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ON COVERINGS OF RANDOM GRAPHS

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Abstract: It is shown that almost all graphs have the property that almost all edges can be covered by edge disjoint triangles. Various generalizations of this statement are considered.

Key words: Random graph, covering.

Classification: 05C99

Many papers have dealt recently with the problem of decomposing a graph into isomorphic subgraphs. In this note we investigate related questions concerning random graphs. Let n be a positive integer; is it true that the majority of graphs with n vertices can be decomposed into edge disjoint triangles (or more generally into edge disjoint copies of a given graph F) so that only relatively few edges are left?

We prove, provided n is sufficiently large that it is so. (For the more detailed definitions concerning random graphs see [2].)

Theorem. Let ε be a positive, $\varepsilon < 1$ and $G = (V, \mathcal{E})$ a random graph with n vertices, such that each edge is present with the prescribed probability p, independently of the presence or absence of any other edges. Then, with probability tending to one (as $n \to \infty$) there exists a system $T(\mathcal{G})$ of edge disjoint triangles in $\mathcal{G}$ so that all but at most εn^2 edges are covered by some triangle from $T(\mathcal{G})$.

Proof: A) We can clearly suppose without loss of generality that $n = 6m + 1$ or $6m+3$. Let $K = K_n$ be a complete graph with the vertex set V. From the existence of Steiner triple systems with n vertices ($n \equiv 1$ or $3 \pmod 6$) it immediately follows that there exists a covering C_0 of the edges of complete graph $K = K_n$ by edge disjoint triangles. Let $\pi_1, \pi_2, \dots, \pi_N$ be independent random permutations of the vertices in V, N will be chosen later. We assume that these permutations are also independent of the random graph $\mathcal{G}$. (In other words, we work on a product space $\{0,1\}^{\binom{n}{2}} \times \pi^N$ with the product measure $P = P_e^{\binom{n}{2}} \times \mu^N$) where π is the set of all permutations of $\{1,\dots,n\}$ each one having μ-measure $1/_{n!}$, and $P(1) = p$, $P(0) = 1-p$.) We define the independent coverings $C_1,\dots,C_N$ as follows: a triangle $\{v_1,v_2,v_3\}$ belongs to C_i if $\{\pi_i v_1, \pi_i v_2, \pi_i v_3\}$ belongs to C_0.

Now our algorithm goes as follows. Select all triangles in G that appear in C_1, then all triangles appearing in C_2 that are edge disjoint from the ones selected before, etc. This way we cover some portion of the edges of $\mathcal{G}$ by edge disjoint triangles, and hopefully a large portion.

Define the indicator variables

$$\chi_e = \begin{cases} 1 & \text{if } e \in \mathcal{E}\text{, nevertheless e has not been covered in our procedure} \\ 0 & \text{otherwise} \end{cases}$$

and set $d = \mathbf{E}\chi_e$, where E denotes the expectation of random variable χ_e. ($E\chi_e$ does not depend on e because of complete symmetry.) For the number D of edges not covered we have

$$D = \sum_{e \in \binom{V}{2}} \chi_e \quad (\binom{V}{2} \text{ is a set of all pairs of V}), \quad ED = \binom{n}{2} d.$$

Now

$$P(D > \varepsilon\binom{n}{2} p) \le ED/_{\varepsilon\binom{n}{2} p} = d/_{\varepsilon p}$$

and

$$P(|\mathcal{C}| < \tfrac{1}{2}\binom{n}{2} p) = o(1)$$

(if only $\binom{n}{2}p \to \infty$), thus in order to show that $D/_{|\mathcal{C}|} \to 0$ it is sufficient to show that $d/_p \to 0$.

B) Define the numbers p_i recursively as follows

(1)
$$p_o = 0$$
$$p_{k+1} = p_k + (p - p_k)^3$$

Taking $d_k = p-p_k$ we have thus $d_o = p$, $d_{k+1} = d_k - d_k^3$.
It is easy to see that $d_k \to 0$ (actually $d_k \sim 1/\sqrt{2k}$). Moreover, since d_k is decreasing we have $0 < d_k < p-kd_k^3$ whence

(2) $$d_k < (p/_k)^{1/3}, \quad k=1,2,\ldots .$$

Now we are going to prove

(3) $$d < d_N + 9^N/_n$$

and thus $d/_p \to 0$ if only $9^N/_{np} \to 0$ and $Np^2 \to \infty$ which holds if $p\sqrt{\log n} \to \infty$ (choose $N = \frac{1}{10}\log n$).

