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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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Abstract: We prove new fixed point theorems for multivalued mappings. Moreover, we construct a simple example which shows that the conjecture of J. P. Penot, stated in [8], is false.

Key words: Metric space, Banach space, fixed point theorems, multivalued mappings.

Classification: Primary 47H10, 47H15
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1. A fixed point theorem for multivalued mappings in complete metric spaces.

Let M be a metric space with metric d, A, B being subsets of M, $x_0 \in M$. Put: $d(x_0,A) = \inf\{d(x_0,x) : x \in A\}$,
$D(A,B) = \{\lambda > 0 : A \subseteq V_\lambda(B)$ and $B \subseteq V_\lambda(A)\} = \max\{\sup\{d(x,B) : x \in A\},$ $\sup\{d(y,A) : y \in B\}\}$, where $V_\lambda(A) = \{y \in M, d(y,Y) \leq \lambda\}$ for $\lambda > 0$.

Definition 1. Let M be a metric space with metric d. We say that a map $f: M \to M$ satisfies the Caristi's condition if there exists a lower semicontinuous function $h: M \to R_+ = [0,\infty)$ such that $d(x,f)) \leq h(x) - h(f(x))$ for all $x \in M$.

Theorem 1. Let M be a complete metric space, $F: M \to M$ be a multivalued mapping of M into the family of all nonempty compact subsets of M such that $D(F(x),F(y)) < d(x,y)$ for all

$x \neq y \in M$. Suppose that there exists a single-valued map $f:$ $:M \to M$ satisfying the Caristi's condition such that:

1) $d(x,F(x)) \geq \inf \{ d(f^n(x)),F(f^n(x)): n=1,2,\dots \}$ for all $x \in M$, where $f^n(x) = (\underbrace{f \circ f \circ \dots \circ f}_{n\text{-times}})(x)$,

2) $K = \{ x \in M,\ f(x) = x \}$ is precompact.

Then F has a fixed point in M.

<u>Proof</u>. We claim that for each $z \in M$ there exists a $z_0 \in K$ such that $d(z_0,F(z_0)) \leq d(z,F(z))$. Let $h:M \to R_+$ be a lower semicontinuous function such that $d(x,f(x)) \leq h(x) - h(f(x))$ for all $x \in M$. We write $x \leq y$ iff $d(x,y) \leq h(x) - h(y)$. Then $\leq$ is a partial order on M. Let z be an arbitrary fixed point in M. Put $M_z = \{ x \in M: d(x,F(x) \leq d(z,F(z))\}$. Then M_z is a non-empty ($z \in M_z$) closed subset of M, since $d(x,F(x))$ is a continuous function on M. Therefore M_z is complete. Using the same argument as in [8] one can prove that there exists a maximal element z_0 in M_z (i.e. if $x \in M_z$ and $x \geq z_0$ then $x = z_0$).

Suppose that there exists an $n \in N$ such that

$d(f^n(z_0),\ F(f^n(z_0))) \leq d(z_0,F(z_0)) \leq d(z,F(z))$

Then $f^n(z_0) \in M_z$. On the other hand, we have:

$d(z_0,f(z_0)) \leq h(z_0) - h(f(z_0))$, $d(f(z_0),\ f^2(z_0)) \leq h(f(z_0)) - h(f^2(z_0))$ ..., $d(f^{n-1}(z_0),\ f^n(z_0)) \leq h(f^{n-1}(z_0)) - h(f^n(z_0))$.

Hence

$$d(z_0,f^n(z_0)) \leq \sum_{i=1}^{n} d(f^{i-1}(z_0),f^i(z_0)) \leq h(z_0) - h(f^n(z_0)),$$

where $f^0(z_0) = z_0$. This implies $f^n(z_0) \geq z_0$, $f^n(z_0) \in M_z$. Hence $f^n(z_0) = z_0$ and it is clear that $f(z_0) = z_0 \in K \cap M_z$.

Now suppose that $d(f^n(z_0),F(f^n(z_0)) > d(z_0,F(z_0))$ for all n. Then there exists a subsequence $\{n_i\}$ such that $\lim_i d(f^{n_i}(z_0),\ F(f^{n_i}(z_0))) = d(z_0,F(z_0))$. It is easy to see that $\{f^n(z_0)\}$ is a Cauchy sequence in M. Then there exists a point $z_\infty \in M$ such that $z_\infty = \lim f^{n_i}(z_0)$, since M is complete. Hence

$$d(z_0,z_\infty) = \lim d(z_0,f^{n_i}(z_0)) \leq h(z_0) - \lim h(f^{n_i}(z_0)) \leq$$
$$\leq h(z_0) - h(z_\infty),$$
$$d(z_\infty,F(z_\infty)) = \lim_i d(f^{n_i}(z_0),\ F(f^{n_i}(z_0))) = d(z_0,F(z_0)) =$$
$$\leq d(z,F(z)).$$

This means that $z_\infty \in M_z$ and $z_\infty \geq z_0$. Therefore $z_\infty = z_0$ and $h(z_\infty) = h(f(z_0)) = h(z_0)$. Hence $d(f(z_0),F(f(z_0))) = = d(z_0,F(z_0))$. This contradicts the assumption $d(f^n(z_0),F(f^n(z_0))) > d(z_0,F(z_0))$ for all n=1,2,... . This proves our claim.

