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FIXED POINTS FOR GENERALIZED NONEXPANSIVE MAPPINGS
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Abstract: The aim of the present paper is to prove the existence of fixed points for generalized nonexpansive mappings in convex metric spaces. Such spaces, introduced by Takahashi, included Banach spaces and results of this paper generalize those of Dotson, Rhoades and others.

Key words and phrases: Convex metric spaces, star-shaped metric spaces, normal structure, nonexpansive, generalized nonexpansive mappings, quasi-contraction, locally contractive, uniformly locally contractive mappings, fixed points.

Classification: 47H10, 52H25

Introduction. In 1970 Takahashi [13] introduced a notion of convexity in metric spaces (see Definition 1.1) and extended some fixed point theorems to convex metric spaces. Subsequently Itoh [5], Machado [7], Tallman [12], Naimpally and Singh [8], Naimpally, Singh and Whitfield [9] studied convex metric spaces and fixed point theorems. This paper is a continuation of these investigations.

In section 1 a fixed point theorem is proved for generalized nonexpansive mappings. Section 2 deals with the existence of fixed points in nonconvex domains.

Throughout this paper X and I denote, respectively, a metric space X, with a metric d, and the unit interval [0,1].

The following definition was introduced by Takahashi [13].

Definition 1.1. Let X be a metric space. A mapping W: $:X \times X \times I \longrightarrow X$ is said to be a _convex structure_ if, for each $x,y \in X$ and $\lambda \in I$, the following condition is satisfied:

$$d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $u \in X$. A metric space with a convex structure will be called a _convex metric space_.

If X is a Banach space, then, as a metric space with $d(x,y) = \| x-y \|$, the mapping $W:X \times X \times I \longrightarrow X$ defined by $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ is a convex structure. Thus a Banach space, or any convex subset of a Banach space, is a convex metric space. There are many convex metric spaces which cannot be imbedded in any Banach space [13, Examples 1.1 and 1.2].

Definition 1.2. Let X be a convex metric space. Let K be a nonempty subset of X. K is called _convex_ if $W(x,y,\lambda)$ belongs to K for all $x,y \in K$ and $\lambda \in I$.

Definition 1.3. A metric space X is said to be _star-shaped_ if there exists an $x_0 \in X$ and a mapping $W:X \times \{x_0\} \times I \longrightarrow X$ such that, for each $x,y \in X$ and $\lambda \in I$,

$$d(x,W(y,x_0,\lambda)) \leq \lambda d(x,y) + (1-\lambda)d(x,x_0).$$

Star-shaped metric spaces are generalizations of star-shaped subsets of Banach spaces, where a subset K of a Banach space is star-shaped if there exists an $x_0 \in K$ such that for each $x \in K$, $\lambda \in I$, $\lambda x + (1-\lambda)x_0 \in K$. It is obvious that convex metric spaces are star-shaped metric spaces, but not conversely.

Definition 1.4. A star-shaped metric space X is said to satisfy _condition_ (I) if for any $x,y \in X$, $\lambda \in I$,

$$d(W(x,x_0,\lambda), W(y,x_0,\lambda)) \leq \lambda d(x,y).$$

The condition (I) is always satisfied in any normed line-
ar space. Indeed, let $W(x,x_0, \lambda) = \lambda x + (1-\lambda)x_0$ and
$W(y,x_0,\lambda) = \lambda y + (1-\lambda)x_0$. Then $d(W(x,x_0,\lambda), W(y,x_0,\lambda)) =$
$= \|W(x,x_0,\lambda) - W(y,x_0,\lambda)\| = \|\lambda x - \lambda y\| = \lambda\|x - y\| = \lambda d(x,y)$.

Definition 1.5. A mapping $T:X \longrightarrow X$ is said to be nonex-
pansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$. T is called gene-
ralized nonexpansive if for all $x,y \in X$, $d(Tx,Ty) \leq \max \{d(x,y),$
$d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$. T is called a quasi-con-
traction if there exists a constant k, $0 \leq k < 1$, such that for
all $x,y \in X$, $d(Tx,Ty) \leq k \max \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty),$
$d(y,Tx)\}$.

A generalized nonexpansive mapping may or may not be con-
tinuous. Indeed, let $X = [0,5]$ with the usual metric. Define
$T:X \longrightarrow X$ by $T(x) = x/2$, $0 \leq x < 4$ and $T(x) = -2x + 10$, $4 \leq x \leq 5$.
Then T is a continuous generalized nonexpansive mapping, which
is not nonexpansive (take $x = 4$, $y = 5$). On the other hand, if
$X = [0,1]$, and $T(x) = 0$, $0 \leq x \leq 1/2$, $T(x) = 1/2$, $1/2 < x \leq 1$, then
T is a discontinuous generalized nonexpansive mapping.

The following is proved in [10, Theorem 2].

