{\star} = \{X \in \mathcal{X}(M) : i(X)\eta \text{ is exact}\}$.

d) *If* $m = \dim M > 1$, $A_\eta^{\star} \in \mathcal{A}_M$; $A_\eta \in \mathcal{A}_M$ *if and only if* $A_\eta = A_\eta^{\star}$ *that is if and only if* $H^1(M) = 0$ (resp. $H^{m-1}(M) = 0$).

Proof. It is known that $[A_\eta, A_\eta] = [A_\eta^{\star}, A_\eta^{\star}] = A_\eta^{\star}$ and that $A_\eta, A_\eta^{\star}$ satisfy (II) and (III) (see[2,8]).

If η is a symplectic form or a volume element, $\mu : X \to i(X)\eta$ is a linear bijection from A_η (resp. $A_\eta^{\star}$) onto the space $Z^p(M)$ of closed p-forms (resp. the space $B^p(M)$ of exact p-forms) of M, for $p = 1$ or $m - 1$.

Setting $X_u = \mu^{-1}u$, if $X_u \in A_\eta^{\star}$, $u = dv$; for each open cover $\mathcal{O}$, take a partition of the unity subordinate to $\mathcal{O}$, $\alpha_\omega (\omega \in \mathcal{O})$. Then $X = \sum_{\omega\in\mathcal{O}} X_{d\alpha_\omega v}$. Hence A_η verifies (I).

Suppose now that A_η verifies (I). Let K_i be compact subsets of M such that $\overset{\circ}{K}_i \uparrow M$ and $K_i \subset \overset{\circ}{K}_{i+1}$. Covering each K_i by open subsets homeomorphic to $\mathbb{R}^n$, it is easily seen that for each $u \in Z^p(M)$ ($p=1$ or $m-1$), $u = du_i + v_i$ with $\operatorname{supp} u_i \subset K_{i+1}$, $\operatorname{supp} v_i \subset CK_i$, $v_i \in Z^p(M)$. Thus $du_i \to u$ in the natural topology of the exterior algebra $\Lambda(M)$ of M. It follows from the next lemma that u is exact. Hence d).

Lemma 2.3. *The subspace* B(M) *is closed in* Z(M) *for the natural topology of* $\Lambda(M)$.

Proof. Suppose first that M is orientable, fix an orientation of M and consider the map

$$T : \alpha \to T_\alpha(\beta) = \int_M \alpha \wedge \beta$$

from $\Lambda^p(M)$ into $[Z^{m-p}(M)_c]^{\star}$. The space $\Lambda(M)$ is equipped with the natural topology (of uniform convergence on compact subsets; see for instance [3]). It is clear that each $T_\alpha(\beta)$ is continuous with respect to α. The Poincaré duality theorem states that $\alpha \in Z^p(M)$ is exact if and only if each $T_\alpha(\beta)$ is vanishing. Thus $B^p(M)$ is a closed subspace of $Z^p(M)$.

If M is not orientable, let $\pi : \hat{M} \to M$ be the oriented two-folded covering manifold of M. The argument above may be used replacing T by

$$\hat{T} : \alpha \to \hat{T}_\alpha(\beta) = \int_{\hat{M}} \pi^* \alpha \wedge \beta \quad (\beta \in Z^{m-p}(\hat{M})_c)$$

and noting that $\alpha \in B^p(M) \Leftrightarrow \pi^*\alpha \in B^p(\hat{M})$. Hence the lemma.

3. On some maximal ideals of localizable subalgebras

Let $A \in \mathcal{A}_M$ be given. For each $x \in M$, denote by $I^x(A)$ the set of $X \in A$ such that for all $p \in \mathbb{N}$ and all $X_1,\dots,X_p \in A$, $L_{X_1} \circ \dots \circ L_{X_p} X$ vanishes at x, where L denotes the Lie derivative.

It is easily seen [8] that $I^x(A)$ belongs to the class $\mathcal{I}_A$ of all proper maximal ideals of A not containing the derived ideal [A,A] . If $A \subset \mathcal{H}(M)_c$, the converse is known to be true [8] ; when $A \not\subset \mathcal{H}(M)_c$, the elements of $\mathcal{I}_A$ of the form $I^x(A)$ are characterized as follows :

Theorem 3.1. *Let* $A \in \mathcal{A}_M$ *and* $I \in \mathcal{I}_A$ *be given. The following statements are equivalent :*

(a) *there exists* $x \in M$ *such that* $I = I^x(A)$,

(b) *there exists a relatively compact open subset* ω *of* M *such that* $N_\omega(A) \subset I$,

(c) $A_c \not\subset I$.

