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THE EPIS OF POS (Z)

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Abstract: In [4] and [5] J. Meseguer conjectured that in POS(ω) epis are exactly the dense maps. In [3] D. Lehmann and A. Pasztor gave an example of an epi which is not dense. This paper provides the exact characterization of all epis of POS(Z), for arbitrary Z.

Key words: Epimorphism, Z-complete poset.

Classification: 06A10, 18A20, 18B99, 68A05

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1. Introduction. For an arbitrary subset system Z (see [1]) let POS(Z) be the category of Z-complete posets (i.e. posets in which any Z-set has a sup) and Z-continuous maps (i.e. maps preserving the sups of Z-sets). If $X \subseteq P$ and $\underline{P} \in |POS(Z)|$ then cl(X) is the least subset Y of P which contains X and in which every Z-set has its sup (which has to exist in P) in Y. A map $f:\underline{P} \longrightarrow \underline{Q} \in Mor$ POS(Z) is dense if Q = cl (f(P). In [4] and [5] J. Meseguer conjectured that in POS(ω) epis are exactly the dense maps (and hence extremal monos coincide with full monos). In [3] D. Lehmann and myself gave a counterexample to this conjecture by constructing an epi which is not even dense. What makes it to be an epi?
In order to answer this question let us consider the following domain $\underline{D}$ of figure 1, described in Meseguer [4]. Then let $B := \{b_n : n \in \omega\}$. Meseguer proved that for any

$\varphi, \psi : \underline{D} \longrightarrow \underline{P} \in \mathrm{MorPOS}(\omega)$, if $\varphi \restriction B = \psi \restriction B$, then $\varphi(a_\infty) = \psi(a_\infty)$ (since e.g. $\varphi(a_\infty) \geq \varphi(b_n) = \psi(b_n) \geq \psi(a_n)$ $\forall n \in \omega$ and hence by $\psi(a_\infty) = \sup(\psi(a_n))_{n \in \omega}$ we get $\varphi(a_\infty) \geq \psi(a_\infty)$). But notice that $a_\infty \notin \mathrm{cl}\,(B)$.

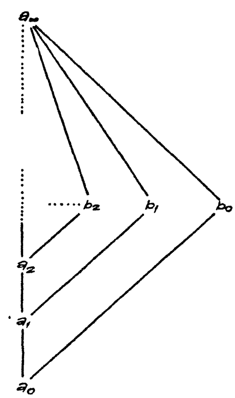


figure 1: domain $\underline{D}$

This leads to the counterexample, which is an embedding i of $B := \{(w,b) : w \in \omega^+\}$ into the domain $\underline{E}$ of figure 2. Since every element of E-B plays the role of the a_∞ of figure 1, we get that for any $\varphi, \psi : \underline{E} \longrightarrow \underline{P}$, $\varphi \restriction B = \psi \restriction B$ implies $\varphi = \psi$, which makes i to be an epi in POS(ω), although it is <u>not</u> dense!

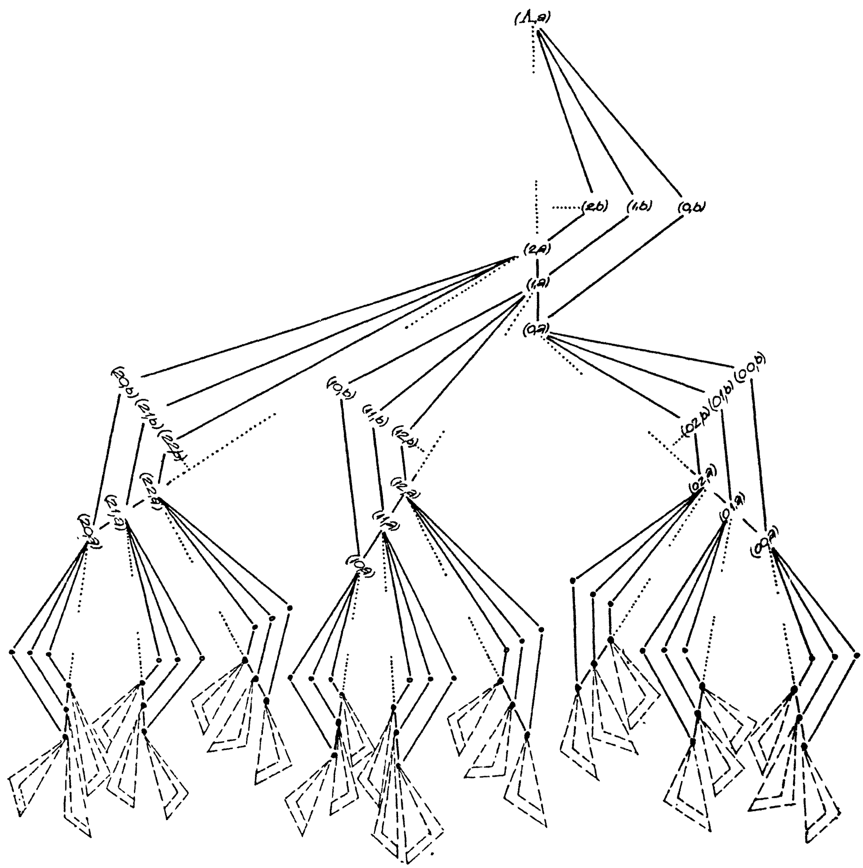


figure 2: domain **E**

My aim is to generalize the properties of i in order to get an exact characterization of all epis of POS(Z), for arbitrary subset system Z. Note throughout the paper that the presence of a bottom $\perp$ would not change any of the results. Thus every result holds for $POS_{\perp}(Z)$, too. Here I would like to thank J.Reiterman for helpful conversations on the topic of this paper.