Theorem R. Let X be a complete reflexive Banach space, K
a closed, bounded, convex subset of C, T a selfmap of K satis-
fying

$\|Tx-Ty\| \leq a(x,y) \|x-y\| + b(x,y) \|x-Tx\| +$

$b(y,x) \|y-Ty\| + c(x,y) \|x-Ty\| + c(y,x) \|y-Tx\|$,

where a, b, c are nonnegative real valued mappings from $K \times K$
satisfying $(a+b+c)(x,y) + (b+c)(y,x) \leq 1$ for all $x,y \in K$. If in
addition, $\sup\limits_{x,y \in K} (b(x,y) + c(y,x)) < 1$.

Then T has a fixed point in K.

In Theorem R, the mapping T is not necessarily continuous.

A theorem similar to Theorem R, for discontinuous genera-
lized nonexpansive mappings is not possible even in uniformly
convex Banach spaces, as can be seen from the following exam-
ple.

Example 1.1. Let $X = R$ and $K = [0,1] \subset R$. Define $T:K \longrightarrow$
$\longrightarrow K$ by $T(x) = 3/4$, $0 \le x \le 1/2$, $T(x) = 1/2$, $1/2 < x \le 1$. Clear-
ly T is a generalized nonexpansive mapping without a fixed
point.

However, for continuous generalized nonexpansive mappings
the following can be proved.

Theorem 1.1. Let X be a compact star-shaped metric spa-
ce satisfying condition (I). Let $T:X \longrightarrow X$ be a continuous ge-
neralized nonexpansive mapping. Then T has a fixed point.

Proof. Let $x_0 \in X$. For $0 < k < 1$, define the mapping T_k as
follows: $T_k(x) = W(Tx,x_0,k)$. Since X is star-shaped, it fol-
lows that T_k maps X into itself. Using condition (I), T_k is a
quasi-contraction. Indeed, $d(T_k(x),T_k(y)) = d(W(Tx,x_0,k),$
$W(Ty,x_0,k)) \le kd(Tx,Ty) \le k \max \{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),$
$d(y,Tx)\}$ for all $x,y \in X$. It follows from [2, Theorem 1] that
there exists an $x_k \in X$ such that $T_k(x_k) = x_k$. Moreover,

$$d(T(x_k),x_k) = d(Tx_k), T_k(x_k)) = d(T(x_k), W(Tx_k,x_0,k))$$

$$\le kd(T(x_k), T(x_k)) + (1-k)d(T(x_k),x_0),$$

which approaches zero as $k \longrightarrow 1$, since X is bounded. Hence
$\lim_k d(T(x_k),x_k) = 0$. Since X is compact, $\{x_k\}$ has a conver-

gent subsequence $\{x_{k_i}\}$ converging to some point $x \in X$. Thus $\lim_i d(x_{k_i}, T(x_{k_i})) = 0$. By the continuity of T, it follows that $T(x_{k_i}) \longrightarrow T(x)$. From the triangular inequality, $d(x, Tx) \leq$ $\leq d(x, x_{k_i}) + d(x_{k_i}, T(x_{k_i})) + d(T(x_{k_i}), T(x))$. Taking the limit as $i \longrightarrow \infty$ yields $x = Tx$.

An immediate consequence of Theorem 1.1 is the following.

Corollary 1.1. [3, Theorem 1.] Let X be a Banach space and K be a nonempty compact star-shaped subset of X. Let $T:K \longrightarrow K$ be a nonexpansive mapping. Then T has a fixed point in K.

Definition 2.1. A mapping T of X into itself is said to be _locally contractive_ if for each $x \in X$ there exist ε and λ ($\varepsilon > 0$, $0 \leq \lambda < 1$), which may depend on x, such that $p, q \in S(x, \varepsilon) =$ $= \{y: d(x,y) < \varepsilon\}$ implies $d(T(p), T(q)) \leq \lambda d(p,q)$. T is called $(\varepsilon - \lambda)$_-uniformly locally contractive_ if it is locally contractive and both ε and λ are independent of x.

Definition 2.2. A mapping T of X into itself is called a **Banach operator** if there exists a constant k, $0 \leq k < 1$, such that for all $x \in X$, $d(T^2(x), T(x)) \leq k d(Tx, x)$.

A Banach operator may not be continuous, and may have more than one fixed point. The following lemma, whose easy proof is omitted, will be needed.

Lemma 2.1. Let X be a complete metric space, $T:X \longrightarrow X$ a continuous Banach operator. Then T has a fixed point.

Definition 2.3. Let X be a convex metric space. X satisfies condition (II) if for all $x, y, z \in X$ and $\lambda \in I$,

$$d(W(x, z, \lambda), W(y, z, \lambda)) \leq \lambda d(x, y).$$

Condition (II) is always satisfied in any normed linear space.