Proof. We first note that

for each proper cover $\mathcal{O}$ *of* M, *there exists* $\omega \in \mathcal{O}$ *such that* $N_\omega(A) \subset I$. (1)

Otherwise, there is a proper cover $\mathcal{O}$ such that $I \subsetneq I + N_\omega(A)$ for each $\omega \in \mathcal{O}$ and, thus, by the maximality of I, such that $A = I + N_\omega(A)$ for each $\omega \in \mathcal{O}$. Let X, Y $\in$ A be given; by (I), choose $X_\omega \in A_\omega$ such that $X = \sum_{\omega\in\mathcal{O}} X_\omega$; by (II) for each $\omega \in \mathcal{O}$, choose $Y_\omega \in I$ and $Y'_\omega \in N_\omega(A)$ such that $Y = Y_\omega + Y'_\omega$.

Since $[X_\omega, Y'_\omega] = 0$, it follows that

$$[X,Y] = \sum_{\omega \in \mathcal{O}} [X_\omega, Y_\omega] \in I$$

which contradicts the fact that $[A,A] \not\subset I$.

We are now in position to prove the theorem.

Suppose first that there is no relatively compact open subset ω such that $N_\omega(A) \subset I$. Then $A_c \subset I$. Indeed, let $X \in A_c$ and choose a relatively compact open neighborhood ω of $K = \text{supp } X$. Since $N_\omega(A) \not\subset I$, it follows from (1) that $N_{CK}(A) \subset I$ so that $X \in I$, thus (c) $\Rightarrow$ (b).

Suppose that $N_\omega(A) \subset I$ for some relatively compact open subset ω of M. Choose a relatively compact neighborhood Ω of $\bar{\omega}$ and $X \in A \setminus I$. Then $X = X' + X''$ with $X' \in A_\Omega$ and $X'' \in A_{C\bar{\omega}} \subset I$. Thus if $K = \text{supp } X'$, $N_{CK}(A) \not\subset I$. By (1), it follows that, for every finite open cover $\mathcal{O}$ of K, $N_\omega(A) \subset I$ for some $\omega \in \mathcal{O}$. This implies that $\{\omega \text{ open} : N_\omega(A) \not\subset I\}$ does not cover K and hence, that there exists $x \in K$ such that $x \in \omega \Rightarrow N_\omega(A) \subset I$.

For such an $x, I_x = 0$. Otherwise, by (II), $A \subset L_X A + N_\omega(A)$ for some $X \in I$ and $x \in \omega$, thus $A \subset I$. Since I is an ideal, $I \subset I^{(x)}(A)$ and, by the maximality of I, $I = I^{(x)}(A)$. Hence (b) $\Rightarrow$ (a).

It is clear from the definition of localizable subalgebras that $A_c \not\subset I^x(A)$ whatever be $x \in M$, thus (a) $\Rightarrow$ (c) and the theorem follows.

Remark 3.2. Let $x,y \in M$ be distinct. Then by (1), one has $A = I^x(A) + I^y(A)$. In view of (I), this implies that for any $I \in \mathcal{I}_A$, there is *at most one* $x \in M$ such that $I = I^x(A)$.

We will need in the sequel a characterization of the zeroes of the elements of a localizable subalgebra. The proof of the following proposition is easily obtained from its analogúe in [5],[8] and [9].

Proposition 3.3. *Let* $A \in \mathcal{A}_M$, $X \in A$ *and* $x \in M$. *One has*

$$X_x \neq 0 \Leftrightarrow A = L_X A + I^x(A).$$

4. Isomorphisms of localizable subalgebras

Our main result is the following :

Theorem 4.1. *Let* M *and* M' *be manifolds and let* $A \in \mathcal{A}_M$ *and* $A' \in \mathcal{A}_{M'}$ *be given. If* A *and* A' *are transitive, then for each isomorphism of Lie algebras* $\phi : A \to A'$, *there exist an open dense subset* Ω *of* M , *an open dense subset* Ω' *of* M' *and a diffeomorphism* $f : \Omega \to \Omega'$ *such that*

$$\phi(X)_{f(x)} = f_{\star}X_x , \qquad \forall x \in \Omega, \ \forall X \in A.$$

In this statement, $f_{\star}$ denotes the differential of f at x. Recall moreover that a subspace $E \subset \mathcal{H}(M)$ is said to be *transitive* if $E_x = T_xM$ for each $x \in M$.