2. A new closure operator

Notation: Ord denotes the class of all ordinals.

Definition 1: Let $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha \in$ Ord be arbitrary. Then "a is α-connected with b through X" - in symbols $a \overset{\alpha,X}{\longmapsto} b$ - if

1) For $\alpha = 0$

$\exists Y_b \subseteq P : b \in cl(Y_b)$ & $\forall y \in Y_b \quad \exists x_y \in X : a \geq x_y \geq y$, and

2) For $\alpha > 0$

$\exists Y_b \subseteq P : b \in cl(Y_b)$ & $\forall y \in Y_b \quad \exists b_y \in P \quad \exists \alpha_y \in$ Ord:

$y \leq b_y$ & $\alpha_y < \alpha$ & $a \overset{\alpha_y,X}{\longmapsto} b_y$.

Remark: Note that in the case of the counterexample in [3] for every $a \in (E \setminus B)$ $a \overset{0,B}{\longmapsto} a$ holds!

For the illustration of Definition 1 see the figure 3 at the end of this paragraph.

The following Lemmas 1-5 give some of the most important properties of the relation "to be connected through X" defined in Definition 1.

Lemma 1: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha, \beta \in$ Ord, if $\alpha < \beta$ then $a \overset{\alpha,X}{\longmapsto} b$ implies $a \overset{\beta,X}{\longmapsto} b$.

Proof: If $\beta = 0$ then evident.

Let $\beta > 0$. Then let $Y_b = \{b\}$. Since $b \in cl\,(Y_b)$ and $\forall y \in Y_b \quad \exists \beta_y = \alpha < \beta \quad \exists b_y = b \in P : b_y \geq y$ & $a \overset{\beta_y,X}{\longmapsto} b_y$, $a \overset{\beta,X}{\longmapsto} b$. □

Lemma 2: For any $\underline{P} \in |POS(Z)|$, $X, Y \subseteq P$, $\alpha \in$ Ord and $a, b \in P$, if $X \subseteq Y$ then $a \overset{\alpha,X}{\longmapsto} b$ implies $a \overset{\alpha,Y}{\longmapsto} b$.

Proof: by transfinite induction.

If $\alpha = 0$ then $a \overset{\alpha, X}{\longmapsto} b$ means $\exists Y_b \subseteq P: b \in cl\ (Y_b)$ & & $\forall y \in Y_b\ \ \exists x_y \in X \subseteq Y: a \geq x_y \geq y$. Hence $a \overset{\alpha, Y}{\longmapsto} b$.

Let $\alpha > 0$ and suppose that $\forall \beta < \alpha$ the Lemma holds. Then $a \overset{\alpha, X}{\longmapsto} b$ means $\exists Y_b \subseteq P: b \in cl\ (Y_b)$ & $\forall y \in Y_b\ \ \exists b_y \in P$ $\exists \alpha_y \in Ord: \alpha_y < \alpha$ & $b_y \geq y$ & $a \overset{\alpha_y, X}{\longmapsto} b_y$. Since by the induction hypothesis $\forall y \in Y_b$: $a \overset{\alpha_y, Y}{\longmapsto} b_y$, we get by Definition 1 $a \overset{\alpha, Y}{\longmapsto} b$. □

<u>Lemma 3</u>: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a, b, c \in P$ and $\alpha \in$ $\in$ Ord, if $a \geq b$ and $b \overset{\alpha, X}{\longmapsto} c$ then $a \overset{\alpha, X}{\longmapsto} c$.

<u>Proof</u>: by transfinite induction on α.

If $\alpha = 0$ then $b \overset{\alpha, X}{\longmapsto} c$ means that $\exists Y_c \subseteq P: c \in cl\ (Y_c)$ & $\forall y \in Y_c\ \ \exists x_y \in X: b \geq x_y \geq y$. Since $a \geq b$, $\forall y \in Y_c: a \geq x_y \geq y$, hence $a \overset{\alpha, X}{\longmapsto} c$.