It is easy to see that $\inf\{d(x,F(x)):x\in M\} = = \inf\{d(x,F(x)):x\in\overline{K}\}$. Since $\overline{K}$ is compact, there exists a point $x_0\in\overline{K}$ such that $d(x_0,F(x_0)) = \inf\{d(x,F(x)):x\in M\}$. If $r = d(x_0,F(x_0)) > 0$, take a $y\in F(x_0)$ such that $d(x_0,y) = = d(x_0,F(x_0)) = r$. Then $d(y,F(y)) \leq D(F(x_0),F(y)) < d(x_0,y) = r$. This contradicts the assumption $d(x_0,F(x_0)) = \inf\{d(x,F(x)):x\in M\}$. Hence $d(x_0,F(x_0)) = 0$ and $x_0\in F(x_0)$. This completes the proof.

Remark: In [8] J.P. Penot has stated the following problem: Let M be a complete metric space, $h:M \to R_+$ be a lower semicontinuous function and $F:M \to M$ be a multivalued mapping of M into the family of all nonempty closed subsets of M satisfying the following condition:
$d(x,F(x)) \leq h(x) - \inf\{h(y):y\in F(x)\}$. Does F have a fixed point in M ?

The following simple example shows that this conjecture is false.

Put $M = [0,\infty)$ with the usual metric. Put $h(x) = \frac{1}{1+x}$ $F(x) = [x + \frac{1}{2(1+x)}, 2x+1]$ for all $x \in M$. Then M is a complete metric space, $h:M \to R_+$ is continuous, F satisfies the condition $d(x,F(x)) \; h(x) - \inf\{h(y): y\in F(x)\}$, but F has not any fixed point in M.

<u>Proposition I</u>. Let M be a complete metric space, $h: :M \to R_+$ be a lower semicontinuous function, $F:M \to M$ be a multivalued mapping which maps M into the family of all nonempty closed subsets of M. Suppose F satisfies the following condition $\inf\{d(x,y) + h(y): y\in F(x)\} \leq h(x)$ for all $x \in M$. Then F has a fixed point in M.

<u>Proof</u>. We claim that for each $x \in M$ there exists an $f(x) \in F(x)$ such that $d(x,f(x)) \leq 2\,h(x) - 2\,h(f(x))$. If $d(x,F(x)) = 0$, put $f(x) = x$. If $d(x,F(x)) > 0$, then
$d(x,F(x)) + \inf\{d(x,y) + 2h(y): y\in F(x)\} \leq 2\inf\{d(x,y) + h(y): y \in F(x)\} \leq 2h(x)$.
It follows that $\inf\{d(x,y) + 2h(y): y\in F(x)\} < 2h(x)$. Then there exists a point $f(x) \in F(x)$ such that $d(x,f(x)) + 2h(f(x)) \leq 2h(x)$. This proves our claim.
According to Christi's Theorem there exists a point $x_0 \in M$ such that $x_0 = f(x_0) \in F(x_0)$. This completes the proof.

<u>Corollary 1</u>(S.B. Nadler [7]). Let M be a complete metric space. If $F:M \to M$ is a multivalued contraction mapping which maps M into the family of all nonempty closed subsets of M, then F has a fixed point.

<u>Proof</u>. Let $D(F(x),F(y)) \leq kd(x,y)$, where $0 \leq k < 1$. Put

$h(x) = \frac{1}{1-k} d(x, F(x))$. Then

$\inf\{d(x,y) + h(y) : y \in F(x)\} = \inf\{d(x,y) + \frac{1}{1-k} d(y, F(y)) :$
$: y \in F(x)\} = \inf\{d(x,y) + \frac{1}{1-k} \cdot D(F(x), F(y)) : y \in F(x)\} \leq$
$\leq \inf\{d(x,y) + \frac{1}{1-k} k\, d(x,y) : y \in F(x)\} = \frac{1}{1-k} d(x, F(x)) = h(x)$.

By Proposition 1, F has a fixed point in M.

Corollary 2. Let M, h, F be as in Proposition 1.

1. If $d(x, F(x)) \leqq h(x) - \sup\{h(y) : y \in F(x)\}$, then F has a fixed point in M.

2. If $D(\{x\}, F(x)) \leq h(x) - \inf\{h(y) : y \in F(x)\}$, then there exists an $x_0 \in M$ such that $f(x_0) = \{x_0\}$.

