Alberto Capozzi; Addolorata Salvatore

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PERIODIC SOLUTIONS FOR NONLINEAR PROBLEMS WITH STRONG RESONANCE AT INFINITY *)

A. CAPOZZI, A. SALVATORE

Abstract: In this paper we are looking for periodic solutions of the equations $-\ddot{x} = \nabla U(x,t)$. We suppose that the problem is asymptotically linear and that 0 belongs to the spectrum of linearized operator at infinity. We obtain multiplicity results. The proof of the theorem is based on a recent abstract theorem, that has been proved for a functional that satisfies a weaker condition than Palais-Smale condition.

Key words: Variational problem, Resonance, Periodic solutions

Classification: Primary 34C25
Secondary 47H15, 49G99

0. Introduction. The aim of this paper is to look for solutions $x(t) \in C^2(R,R^n)$ of the equations

$$(0.1)\quad \begin{cases} -\ddot{x} = \nabla U(x,t) \\ x(o) = x(T) \\ \dot{x}(o) = x(T) \end{cases}$$

where $T>o$ is a given period, $U(x,t) \in C^2(R^n \times R,R)$, $U(x,t) = U(x,t+T)$ $\forall x \in R^n$ $\forall t \in R$.

The problem (0.1) has been studied by many authors under different assumptions on the function U. We refer to Benci [2] and Thews [5] for a rather complete bibliography. If the pro-

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blem is subquadratic (cf. [2]) multiplicity results for problem (0.1) have been obtained in the non resonant case (i.e. if 0 does not belong to the spectrum of linearized operator at infinity). It is well known that the solutions of (0.1) are the critical points of the functional of the action in a suitable function space. In the non resonant case this functional satisfies the well known Palais-Smale condition. The interest of the resonant case lies in the fact that the Palais-Smale condition is not always satisfied. Recently some techniques have been developed for studying non linear problems, having a variational structure, with a "strong resonance" at infinity (cf. [1]). Our purpose is to use these techniques for solving the problem (0.1).

We denote by $U_{xx}(x,t)$ the Hessian matrix of $U(x,t)$ with respect to the space variables and we assume that there exists

$$\lim U_{xx}(x,t) = M(t) \quad \text{as} \quad |x| \longrightarrow \infty \ \forall t \in [o,T]$$

where $M(t)$ is an $[n\times n]$ symmetric matrix with elements continuous in $[o,T]$.

If we set

$$\nabla U(x,t) = M(t)x - \nabla V(x,t),$$

the problem (0.1) becomes

$$(0.2) \quad \begin{cases} -\ddot{x} = M(t)x - \nabla V(x,t) \\ x(o) = x(T) \\ \dot{x}(o) = \dot{x}(T). \end{cases}$$

We denote by $\mathcal{L}$ the self-adjoint realization in $L^2((o,T),R^n)$ of the operator $x \longrightarrow -\ddot{x} - M(t)x$ with periodic conditions. We assume that

(I_1) $\quad \nabla V(o,t) = o \quad \forall t \in R, \quad o \in \sigma(\mathcal{L})$

(I_2) $\quad V(x,t) \longrightarrow o$ as $|x| \longrightarrow \infty$ uniformly in $t \in R$

$\qquad (\nabla V(x,t),x) \longrightarrow o$ as $|x| \longrightarrow \infty$ uniformly in $t \in R$.

We observe that if $(I_1),(I_2)$ hold, the problem (0.2) has a "strong resonance" at infinity.

We denote by $\mu(t)$ the smallest eigenvalue of $V_{xx}(o,t)$ and we also assume that

(I_3) $\quad \mu = \inf\limits_{[o,T]} \mu(t) > o$,

(I_4) there exists $\lambda_h \in \sigma(\mathcal{L}) \quad \lambda_h < o$ s.t. $\lambda_h + \mu > o$,

(I_5) $\quad V(x,t) = V(-x,t) \quad \forall x \in R^n, \quad \forall t \in R$.

We consider the operator $x \longrightarrow -\ddot{x} - \nabla U(x,t)$ linearized at infinity and at origin and we set

$$L_\infty x = -\ddot{x} - M(t)x$$

$$L_o x = -\ddot{x} - M(t)x + V_{xx}(o,t)x.$$

We denote by m_∞ (resp. m_o) the maximal dimension of subspaces where L_∞ (resp. L_o) is negative semidefinite.

The following theorem holds:

<u>Theorem 0.1</u>. - If $(I_1),(I_2),(I_3),(I_4),(I_5)$ hold, then the problem (0.2) possesses at least m distinct pairs of non-trivial solutions with

$$m = m_\infty - m_o.$$

The proof of Theorem (0.1) is based on the abstract theorem (2.4) in [1].