Let $\alpha > 0$ and suppose that $\forall \beta < \alpha$ the Lemma holds. Then $b \overset{\alpha, X}{\longmapsto} c$ means that $\exists Y_c \subseteq P: c \in cl\ (Y_c)$ & $\forall y \in Y_c\ \ \exists \alpha_y \in Ord$ $\exists b_y \in P$: $\alpha_y < \alpha$ & $b_y \geq y$ & $b \overset{\alpha_y, X}{\longmapsto} b_y$. Then by induction hypothesis $\forall y \in Y_c: a \overset{\alpha_y, X}{\longmapsto} b_y$ hence by Definition 1 $a \overset{\alpha, X}{\longmapsto} c$. □

<u>Lemma 4</u>: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a, b, c \in P$ and $\alpha \in Ord$, if $a \overset{\alpha, X}{\longmapsto} b$ and $b \geq c$, then $a \overset{\alpha+1, X}{\longmapsto} c$.

<u>Proof</u>: Let $Y_c = \{c\}$. Then $c \in cl\ (Y_c)$ & $\forall y \in Y_c$ $\exists \alpha_y = \alpha \in Ord\ \ \ \exists b_y = b \in P$: $\alpha_y < \alpha + 1$ & $b_y \geq y$ & & $a \overset{\alpha_y, X}{\longmapsto} b_y$. Hence by Definition 1 $a \overset{\alpha+1, X}{\longmapsto} c$. □

<u>Lemma 5</u>: For any $\underline{P} \in |POS(Z)|$, for any $X \subseteq P$ and for any $a \in cl\ (X)$: $a \overset{0, X}{\longmapsto} a$.

<u>Proof</u>: Let $Y_a := \{x \in X: a \geq x\}$. Then $a \in cl\ (Y_a)$ & & $\forall y \in Y_a\ \ \exists x_y (=y) \in X: a \geq x_y \geq y$. Hence by Definition 1

$a \overset{0,X}{\longmapsto} a.$ □

Remember that our aim essentially is to find the co-congruence relation of maps of POS(Z). In the following we will see how the relation "To be connected through X" leads to this aim.

Proposition 1: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a,b \in P$, $\alpha \in$ Ord and $\varphi, \psi : \underline{P} \longrightarrow \underline{Q} \in$ Mor POS(Z), if $a \overset{\alpha,X}{\longmapsto} b$ and $\varphi \upharpoonright X = \psi \upharpoonright X$, then $\varphi(a) \geq \psi(b)$ and $\psi(a) \geq \varphi(b)$.

Proof: by transfinite induction on α.

If $\alpha = 0$ then $a \overset{\alpha,X}{\longmapsto} b$ means $\exists Y_b \subseteq P : b \in cl\ (Y_b)$ & & $\forall y \in Y_b \quad \exists x_y \in X : a \geq x_y \geq y$. Then $\forall y \in Y_b \quad \varphi(a) \geq \varphi(x_y) = \psi(x_y) \geq \psi(y)$ resp. $\psi(a) \geq \psi(x_y) = \varphi(x_y) \geq \varphi(y)$. Since $b \in cl\ (Y_b)$ implies $\psi(b) \in cl\ (\psi(Y_b))$ resp. $\varphi(b) \in cl\ (\varphi(Y_b))$ (since φ and ψ are Z-continuous), we get $\varphi(a) \geq \psi(b)$ resp. $\psi(a) \geq \varphi(b)$.

Let $\alpha > 0$ and suppose that $\forall \beta < \alpha$ the proposition is true. Then $a \overset{\alpha,X}{\longmapsto} b$ means that $\exists Y_b \subseteq P : b \in cl\ (Y_b)$ & $\forall y \in Y_b$ $\exists \alpha_y \in Ord \quad \exists b_y \in P \quad \alpha_y < \alpha$ & $b_y \geq y$ & $a \overset{\alpha_y,X}{\longmapsto} b_y$. By the induction hypothesis

$$\forall y \in Y_b : \varphi(a) \geq \psi(b_y) \text{ resp. } \psi(a) \geq \varphi(b_y).$$

Then $\forall y \in Y_b : \varphi(a) \geq \psi(b_y) \geq \psi(y)$ resp. $\psi(a) \geq \varphi(b_y) \geq \varphi(y)$ and since $\psi(b) \in cl\ (\psi(Y_b))$ resp. $\varphi(b) \in cl\ (\varphi(Y_b))$, we get

$$\varphi(a) \geq \psi(b) \text{ resp. } \psi(a) \geq \varphi(b).$$ □

Corollary 1: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a,b \in P$ and $\alpha \in$ Ord, $a \overset{\alpha,X}{\longmapsto} b$ implies $a \geq b$.

Proof: Let $\varphi = \psi = id_P$. □

Corollary 2: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a \in P$, $\alpha \in Ord$ and $\varphi, \psi : \underline{P} \longrightarrow \underline{Q} \in Mor\ POS(Z)$, if $a \overset{\alpha,X}{\longmapsto} a$ and $\varphi \upharpoonright X = \psi \upharpoonright X$ then $\varphi(a) = \psi(a)$!

Definition 2: Let $\underline{P} \in |POS(Z)|$ and $X \subseteq P$ be arbitrary. Then $CL(X) := \{a \in P \mid \exists \alpha \in Ord: a \overset{\alpha,X}{\longmapsto} a\}$.

Having arrived at this point we know that CL(X) is contained in the co-congruence relation of any map of POS(Z) with image X (see Cor. 3 of Prop. 2).
Now let us prove some properties of CL(X).