Proof. Set $I_A^o = \{I^x(A) \mid x \in M\}$, $\Omega = \{x \in M : \phi(I^x(A) \in I_{A'}^o\}$ and $\Omega' = \{x' \in M' : \phi^{-1}(I^{x'}(A')) \in I_A^o\}$. It follows from Remark 3.2. that there is a bijection $f : \Omega \to \Omega'$ such that $\phi(I^x(A)) = I^{f(x)}(A')$ for each $x \in \Omega$.

Let us prove that Ω and Ω' are open and dense. Observe first that

(a) *If* $I \in I_A \setminus I_A^o$ *and if* $A = L_XA + I$, *then* $X \notin A_c$.

Indeed, by thm. 3.1., $A_c \subset I$ so, if $X \in A_c$, $A \subset A_c + I \subset I$.

Let now $x_o \in \Omega$ and $x'_o = f(x_o)$ be given. We may choose $X \in A$ such that $\phi(X)_{x'_o} \neq 0$ and that $\phi(X) \in A'_c$. In view of prop. 3.3., $X_{x_o} \neq 0$ so that $\omega = \{x \in M : X_x \neq 0\}$ is an open neighborhood of x_o in M. Take then $x \in \omega$. One has $A = L_XA + I^x(A)$ and, thus $L_{\phi(X)}A' + \phi(I^x(A)) = A'$. Since $\phi(X) \in A'_c$, it follows from (a) that $\phi(I^x(A)) \in I_{A'}^o$. Th s $\omega \subset \Omega$ and $f(\omega) \subset \text{supp}\,\phi(X)$. This shows that Ω and Ω' are open an oreover, that $f : \Omega \to \Omega'$ is an homeomorphism.

Suppose now that $(C\Omega)^o \neq \emptyset$. Then we may choose $X \in A_c \setminus \{0\}$ such that supp X $\subset C\Omega$. It is clear by prop. 3.3., that supp $\phi(X) \subset C\Omega'$. If $\phi(X)_{x'} \neq 0$ for some $x' \notin \Omega'$, then $L_{\phi(X)}A' + I^{x'}(A') = A'$ and $L_XA + \phi^{-1}(I^{x'}(A')) = A$. Since $\phi^{-1}(I^{x'}(A')) \in I_A \setminus I_A^o$, it follows from (a) that $X \notin A_c$. Thus $\phi(X)$ vanishes on $C\Omega'$ so that $\phi(X) = 0$ d $X = 0$. This contradicts our choice of X. Hence $(C\Omega)^o = \emptyset$.

This proves that Ω and Ω' are dense subsets.

We shall now prove the following lemma.

(b) *Let* $x \in \Omega$, $x' = f(x)$ *and* $h \in T_x M \setminus \{0\}$ *be given. There exists* $X_1,\dots,X_m \in A$ *and* $\lambda_1,\dots,\lambda_m \in C_\infty(M)$ *such that* $\{X_{i,x} | i \leq m\}$ *is a frame of* $T_x M$, $\sum_{i \leq m} \lambda_i X_i \in A$, $X_{1,x} = h$ *and* $X_{1,x}.\lambda_i \neq 0$ *for some* $i \leq m$.

Since A is transitive, we may find $X_1,\dots,X_m \in A$ such that $\{X_{i,x} | i \leq m\}$ is a frame of $T_x M$ and such that $X_{1,x} = h$. In view of (II), we may choose an open neighborhood ω of x such that $A = L_{X_1} A + N_\omega(A)$ and such that $\{X_{i,y} | i \leq m\}$ is a frame of $T_y M$ for each $y \in \omega$. For each $i \leq m$, there exist $Y_i \in A$ and $Y_i' \in N_\omega(A)$ such that

$$X_i = [X_1, Y_i] + Y_i' \; . \tag{2}$$

Using (I), we may moreover assume that $Y_i = Y_i'' + Y_i'''$ where $Y_i'' \in A$ has compact support in ω and $Y_i''' \in A$ vanishes on some neighborhood of x. There exist then $\lambda_{ij} \in C_\infty(M)$ $(i,j \leq m)$ such that $Y_i'' = \sum_{j \leq m} \lambda_{ij} X_j$ and evaluating (2) at x, we get

$$X_{i,x} = \sum_{j \leq m} (X_{1,x}.\lambda_{ij}) X_{j,x} + \sum_{1<j\leq m} \lambda_{ij}(x)\; [X_1, X_j]_x \; .$$

Therefore, there are some $i,j \leq m$ for which $X_{1,x}.\lambda_{ij} \neq 0$ for, otherwise, $[X_1, X_j]_x$ $(i<j\leq m)$ would span $T_x M$.