1. <u>Notations and preliminaries</u>. We set $L^2 = L^2((o,T),R^n)$, $H^1 = H^1((o,T),R^n)$ and denote by

$$(\cdot,\cdot),\ (\cdot,\cdot)_{L^2},\ (\cdot,\cdot)_{H^1}$$

respectively the scalar product on R^n, L^2, H^1.

We set $H = \{u \in H^1 \mid u(o) = u(T)\}$ equipped with the scalar product

$$(u,v)_H = (u,v)_{H^1}.$$

If X is a real Banach space, we denote by X' its dual and by $\langle\cdot,\cdot\rangle$ the pairing between X' and X. In the sequel we shall use the unique symbol $\|\cdot\|$ for the norms in X and X'. If $R>o$ we set $B_R = \{u \in X \mid \|u\| \leq R\}$ and $S_R = \{u \in X \mid \|u\| = R\}$.

If $f \in C^1(X,R)$, we denote by $f'(u)$ the Fréchet derivative of f at $u \in X$.

We recall the following definition [1],[3], which is a weaker version of the well-known Palais-Smale condition.

Definition 1.1. - We shall say that $f \in C^1(X,R)$ satisfies the condition (I) in $]c_1,c_2[$, $(-\infty \leq c_1 < c_2 \leq +\infty)$, if

(I)
- (i) every bounded sequence $\{u_k\} \subset f^{-1}(]c_1,c_2[)$, for which $\{f(u_k)\}$ is bounded and $f'(u_k) \to o$, possesses a convergent subsequence
- (ii) $\forall\ c \in]c_1,c_2[\ \exists \sigma, R, \alpha > o$ s.t. $[c-\sigma, c+\sigma] \subset]c_1,c_2[$ and $\forall u \in f^{-1}([c-\sigma,c+\sigma])$, $\|u\| \geq R$: $: \|f'(u)\|\ \|u\| \geq \alpha$.

We shall need the following abstract theorem for a real functional f on a real Hilbert space M ([1], th. 2.4).

Theorem 1.1. - Suppose that $f \in C^1(M,R)$ satisfies the following properties:

f_1) f satisfies condition (I) in $]0,+\infty[$;

f_2) there exist two closed subspaces M^+, M^- of M, with codim $M^+ < +\infty$, and two constant $c_\infty > c_0 > f(0)$ such that

a) $f(u) > c_0 \quad \forall u \in S_\rho \cap M^+$

b) $f(u) < c_\infty \quad \forall u \in M^-$

f_3) f is even.

Then, if dim $M^- \geq$ codim M^+, f possesses at least m = dim M^- - codim M^+ distinct pairs of critical points whose corresponding critical values belong to $[c_0, c_\infty]$.

2. - Proof of the Theorem. Standard arguments in the calculus of variations show that the classical solutions of (0.2) correspond to the critical points of the functional

$$(2.1) \quad f(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T (M(t)u(t),u(t))dt + \int_0^T V(u(t),t)dt$$

defined on H. Clearly $f \in C^2(H,R)$ and $\forall u \in H$

$$(2.2) \quad \langle f'(u),h\rangle = \int_0^T (\dot{u},\dot{h})dt - \int_0^T (M(t)u,h)dt + \int_0^T (\nabla V(u,t),h)dt \qquad \forall h \in H$$

$$(2.3) \quad f''(u)[h,s] = \int_0^T (\dot{h},\dot{s})dt - \int_0^T (M(t)h,s)dt + \int_0^T (V_{xx}(u,t)h,s)dt \qquad \forall h,s \in H.$$

We denote by $\beta(t)$ the largest eigenvalue of $M(t)$ and by I_n the identity matrix in R^n, and we set

$$\beta = \sup_{[0,T]} \beta(t) \qquad M_1(t) = M(t) + I_n.$$

Let $a(u,v): H \times H \longrightarrow R$ be the bilinear form defined by

$$a(u,v) = \int_0^T [(\dot u,\dot v)+(u,v)]\,dt - \int_0^T (M_1(t)u,v)\,dt + $$
$$+ \beta\int_0^T (u,v)\,dt.$$