Lemma 6: For any $\underline{P} \in |POS(Z)|$ and $X \subseteq P$, $cl(X) \subseteq CL(X)$.

Proof: by Lemma 5. □

Corollary: For any $\underline{P} \in |POS(Z)|$ and $X \subseteq P$, $X \subseteq CL(X)$.

Proposition 2: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$, $a, b \in P$ and $\alpha \in Ord$, $a \overset{\alpha,CL(X)}{\longmapsto} b \Longrightarrow \exists \beta \in Ord: a \overset{\beta,X}{\longmapsto} b$.

Proof: by transfinite induction on α.
Let $\alpha = 0$. Then $b \in cl\ (Y_b)$ & $\forall y \in Y_b \quad \exists x_y \in CL(X): a \geq x_y \geq y$. Since $x_y \in CL(X) \quad \exists \alpha_y \in Ord: x_y \overset{\alpha_y,X}{\longmapsto} x_y$. Then by Lemma 3 $a \overset{\alpha_y,X}{\longmapsto} x_y$. Now $b \in cl\ (Y_b)$ & $\forall y \in Y_b \quad \exists x_y \in P \quad \exists \alpha_y \in Ord$: $\alpha_y < \beta := \sum_{y \in Y_b} (\alpha_y + 1)$ & $x_y \geq y$ & $a \overset{\alpha_y,X}{\longmapsto} x_y$. Then by Definition 1 $a \overset{\beta,X}{\longmapsto} b$.

Let $\alpha > 0$ and suppose that for any $\delta < \alpha$ the proposition holds. Then $a \overset{\alpha,CL(X)}{\longmapsto} b$ means that $\exists Y_b \subseteq P: b \in cl\ (Y_b)$ & & $\forall y \in Y_b \quad \exists b_y \in P \quad \exists \alpha_y \in Ord: \alpha_y < \alpha$ & $b_y \geq y$ & $a \overset{\alpha_y,CL(X)}{\longmapsto} b_y$ but then by induction hypothesis $\forall y \in Y_b \quad \exists \beta_y \in Ord$: $: a \overset{\beta_y,X}{\longmapsto} b_y$.

Then by Definition 1 for $\beta := \sum_{y \in Y_b} (\beta_y + 1)$ $a \overset{\beta, X}{\longmapsto} b$. □

<u>Corollary 1</u>: For any $\underline{P} \in |POS(Z)|$ and $X \subseteq P$: $CL(CL(X)) = CL(X)$.

<u>Proof</u>: by Corollary of Lemma 6 and by Proposition 2 for $a = b$. □

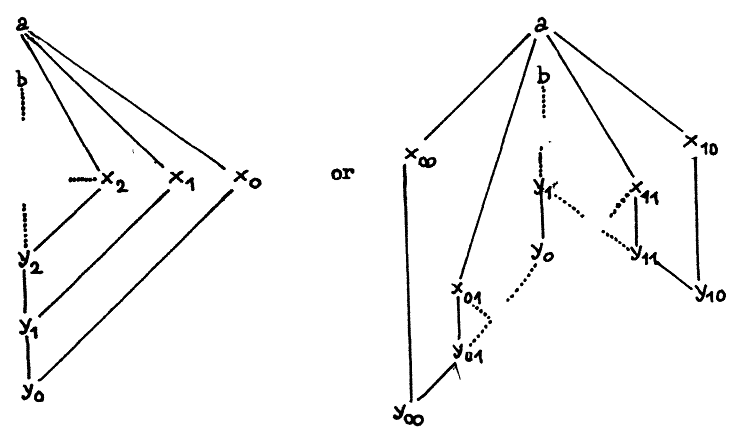
<u>Corollary 2</u>: For any $\underline{P} \in |POS(Z)|$ the operator $CL: \mathcal{P}(P) \longrightarrow \mathcal{P}(P)$, which assigns to each $X \subseteq P$ $CL(X)$, is a closure operator.

<u>Proof</u>: By Corollary of Lemma 6 $X \subseteq CL(X)$, by Lemma 2 if $X \subseteq Y$ then $CL(X) \subseteq CL(Y)$ and by the above Corollary 1 $CL(CL(X)) = CL(X)$. □

<u>Corollary 3</u>: For any $\underline{P} \in |POS(Z)|$, $X \subseteq P$ and $\varphi, \psi : \underline{P} \longrightarrow \underline{Q} \in Mor\ POS(Z)$, if $\varphi \restriction X = \psi \restriction X$ then $\varphi \restriction CL(X) = \psi \restriction CL(X)$.

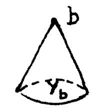
<u>Proof</u>: by Corollary 2 of Proposition 1. □

For $\alpha = 0$ and for $Z = \omega$

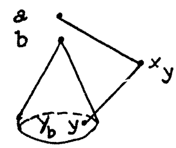


etc.