In order to achieve the proof of thm. 4.1., we need one more remark, namely

(c) *Let* $X_1,\dots,X_p \in A$ *and* $\lambda_1,\dots,\lambda_p \in C_\infty(M)$ *be given. If* $X = \sum_{i \leq p} \lambda_i X_i \in A$, *then*

$$\phi(X)_{x'} = \sum_{i \leq p} \lambda_i \circ f^{-1}(x') \; . \; \phi(X_i)_{x'} \; , \qquad x' \in \Omega'.$$

Indeed, for each $x \in \Omega$, $X - \Sigma \lambda_i(x) X_i \in A$ vanishes at x. It follows then from prop. 3.3. that $\phi(X) - \Sigma \lambda_i(x)\ \phi(X_i)$ vanishes at $f(x)$. This proves (c).

In order to prove that f is a diffeomorphism, choose $h_1 \in T_xM\setminus\{0\}$ and let $X_1,\ldots,X_m$ and $\lambda_1,\ldots,\lambda_m$ be as in lemma (b). From (c), we have

$$\sum_{i\leq m} (\lambda_i \circ f^{-1})\phi(X_i) = \phi(\sum_{i\leq m} \lambda_i X_i)$$

on Ω'. Since $\{X_{i,x} | i\leq m\}$ is a frame of T_xM, it follows from prop. 3.3. that $\{\phi(X_i)_{y'} | i\leq m\}$ is a frame of $T_{y'}M$ in a neighborhood of x' in M'. Therefore the functions $\lambda_i \circ f^{-1}$ are smooth on a neighborhood of x'.

Moreover, applying ϕ to both members of the identity

$$[X, \sum_{i\leq m} \lambda_i X_i] = \sum_{i\leq m} (X.\lambda_i)X_i + \sum_{i\leq m} \lambda_i[X,X_i]$$

and using (c),

$$\sum_{i\leq m} (\phi(X).(\lambda_i \circ f^{-1}))\phi(X_i) = \sum_{i\leq m} ((X.\lambda_i) \circ f^{-1})\phi(X_i),\ \forall X \in \mathcal{X}(M),$$

so that, the $\phi(X_i)$'s being linearly independent at x',

$$\phi(X) . \lambda_i \circ f^{-1}(x') = (X.\lambda_i) \circ f^{-1}(x'), \quad \forall i \leq m. \tag{3}$$

Recall that for one of the λ_i's (call it μ_1), $\langle h_1, d\mu_1\rangle_x \neq 0$. Choose then $h_2 \in T_xM\setminus\{0\}$ such that $\langle h_2, d\mu_1\rangle_x = 0$ and repeat the argument. We obtain by a finite induction $h_1,\ldots,h_m \in T_xM$ and $\mu_1,\ldots,\mu_m \in C_\infty(M)$ such that $\langle h_i, d\mu_j\rangle = 0$ for $i < j$, $\neq 0$ for $i = j$ and such that $\mu_j \circ f^{-1}$ is smooth on a neighborhood of x'. It is clear that differentials $d\mu_{j,x'}$ $(j\leq m)$ are linearly independent thus the μ_j's define a system of local coordinates of M' around x' and therefore f^{-1} is smooth in a neighborhood of x'. It follows that f is a diffeomorphism. Moreover (3) shows that its differential at $x \in \Omega$ is ϕ. Hence the conclusion.

Remarks 4.2. (i) In the above proof, the transitivity of A and A' has only been used to show that f is smooth and that $\phi = f$ on Ω. If $A \in \mathcal{A}_M$ and $A' \in \mathcal{A}_{M'}$ are no longer assumed to be transitive, the proof shows that $f : \Omega \to \Omega'$ is an homèomorphism; if moreover A is a $C_\infty(M)$-module and A' a $C_\infty(M')$-module, a standard argument [1],[8],[9] show that f is a diffeomorphism and that $\phi = f$ on Ω.

(ii) Without additional assumptions, it is impossible to extend f to a diffeomorphism of M onto M'. Take for instance $M' = M\setminus\{x_o\}$, $A = \{X|_{M'} : X \in \mathcal{H}(M)\}$, $A' = \mathcal{H}(M)$ and $\phi^{-1} : X \to X|_{M'}$. It is easily checked that $A' \in \mathcal{A}_{M'}$ and that ϕ is the differential of the embedding of M' into M.

Let us now try to find conditions insuring that $\Omega = M$ and $\Omega' = M'$.