Theorem 2.1. Let X be a convex metric space satisfying condition (II). Let K be a compact subset of X. Let $T:K \longrightarrow K$ be a continuous mapping. Suppose,

(i) there exists $q \in K$ and a fixed sequence of positive reals $k_n (k_n < 1)$ converging to 1, such that $W(Tx,q,k_n) \in K$ for all $x \in K$; further for each $x \in K$ and k_n, $d(T(W(Tx,q,k_n)),T(x)) \leq$
$\leq d(W(Tx,q,k_n,x)$,

or

(ii) K is star-shaped and $d(Tx,Ty) \leq d(x,y)$ whenever $d(x,y) < \varepsilon$ $(x,y \in K)$ for $\varepsilon > 0$. Then T has a fixed point.

Proof. Define the map T_n by

$$T_n(x) = W(Tx,q,k_n).$$

Then each T_n is a continuous Banach operator. Indeed, using condition (i) and (II),

$$d(T_n^2(x), T_n(x)) = d(T_n(W(Tx,q,k_n)), W(Tx,q,k_n))$$

$$= d(W(T(W(Tx,q,k_n)),q,k_n)), W(Tx,q,k_n))$$

$$\leq k_n d(T(W(Tx,q,k_n),Tx)$$

$$\leq k_n d(W(Tx,q,k_n),x)$$

$$= k_n d(T_n(x),x).$$

By hypothesis it follows that each T_n maps K into itself. It follows from Lemma 2.1 that there exists a $y_n \in K$ such that $T_n(y_n) = y_n$.

$$d(T(y_n),y_n) = d(T(y_n), W(Ty_n,w,k_n)) \leq$$

$$\leq k_n d(T(y_n), \ T(y_n)) \ + \ (1-k_n) d(T(y_n),q)$$

which approaches zero as $k_n \longrightarrow 1$. Hence $\lim_n d(y_n, T(y_n)) = 0$.
Since K is compact, $\{y_n\}$ has a convergent subsequence $\{y_{n_i}\}$
converging to some point $y \in X$ and, from the above,
$d(y_{n_i}, T(y_{n_i})) \longrightarrow 0$. By the continuity of T it follows that
$T(y_{n_i}) \longrightarrow T(y)$. Finally,

$$d(y,Ty) \leq d(y,y_{n_i}) \ + \ d(y_{n_i}, T(y_{n_i}) \ + \ d(T(y_{n_i}),T(y)).$$

Taking the limit as $i \longrightarrow \infty$ yields $y = T(y)$.

If (ii) holds, then each map T_n defined by $T_n(x) =$
$= W(Tx,q,k_n)$ is (ϵ, k_n)-uniformly locally contractive. Moreo-
ver, K being compact and star-shaped is complete and ϵ-chain-
able. It follows from [4, Theorem 5.2] that each T_n has a fix-
ed point y_n. The rest of the proof is the same as in (i).

Remark 2.1. If X is a Banach space and K is a star-shap-
ed subset of X (in particular if K is convex) then, for a non-
expansive self mapping T of K, $(1-k_n)q + k_n T(x) \in K$, for each
$x \in K$ and any sequence $\{k_n\}$ converging to $1 (k_n < 1)$, where q is
the star center of K. Moreover, from the nonexpansiveness of
T it follows that $\| T((1-k_n)q + k_n T(x)) - T(x) \| \leq \| (1-k_n)q +$
$+ k_n Tx - x \|$.

Remark 2.2. If K is star-shaped, then Theorem 1 of [3]
follows from Theorem 2.1. Hypothesis (i) above weakens the
star-shaped assumption, as can be seen from the following ex-
ample.

Example 2.1. Let K be the set $\{(0,y):y \in [-1,1]\} \ \cup$
$\cup \ \{(1-\frac{1}{n},0):n \in N\} \cup \{(1,0)\}$ with the metric induced by the

norm $\|(x,y)\| = |x| + |y|$. Define the map $T:K \longrightarrow K$ as fol-
lows: $T(0,y) = (0,-y)$, $T(1-\frac{1}{n},0) = (0,1-\frac{1}{n})$, $T(1,0) = (0,1)$.
Then T satisfies condition (i) of Theorem 2.1 with the choice
$q = (0,0)$, $k_n = 1 - \frac{1}{n}$, $n = 1,2,\ldots$, although K is not star-
shaped.

Remark 2.3. Let X be a reflexive Banach space and K be
a nonempty weakly compact convex subset of X having normal
structure. It is clear from Example 1.1 that a discontinuous
generalized nonexpansive selfmap of K may not have a fixed
point. Also, if X is a Banach space and K is a nonempty weak-
ly compact convex subset of X, then a continuous generalized
nonexpansive map need not have a fixed point (see, e.g., [1]
and [11]). It is known (6, Exercises 1.12, 1.13, 1.14) that a
continuous generalized nonexpansive selfmap of a closed, boun-
ded, convex subset of certain Banach spaces is fixed point free.

Problem. Must a continuous generalized nonexpansive map-
ping of a weakly compact convex subset of a reflexive Banach
space with normal structure have a fixed point?

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