Further on we shall symbolize $b \in cl\ (Y_b)$ by



i.e. for $\alpha = 0$ the figure looks like that:



For $\alpha > 0$ we illustrate then Definition 1 by:

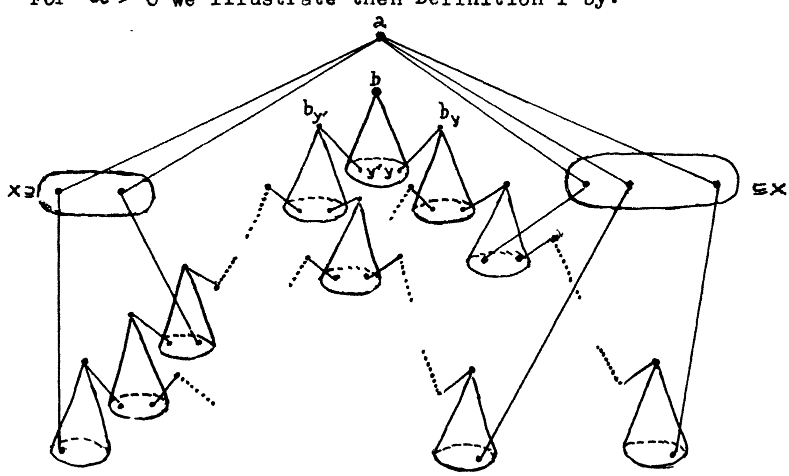


figure 3: Illustration of Definition 1.

3. <u>The epis of POS(Z)</u>. Now we are ready to give the exact characterization of epis in POS(Z):

<u>Theorem</u>: Any $f:\underline{P} \longrightarrow \underline{Q} \in$ Mor POS(Z) is an epi iff $CL(f(P)) = Q$.

<u>Proof</u>: If $CL(f(P)) = Q$ then f epi follows immediately from Corollary 3 of Proposition 2.

Let $f:\underline{P} \longrightarrow \underline{Q} \in$ Mor POS(Z) be an epi in POS(Z). Denote

$CL(f(P)) = Q_0$ and suppose $Q - Q_0 \neq \emptyset$. Then we construct $\hat{\varphi}, \hat{\psi}: \underline{Q} \longrightarrow \underline{\hat{R}} \in \text{Mor POS}(Z)$ with $\hat{\varphi} \cdot f = \hat{\psi} \cdot f$ but $\hat{\varphi} \neq \hat{\psi}$ which contradicts the epiness of f. First we define the maps φ and ψ, namely $\varphi := id_Q$ and $\psi := id_{Q_0} \dot{\cup} \wp$, where $\wp: Q - Q_0 \hookrightarrow\!\!\!\!\twoheadrightarrow im\,\wp$ is a bijection ($Q - Q_0 \neq \emptyset$) with $im\,\wp \cap (Q - Q_0) = \emptyset$. Let the set $R := Q \dot{\cup} im\,\wp$.

We are going to define on R the relation $\leq_R$: Let $a, b \in R$, then

$$a \leq_R b \text{ iff } \begin{cases} A\colon a \leq_Q b \text{ if } a, b \in Q, \\ B\colon \wp^{-1}(a) \leq_Q \wp^{-1}(b) \text{ if } a, b \in im\,\wp, \\ C\colon \wp^{-1}(b) \overset{\alpha,\, Q_0}{\longmapsto} a \text{ for an } \alpha \in \text{Ord if } a \in Q, b \in im\,\wp, \\ D\colon b \overset{\alpha,\, Q_0}{\longmapsto} \wp^{-1}(a) \text{ for an } \alpha \in \text{Ord if } a \in im\,\wp,\ b \in Q. \end{cases}$$

In the following we shall prove that $\leq_R$ is a partial order.

1) reflexive: $\forall a \in R: a \in Q \Longrightarrow a \leq_Q a \underset{A}{\Longrightarrow} a \leq_R a$ and $a \in im\,\wp \Longrightarrow \wp^{-1}(a) \leq_Q \wp^{-1}(a) \underset{B}{\Longrightarrow} a \leq_R a$.

2) antisymmetric: let $a \leq_R b$ and $b \leq_R a$. Then

a) $a, b \in Q \underset{A}{\Longrightarrow} a = b$

b) $a, b \in im\,\wp \underset{B}{\Longrightarrow} \wp^{-1}(a) = \wp^{-1}(b) \Longrightarrow a = b$

c) $a \in Q, b \in im\,\wp \underset{C,D}{\Longrightarrow} \wp^{-1}(b) \overset{\alpha_a,\, Q_0}{\longmapsto} a$ and $a \overset{\alpha_b,\, Q_0}{\longmapsto} \wp^{-1}(b) \underset{\text{Cor. 1 of Prop. 1}}{\Longrightarrow} \wp^{-1}(b) \;_Q\!\geq a$ and $a \;_Q\!\geq \wp^{-1}(b) \Longrightarrow$
$\Longrightarrow \wp^{-1}(b) = a$

$\underset{\text{Def. of } \wp}{\Longrightarrow} \underline{a \in Q - Q_0}$. But $a = \wp^{-1}(b) \overset{\alpha_a,\, Q_0}{\longmapsto} a$ implies $\underline{a \in CL(Q_0) = Q_0}$ which is a contradiction. This means:

$a \in Q, b \in im\,\wp \Longrightarrow \neg(a \leq_R b \;\&\; b \leq_R a)$.

d) For $a \in im\,\wp$, $b \in Q$ we get the same.