It is easy to verify that a is continuous and coercive (i.e. $a(u,u)\geq \text{const}\ \|u\|_H^2$) on H. Then by the Lax-Milgram theorem there exists a unique bounded linear operator $S:H \longrightarrow H$ with a bounded linear inverse S^{-1} such that

$$(Su,v)_H = a(u,v) \qquad \forall u,v \in H.$$

We set

$$\mathcal{D}(\mathcal{S}) = \{u \in H \mid Su \in L^2\}$$

and

$$\mathcal{S} = S_{|\mathcal{D}(\mathcal{S})}$$

$\mathcal{S}$ is a linear continuous self-adjoint operator with compact resolvent. Then $\sigma(\mathcal{S})$ consists of a positively divergent sequence of isolated eigenvalues with finite multiplicities. We denote by $s_0 < s_1 \ldots\ldots < s_j < \ldots.$ the eigenvalues of $\mathcal{S}$ and by $\lambda_0 < \lambda_1 \ldots. < \lambda_j < \ldots.$ the eigenvalues of $\mathcal{L}$.

Obviously $\mathcal{L} = \mathcal{S} - \beta i$, where $i:L^2 \longrightarrow L^2$ is the identity map, $\lambda_j = s_j - \beta \quad \forall_j$ and by (I_1) it follows that there exists k such that $\beta = s_k \in \sigma(\mathcal{S})$.

We denote by M_j the sequence of eigenspaces corresponding to the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_j, \ldots.$ If $m \geq o$ is an integer number we set

$H^-(m) = \bigoplus_{j \leq m} M_j$

$H^+(m)$ = closure in H of the linear space spanned by $\{M_j\}_{j \geq m}$

Clearly $H^+(m) \cap H^-(m) = M_m$ and $H = H^-(m) \oplus H^+(m+1)$. For every $u \in H$, we set

$$u = u^+ + u^- + u_o,$$

where $u^+ \in H^+(k+1)$, $u^- \in H^-(k-1)$, $u_o \in M_k$.

Lemma 2.1. - There exist $\eta, \tau, \nu > 0$ such that

(i) $(Su,u^+)_{L^2} - \beta \| u^+\|^2_{L^2} \geq \eta \| u^+\|^2_H \quad \forall u \in H$

(ii) $(Su,u^-)_{L^2} - \beta \| u^-\|^2_{L^2} \geq - \tau \| u^-\|^2_H \quad \forall u \in H$

(iii) $\beta \| u^-\|^2_{L^2} - (Su,u^-)_{L^2} \geq \nu \| u^-\|^2_H \quad \forall u \in H$

Proof. (i) $(Su,u^+)_{L^2} - \beta \| u^+\|^2_{L^2} = (Su^+,u^+)_{L^2} - \beta \| u^+\|^2_{L^2} = \sum_{j=k+1}^{\infty} (s_j - \beta) \| u_j\|^2_{L^2} = \sum_{j=k+1}^{\infty} \frac{s_j-\beta}{s_j} s_j \|u_j\|^2_{L^2} \geq \eta (Su^+,u^+)_{L^2} \geq \eta \| u^+\|^2_H$

(ii) $(Su,u^-)_{L^2} - \beta \|u^-\|^2_{L^2} = (Su^-,u^-)_{L^2} - \beta \| u^-\|^2_{L^2} = \sum_{j=0}^{k-1}(s_j - \beta) \| u_j\|^2_{L^2} \geq (s_o - \beta) \sum_{j=0}^{k-1} \| u_j\|^2_{L^2} = - \tau \|u^-\|^2_{L^2} \geq - \tau \|u^-\|^2_H$

(iii) $\beta \| u^-\|^2_{L^2} - (Su,u^-)_{L^2} = \beta \|u^-\|^2_{L^2} - (Su^-,u^-)_{L^2} = \sum_{j=0}^{k-1}(\beta - s_j) \|u_j\|^2_{L^2} = \sum_{j=0}^{k-1} \frac{\beta - s_j}{s_j} s_j \|u_j\|^2_{L^2} \geq \nu (Su^-,u^-)_{L^2} \geq \nu \| u^- \|^2_H.$

Lemma 2.2. - If $(I_1),(I_2)$ hold, the functional $f(u)$ defined by (2.1) satisfies the condition (I).

Proof. The proof is substantially analogous to the proof of Theorem (3.1) in [1]. It is only necessary to use the Lemma 2.1 and an obvious generalization of Lemma 3.2 in [1].