C) Consider an edge e. Let $T_k = T_k(e)$ denote the triangle in C_k that cover e. Start with $T_N(e)$.
In C_{N-1} there are three triangles (not necessarily different) containing the edges of $T_N(e)$. In C_{N-2} there are nine triangles containing the nine edges that appeared so far, etc.

Let A = A(e) denote the event that the $3 + 3^2 + \ldots + 3^N =$

$= \frac{3}{2}(3^n - 1)$ edges thus appearing are all different, and $B_k =$ $= B_k(e)$ the event that the edge e is covered up to the K-th step of our procedure $(k=1,\dots,N)$.

We fix the covering $C_1,\dots,C_N$ in such a way that A holds, and randomize $\mathcal{G}$. Define the conditional probability

$$P_k = P(B_k | C_1,\dots,C_N)$$

for these fixed coverings.

For the probability $P_{k+1} - P_k$ that e gets covered in exactly the (k+1)-th step, we obviously have

$$P_{k+1} - P_k = (p - P_k)^3, \quad P_1 = p^3$$

since the three edges of $T_k(e)$ have to be drawn in $\mathcal{G}$ and should not have been covered earlier (this explains $p - P_k$), moreover, these three events are independent, for we fixed the C-s in A(e).

Thus P_k, and also their mixture $P(B_k | A)$ satisfy (1), and hence are equal to p_k.

We have

$$d = p - P(B_N) = p - P(B_N | A)P(A) - P(B_N | \bar{A})P(\bar{A}) \leq p - p_N P(A) \leq$$
$$\leq p - p_N + P(\bar{A}) = d_N + P(\bar{A}).$$

Now

$$P(\bar{A}) < \sum_{k=1}^{N-1} 2.9^k/_n < 9^N/_n$$

for up to the k-th step (backwards) in the above argument C) we have 3^k edges altogether, and the probability that the corresponding random 3^k points (one step back) are all different from the $(3^k + 3)/_2$ points obtained so far, is less than $2.9^k/_n$. Q.E.D.

Remark. Here we outline that in our theorem triangle can be replaced by any other graph F. Consider a graph (configuration of edges) F which K_n can be covered by. An important result of R.M. Wilson [1] shows that the trivial necessary conditions for n are also "asymptotically sufficient" and hence K_n can be covered by edge disjoint copies of F for all sufficiently large n satisfying the necessary conditions.

If F contains r edges rather than three, then we have to change (1) to

$$p_{k+1} = p_k + (p - p_k)^r, \; p_o = 0$$

which leads to

$$d_k = p - p_k \sim ((r - 1)k)^{1/r-1}$$

and also (3) to

$$d < d_N + r^{2N}/n$$

which leads to the condition

$$p(\log n/\log r)^{1/r-1} \to \infty .$$

Thus, with p = const (say 1/2), the procedure works for covering with subgraphs with o(log log n) edges, e.g. for o($\sqrt{\log \log}$ n)-gons.

For fixed r we have seen that the procedure works as long as

$$p(\log n)^{1/r-1} \to \infty$$

i.e. as long as the number of edges is much larger than

$$n^2/(\log n)^{1/r-1} .$$

For triangles this is

$$n^2/\sqrt{\log n}\ .$$

A good guess is, however, that even a random graph with

$$\omega(n)n^{3/2},\ \omega(n)\longrightarrow\infty$$

edges can be covered almost perfectly. This would be a strong statement and is completely beyond the power of our method. x)

R e f e r e n c e s

[1] R.M. WILSON: Decompositions of complete graphs into subgraphs isomorphic to a given graph, Proc. 5th British Combinatorial Conf. (Aberdeen 1975), Congressus Numeratium XV, Utilitas Math. Publ. Inc., Winnipeg, 1976, 647-659.

[2] P. ERDŐS; J. SPENCER: Probabilistic Methods in Combinatorics, Akadémiai Kiadó, Budapest 1974.

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x) Added in proofs: Recently we have proved this conjecture.