3) transitive: let $a \leq_R b \leq_R c$. Then:

a) $a, b, c \in Q \underset{A}{\Longrightarrow} a \leq_R c$.

b) $a,b,c \in \operatorname{im}\varphi \underset{B}{\Longrightarrow} a \leq_R c$

c) $a \in Q, bc \in \operatorname{im}\varphi \underset{C,B}{\Longrightarrow} \varphi^{-1}(b) \overset{\alpha,\, Q_0}{\longmapsto} a$ and

$\varphi^{-1}(b) \leq_Q \varphi^{-1}(c) \underset{Lemma\,3}{\Longrightarrow} \varphi^{-1}(c) \overset{\alpha,\, Q_0}{\longmapsto} a \underset{C}{\Longrightarrow} a \leq_R c$.

d) For $a \in \operatorname{im}\varphi, b,c \in Q$ dually by D and A.

e) $a,c \in \operatorname{im}\varphi, b \in Q \underset{D,C}{\Longrightarrow} b \overset{\alpha_a,\, Q_0}{\longmapsto} \varphi^{-1}(a)$ and

$\varphi^{-1}(c) \overset{\alpha_b,\, Q_0}{\longmapsto} b \underset{\text{Cor. 1 of Prop. 1}}{\Longrightarrow} \varphi^{-1}(a) \leq_Q b \leq_Q \varphi^{-1}(c) \underset{B}{\Longrightarrow} a \leq_R c$.

f) For $a,c \in Q$, $b \in \operatorname{im}\varphi$ dually by C, D and A.

g) $a,b, \in Q$, $c \in \operatorname{im}\varphi \underset{A,C}{\Longrightarrow} a \leq_Q b$ and

$\varphi^{-1}(c) \overset{\alpha,\, Q_0}{\longmapsto} b \underset{Lemma\,4}{\Longrightarrow} \varphi^{-1}(c) \overset{\alpha+1,\, Q_0}{\longmapsto} a \underset{C}{\Longrightarrow} a \leq_R c$.

h) For $a,b \in \operatorname{im}\varphi$, $c \in Q$ dually by B and D.

Next we shall prove that φ and ψ are Z-continuous. For this let Y be an arbitrary Z-set in Q with $a = \sup_{\leq_R} Y$. First we show that $\varphi(a) = \sup_{\leq_R} \varphi(Y)$. By A (of the definition of $\leq_R$) it is clear that $a \,{}_R\!\geq y \;\; \forall y \in Y$ and that $\forall b \in Q$, if $y \leq_R b \;\; \forall y \in Y$, then also $a \leq_R b$ holds. Let $b \in \operatorname{im}\varphi$ with $y \leq_R b \;\; \forall y \in Y$. Then by C $\varphi^{-1}(b) \overset{\alpha_y,\, Q_0}{\longmapsto} y \;\; \forall y \in Y$, which by Definition 1 implies $\varphi^{-1}(b) \overset{\alpha,\, Q_0}{\longmapsto} a$, where $\alpha := \sum_{y \in Y} (\alpha_y + 1)$. Again by C we get $a \leq_R b$.
Now we are going to prove that $\psi(a) = \sup_{\leq_R} \psi(Y)$.

1. Suppose $a \in Q_0$ (and hence $\psi(a)=a$). By A $y \leq_R a$ $\forall y \in Y \cap Q_0$ and by D $\varphi(y) \leq_R a \;\; \forall y \in Y \cap (Q-Q_0)$ since for these y-s $a \overset{0,\, Q_0}{\longmapsto} y$. Hence $z \leq_R a \;\; \forall z \in \psi(Y)$. Let $z \leq_R b \;\; \forall z \in \psi(Y)$.

a) if $b \in Q$ then by A $z \leq_Q b \;\; \forall z \in \psi(Y) \cap Q_0$ and by D and Corollary 1 of Proposition 1 $\varphi^{-1}(z) \leq_Q b \;\; \forall z \in \psi(Y) \cap \operatorname{im}\varphi$. This means $y \leq_Q b \;\; \forall y \in Y$, hence $a \leq_Q b$ and hence by A $a \leq_R b$.

b) Suppose $b \in \operatorname{im}\rho$. Then by B $\rho^{-1}(z) \leq_Q \rho^{-1}(b)$ $\forall z \in \psi(Y) \cap \operatorname{im}\rho$ and by C and Corollary 1 of Proposition 1 $z \leq_Q \rho^{-1}(b)$ $\forall z \in \psi(Y) \cap Q_0$, i.e. $z \leq_Q \rho^{-1}(b)$ $\forall y \in Y$. This implies $a \leq_Q \rho^{-1}(b)$. Since $a \in Q_0$ we can write $\rho^{-1}(b) \overset{0,\, Q_0}{\longmapsto} a$, hence by C $a \leq_R b$.