Lemma 4.3. *Let $A \in \mathcal{A}_M$, $A' \in \mathcal{A}_{M'}$ and a isomorphism $\phi : A \to A'$ be given. Let Ω,Ω' and f be defined as in the proof of thm. 4.1.*

If $f(K \cap \Omega)$ is relatively compact in M' whenever K is compact in M , then $\Omega = M$.

Proof. Suppose that $\Omega \neq M$ and take $x \notin \Omega$. For any relatively compact open neighborhood ω of x in M, one has, since Ω is an open dense subset of M,

$$I^x(A) \supset N_\omega(A) = \bigcap_{y\in\omega\cap\Omega} I^y(A) \ ;$$

thus

$$\phi(I^x(A)) \supset \bigcap_{y'\in f(\omega\cap\Omega)} I^{y'}(A') = N_{f(\omega\cap\Omega)}(A') \ ;$$

$f(\omega\cap\Omega)$ is open in M' and $\phi(I^x(A)) \in I_{A'} \setminus I^o_{A'}$. Therefore, in view of thm. 3.1. $f(\omega\cap\Omega)$ is not relatively compact. Thus Ω must be equal to M.

Theorem 4.4. *Let $A \in \mathcal{A}_M$ and $A' \in \mathcal{A}_{M'}$ be given. Suppose that A is a $C_\infty(M)$-module and that A' is a $C_\infty(M')$ -module. If $\phi : A \to A'$ is an isomorphism, then there is a diffeomorphism $f : M \to M'$ such that $\phi = f$.*

Proof. Let Ω,Ω' and f be as in the proof of thm. 4.1. We know that Ω and Ω' are open dense subsets and, by Rem. 4.2., that f is a diffeomorphism and that $\phi = f_{*}$ on Ω.

By lemma 4.3., we are left to show that $f(K \cap \Omega)$ is relatively compact whenever K is compact in M. If it is not true, we can easily construct a sequence $x_m \in \Omega$ which converges to some $x \in M \setminus \Omega$ and such that $f(x_m) \to \infty$ in M'. Choose $X \in A$ such that $X_x \neq 0$. For m large enough, $X_{x_m} \neq 0$. There exists $\lambda \in C_\infty(M')$ such that $\lambda \circ f(x_m).X_{x_m} \to \infty$ in $T(M')$. However $\lambda\phi(X) \in A'$, thus $\phi^{-1}(\lambda\phi(X)) \in A$ and $\lambda \circ f(x_m).X_{x_m} = \phi^{-1}(\lambda\phi(X))_{x_m} \to \phi^{-1}(\lambda\phi(X))_x$. Hence a contradiction.

The above proof can easily be adapted to show the following.

Theorem 4.5. *Let* $A \in \mathcal{A}_M$ *and* $A' \in \mathcal{A}_{M'}$ *be transitive and be stable under locally finite series. If* $\phi : A \to A'$ *is an isomorphism, then there is a diffeomorphism* $f : M \to M'$ *such that* $\phi = f$.

To illustrate the results of this section, we shall mention some corollaries. They are easily deduced and the proofs will not be given. They generalize the results of [4],[5],[8],[9] in which their analogues for vector fields with compact supports were obtained.

Corollary 4.6. *Let* N *(resp.* N'*) be a closed regular submanifold of* M *(resp.*M'*). If* $\phi : A(N) \to A(N')$ *is an isomorphism of Lie algebras, then* ϕ *is the differential of some diffeomorphism* $f : M \to M'$ *such that* $f(N) = N'$.

Corollary 4.7. *Let* η *(resp.* η'*) be a contact form on* M *(resp.* M'*). Let* A_η *(resp.* $A_{\eta'}$*) denote the Lie algebra of all infinitesimal conformal contact transformations of* (M,η) *(resp.* M', η'*)). If* $\phi : A_\eta \to A_{\eta'}$ *is an isomorphism of Lie algebras, then it is the differential of some diffeomorphism* $f : M \to M'$ *such that* $f^{*}\eta' = \lambda\eta$ *for some* $\lambda \in C_\infty(M)$.

Recall the notations A^{*}_η and A_η introduced in §2 for a symplectic structure or a volume element η.

Corollary 4.8. *Let* η *(resp.* η'*) be a symplectic form or a volume element of*

M *(resp.* M'*). If* $\phi : A_\eta \to A_{\eta'}$ *(resp.* $\phi^\circ : A^*_\eta \to A^*_{\eta'}$ *) is an isomorphism of Lie algebras, then* ϕ *(resp.* ϕ°*) is the differential of some diffeomorphism* $f : M \to M'$ *such that* $f^* \eta' = k\eta$ *for some* $k \in \mathbb{R}$.

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