Lemma 2.3. - Suppose that $(I_1),(I_3),(I_4)$ hold, then there exist $\rho > o$ and $\gamma > o$ such that

$$f(u) \geq f(o) + \gamma \qquad \forall u \in H^+(h) \cap S_\rho .$$

Proof. We have $\forall u \in H$

$$(2.4) \quad f(u) = f(o) + \langle f'(o),u\rangle + f''(o) [u,u] + o(\|u\|_H^2).$$

By (2.2) and by (I_1) we have $\forall u \in H$

$$(2.5) \qquad \langle f'(o),u\rangle = o.$$

By (2.3), $(I_3),(I_4)$ we have $\forall u \in H^+(h)$

$$f''(o) [u,u] = (Su,u)_{L^2} - \beta \|u\|^2_{L^2} + \int_0^T (V_{xx}(o,t)u,u)dt =$$

$$= (Su^+,u^+)_{L^2} - \beta \|u^+\|^2_{L^2} + \int_0^T (V_{xx}(o,t)u,u)dt \geq$$

$$\geq \sum_{j=h}^{\infty} s_j \|u_j\|^2_{L^2} - (\beta - \mu) \|u^+\|^2_{L^2} =$$

$$= \sum_{j=h}^{\infty} (s_j - \beta + \mu) \|u_j\|^2_{L^2}.$$

There exist $t > h$ and $\delta > o$ such that

$$s_j - \beta + \mu > \delta s_j \qquad \forall j > t,$$

then

$$\sum_{j=h}^{\infty} (s_j - \beta + \mu) \|u_j\|^2_{L^2} = \sum_{j=h}^{t} \left(\frac{s_j - \beta + \mu}{s_j}\right) s_j \|u_j\|^2_{L^2} +$$

$$+ \sum_{j=t+1}^{\infty} (s_j - \beta + \mu) \|u_j\|^2_{L^2} \geq \text{const} \sum_{j=h}^{t} s_j \|u_j\|^2_{L^2} +$$

$$+ \sum_{j=t+1}^{\infty} \delta s_j \|u_j\|^2_{L} \geq \text{const} \sum_{j=h}^{\infty} s_j \|u_j\|^2_{L^2}.$$

Then we have

(2.6) $f''(o)\,[u,u] \geq \text{const}\ \|u^{+}\|^2_{L^2}$.

Finally, by (2.4),(2.5),(2.6) we have

$$f(u) \geq f(o) + \gamma \quad \text{with } \gamma > o.$$

Lemma 2.4. - There exists $\sigma > o$ such that

$$f(u) < \sigma \quad \forall u \in H^{-}(k).$$

Proof. Let

$$\lambda = \sup_{[o,T]} V(u,t),$$

then $\forall u \in H^{-}(k)$

$$f(u) = (Su^{-},u^{-})_{L^2} - \beta\|u^{-}\|^2_{L^2} + \int_0^T V(u,t)dt) \leq$$
$$\leq \sum_{j=0}^{K} (s_j - \beta)\ \|u_j\|^2_{L^2} + \lambda T \leq \lambda T.$$

Finally we can prove the Theorem (0.1).

Proof of Theorem 0.1. By Lemma 2.2, Lemma 2.3, Lemma 2.4 and by (I_5) we have that the functional f, defined by (2.1), satisfies (f_1),(f_2),(f_3) of the Theorem (1.1). Hence, the problem (0.2) possesses at least

$$m = \dim (M_h \oplus \dots \oplus M_k)$$

distinct pairs of nontrivial solutions.

Obviously

$$m = m_{\infty} - m_0.$$

3. - A particular case. We denote by $\mathcal{M}$ the self-adjoint realization in L^2 of the operator $x \longrightarrow -\ddot{x}$ with periodicity con-

ditions, and we consider the particular case

$$M(t) = \alpha_k I_n, \quad \alpha_k = (\frac{2k\pi}{T})^2, \quad k = o,1,......$$

$\alpha_k \in \sigma(\mathcal{M})$ and the problem (0.2) becomes

$$(3.1) \quad \begin{cases} -\ddot{x} - \alpha_k x + \nabla V(x,t) = o \\ x(o) = x(T) \\ \dot{x}(o) = x(T) \end{cases}$$

If we assume that

$(I_4)'$ there exist $\alpha_h \leq \alpha_k$ s.t. $\alpha_h - \alpha_k + \mu > o$

we have that, if $(I_1),(I_2),(I_3),(I_4)',(I_5)$ hold, then the problem (3.1) possesses at least

$$m = \dim H^-(k) - \operatorname{codim} H^+(h)$$

distinct pairs of nontrivial solutions.

If we assume $\alpha_k = o$, we obtain the case studied by Thews [5].

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Università di Bari, Istituto di Matematica Applicata, Via Re David 200, 70125- Bari, Italia
S.I.S.S.A., Strada Costiera, 11 - 34014 Trieste, Italia

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