2. Let $a \in Q - Q_0$ (and hence $\psi(a) = \rho(a)$). By B $z \leq_R \rho(a)$ $\forall z \in \psi(Y) \cap \operatorname{im}\rho$ and by C also $z \leq_R \rho(a)$ $\forall z \in \psi(Y) \cap$ $\cap Q_0$ since for these z-s $a \overset{0,\, Q_0}{\longmapsto} z$. Now let $z \leq_R b$ $\forall z \in \psi(Y)$

a) If $b \in Q$ then by D $b \overset{\alpha_z,\, Q_0}{\longmapsto} \rho^{-1}(z)$ $\forall z \in \psi(Y) \cap$ $\cap \operatorname{im}\rho$ and by A and the above remark $b \overset{0,\, Q_0}{\longmapsto} z$ $\forall Az \in \psi(Y) \cap$ $\cap Q_0$. Applying Definition 1 we get $b \overset{\alpha,\, Q_0}{\longmapsto} a$, where $\alpha := \sum_{z \in \psi(Y) \cap \operatorname{im}\rho} (\alpha_z + 1)$, which by D means $\rho(a) \leq_R b$.

b) If $b \in \operatorname{im}\rho$ then by B $\rho^{-1}(z) \leq_Q \rho^{-1}(b)$ $\forall z \in \psi(Y) \cap \operatorname{im}\rho$ and by C and Corollary 1 of Proposition 1 $y \leq_Q \rho^{-1}(b)$ $\forall y \in Y \cap Q_0$, i.e. $y \leq_Q \rho^{-1}(b)$ $\forall y \in Y$, hence $a \leq_Q \rho^{-1}(b)$. By B this means $\rho(a) \leq_R b$.

By Banaschewski-Nelson [2] or Meseguer [6] POS(Z) is (full-mono)-reflective in ZPOS - the category of posets and Z-continuous maps. Let η_R denote the (Z-continuous) POS(Z)-reflection of $\underline{R} := (R, \leq_R)$ and let $\hat{\psi} := \eta_{\underline{R}} \cdot \psi$ and $\hat{\varphi} := \eta_R \cdot \varphi$. Then $\hat{\psi} \neq \hat{\varphi}$ since $\varphi \neq \psi$ and $\hat{\psi} \cdot f = \hat{\varphi} \cdot f$ since $\psi \cdot f = \varphi \cdot f$. □

4. Some consequences

Corollary 1: An $m: \underline{P} \longrightarrow \underline{Q} \in \operatorname{Mor} POS(Z)$ is an extremal mono iff it is full (i.e. $m(a) \leq_Q m(b)$ iff $a \leq_P b$ $\forall a, b \in P$) and $CL(m(P)) = m(P)$.

Proof: 1) Let m be an extremal mono and let $m=f\cdot e$, with $e:\underline{P}\longrightarrow\underline{R}$, $R:=CL(m(P))$, $\preceq_R:=\preceq_Q\cap R^2$ and $e:=m$ and $f:\underline{R}\longrightarrow\underline{Q}$, $f:=id_R$. Since by Lemma 6 $cl\,(R) = cl\,(CL(m(P))) = CL(m(P)) = R$, $\underline{R}\in|POS(Z)|$, thus $e,f\in Mor\ POS(Z)$. By the Theorem e is an epi, hence e is an isomorphism. This implies $m(P)=e(P)=R=CL(m(P))$ and $m^2\,(\preceq_P) = e^2\,(\preceq_P) = \preceq_R = \preceq_Q\cap R^2 = \preceq_Q \cap (m(P))^2$.

2) Let m be full and let $CL(m(P)) = m(P)$. Suppose $m = f\cdot e$ with $e:\underline{P}\longrightarrow\underline{R}$, $f:\underline{R}\longrightarrow\underline{Q}\in Mor\ POS(Z)$ and e epi. We have to show that e is an isomorphism.

By the Theorem we know that $CL(e(P)) = R$. Then $f(R) = f(CL(e(P)))$. By the Corollary of the following Lemma 7

$$f(CL(e(P)))\subseteq CL(f(e(P))) = CL(m(P)) = m(P).$$

Before going on let us prove the

Lemma 7: For any $f:\underline{P}\longrightarrow\underline{Q}\in Mor\ POS(Z)$, $X\subseteq P$, $\alpha\in Ord$ and $a,b\in P$, if $a\ \vdash\!\frac{\alpha,\,X}{}\ b$ then $f(a)\ \vdash\!\frac{\alpha,\,f(X)}{}\ f(b)$.

Proof: By transfinite induction.

For $\alpha=0$ $\exists Y_b\subseteq P: b\in cl\,(Y_b)$ & $\forall y\in Y_b\ \exists x_y\in X: a\geq x_y\geq y$ hence $f(a)\geq f(x_y)\geq f(y)$. Since $f(b)\in cl\,(f(Y_b))$ we get by Definition 1 $f(a)\ \vdash\!\frac{0,\,f(X)}{}\ f(b)$.

Let $\alpha>0$ and suppose that for any $\beta<\alpha$ the Lemma holds. Then $a\ \vdash\!\frac{\alpha,\,X}{}\ b$ means that $\exists Y_b\subseteq P: b\in cl\,(Y_b)$ & $\forall y\in Y_b\ \exists b_y\in P$ $\exists\alpha_y\in Ord: \alpha_y<\alpha$ & $b_y\geq y$ & $a\ \vdash\!\frac{\alpha_y,\,X}{}\ b_y$ hence $f(b_y)\geq f(y)$ & $f(a)\ \vdash\!\frac{\alpha_y,\,f(X)}{}\ f(b_y)$. Again, since $f(b)\in cl\,(f(Y_b))$, we then get $f(a)\ \vdash\!\frac{\alpha,\,f(X)}{}\ f(b)$.

☐ Lemma 7.

Corollary: For any $f: \underline{P} \longrightarrow \underline{Q} \in \text{Mor POS}(Z)$ and $X \subseteq P$, $f(CL(X)) \subseteq CL(f(X))$.

Let us continue the proof. We have got that $f(CL(e(P))) \subseteq m(P)$. Then if we knew that $e \cdot m^{-1} \cdot f \in \text{Mor POS}(Z)$ then we would get that $CL(e(P)) = e(P)$, i.e. that e is surjective, since $e \cdot m^{-1} \cdot f = id_R$ (since e is epi and $e \cdot m^{-1} \cdot f \cdot e = e \cdot m^{-1} \cdot m = e \cdot id_P = id_R \cdot e$) and so $\forall a \in CL(e(P))$ $e(m^{-1}(f(a))) = a$, i.e. $a \in e(P)$. But even $m^{-1} \cdot f \in \text{Mor POS}(Z)$ since for any Z-set $A \subseteq R$ with $a = \sup(A)$, $f(A)$ is a Z-set and $f(a) = \sup_{\leq Q} f(A)$ and since $f(R) \subseteq m(P)$ and m is full $m^{-1}(f(A))$ is a Z-set in P, so it must have a supremum and this is $m^{-1}(f(a))$. Since $\forall a,b \in P$ $e(a) \leq e(b) \Longrightarrow f(e(a)) \leq f(e(b)) \Longrightarrow m(a) \leq m(b) \Longrightarrow a \leq b$ e is also full. Thus e is full and surjective, i.e. an isomorphism and hence m is an extremal mono.

☐ Corollary 1.

Corollary 2: POS(Z) is co-(well-powered).

Proof: It is enough to prove the following

Lemma 8: For any $\underline{P} \in |\text{POS}(Z)|$ and $X \subseteq P$, $CL(X) \subseteq \overline{X}$, where $\overline{X}$ is the join-closure of X in P, i.e. $\overline{X} = \{\sup S : S \subseteq X\}$.

Proof: We are going to prove by transfinite induction on α that $a \overset{\alpha, X}{\longmapsto} a$ implies $a = \sup X_a$, where $X_a = \{x \in X : a \geq x\}$. Therefore let $b \geq x$ $\forall x \in X_a$ for some $b \in P$. If $\alpha = 0$ then $\exists Y_a \subseteq P : a \in cl(Y_a)$ & $\forall y \in Y_a$ $\exists x_y \in X : a \geq x_y \geq y$, which immediately implies $b \geq a$.

Now suppose $\alpha > 0$ and that for any $\beta < \alpha$ the statement holds. Then $\exists Y_a \subseteq P : a \in cl(Y_a)$ & $\forall y \in Y_a$ $\exists \alpha_y < \alpha$ $\exists b_y \in P : b_y \geq y$ & & $a \overset{\alpha_y, X}{\longmapsto} b_y$. Since by Cor. 1 of Prop. 1 $\forall y \in Y_a$ $X_{b_y} \subseteq X_a$, by induction hypothesis we get that $\forall y \in Y_a$ $b \geq b_y$. Then $b_y \geq y$ and $a \in cl(Y_a)$ imply $b \geq a$.

□ Lemma 8 and Corollary 2

R e f e r e n c e s

[1] J. ADÁMEK, E. NELSON, J. REITERMAN: Tree constructions of free continuous algebras, preprint (1980).

[2] B. BANASCHEWSKI, E. NELSON: Completions of partially ordered sets as reflection, Comp. Sci. Tech. Rep. No. 79-CS-6, Mc Master University, Hamilton, Ontario (1979).

[3] D. LEHMANN, A. PASZTOR: On a conjecture of Meseguer, Technical report No.170, Technion, Dpt. of Comp. Sci., Haifa, Israel (1980), to appear in TCS.

[4] J. MESEGUER: Completions, Factorizations and Colimits for ω-Posets, Colloquia Mathematica Societatis Janos Bolyai (in press).

[5] J. MESEGUER: Varieties of chain-complete algebras, Journal of Pure and Applied Algebra 19(1980), 347-383.

[6] J. MESEGUER: Order completion Monads, to appear.

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