Proof. It is clear that F has a fixed point in M, because $\inf\{d(x,y) + h(y) : y \in F(x)\} \leq d(x, F(x)) + \sup h(F(x))$ and $\inf\{d(x,y) + h(y) : y \in F(x)\} \leq D(\{x\}, F(x)) + \inf\{h(y) : y \in F(x)\}$.
To prove 2, it is sufficient to note that for each $x \in M$ there exists a point $f(x) \in F(x)$ such that
$D(\{x\}, F(x)) \leqq h(x) - \inf\{h(y) : y \in F(x)\} \leq 2h(x) - 2h(f(x))$.
By Caristi's Theorem there exists a point $x_0 \in M$ such that $x_0 = f(x_0)$. Then $D(\{x_0\}, F(x_0)) \leq 2h(x_0) - 2h(f(x_0)) = 0$. It follows that $F(x_0) = \{x_0\}$. This completes the proof.

2. A fixed point theorem for multivalued mappings in Banach spaces

Definition 2. Let X, Y be topological spaces, $F: X \multimap Y$ be a multivalued mapping. We say that F is upper semicontinuous at $x \in X$ if for each open set $G \subset Y$, $F(x) \subset G$ there exists a neighborhood U of x such that for each $x' \in U$ we have $F(x') \subseteq$ $\subseteq G$.

Theorem 2. Let X be a Banach space, $C \subseteq X$ be a convex closed nonempty bounded subset of X, $F: C \longrightarrow C$ be a multivalued nonexpansive mapping which maps into the family of all nonempty convex closed subsets of C. Suppose that there exist a function $\mu: R_+ \longrightarrow R_+$ which is nondecreasing and $\mu(t) > 0$ for all $t > 0$, a function $\varphi: C-C \longrightarrow R$ weakly continuous at θ, $\varphi(\theta) > 0$ and a mapping $\psi: C-C \to \mathcal{C}(X^*)$, where $\mathcal{C}(X^*)$ denotes the family of all nonempty closed subsets of the dual space X^*, weakly-strongly upper-semicontinuous at θ, $\psi(\theta)$ is compact, such that

$$d(x,F(x)) + d(y,F(y)) \geq \mu(\|x-y\|)\,\varphi(x-y) - \psi_s(x-y)$$

for all $x,y \in C$, where $\psi_s(x) = \sup\{|\langle x^*,x\rangle| \mid x^* \in \psi(x)\}$. Then F has a fixed point in C.

Proof. By the boundness of C, there exists a number $M > 0$ such that $C \subseteq B_M = \{x \in X: \|x\| \leq M\}$. Hence $C-C \subseteq B_{2M}$. By the standard argument there exists a sequence $\{x_n\} \subseteq C$ such that $d(x_n, F(x_n)) < \frac{1}{n}$ for each $n \in N$. Since $\{x_n\}$ is bounded in X, $\{x_n\}$ is weakly precompact. Then there exists a weakly Cauchy subnet $\{x_{\wp(i)}\}_{i \in I}$ of $\{x_n\}$ where $\wp: I \longrightarrow N$. Then it is clear that the net $\{u_{i,j}\}_{(i,j) \in I \times I}$ where $u_{i,1} = x_{\wp(i)} - x_{\wp(j)}$ converges weakly to θ.

We claim that $\lim \|u_{i,j}\| = 0$. Suppose that it is false. There exists a number $r > 0$ such that for any $(i,j) \in I \times I$ there exists an $(i',j') \in I \times I$, $(i',j') \geq (i,j)$ and $\|u_{i,j}\| \geq r$. Since φ is weakly continuous at θ we have $\lim \varphi(u_{i,j}) = \varphi(\theta) = k > 0$. Let $\tau: X \longrightarrow X^{**}$ be a canonical embedding map of X into its bidual space X^{**}. Since $\{\tau(u_{i,j})\}$ is bounded in X^{**}, $\{\tau(u_{i,j})\}$ is an equicontinuous family of mappings from $(X^*, \|\cdot\|)$ into R. Since $\{\tau(u_{i,j})\}$ converges pointwise to Θ on X^* and $\psi(\Theta)$ is a compact subset of X^* by Theorem 4.5 [9, chapt. III] it follows that $\{\tau(u_{i,j})\}$ converges uniformly to Θ on $\psi(\Theta)$. Then there exists an index $(i_0,j_0)\in I\times I$ such that for $(i,j)\in I\times I$, $(i,j)\geq(i_0,j_0)$ we get $\varphi(u_{i,j})\geq\frac{3}{4}\cdot k$ and $|\langle\tau(u_{i,j}),x^*\rangle| = |\langle x^*,u_{i,j}\rangle| \leq \leq\frac{1}{8}k\mu(r)$ for all $x^*\in\psi(\Theta)$. Since ψ is weakly-strongly upper-semicontinuous at Θ and $\{u_{i,j}\}$ converges weakly to Θ, there exists an index $(i_1,j_1)\in I\times I$, $(i_1,j_1)\geq(i_0,j_0)$ such that:

$\psi(u_{i,j})\subseteq\psi(\Theta)+\frac{k\mu(r)}{16M}B_1^*(\Theta)$, where $B_1^* = \{x^*\in X^*: \|x^*\|\leq 1\}$ for all $(i,j)\in I\times I$, $(i,j)\geq(i_1,j_1)$. Then

$$\psi_s(u_{i,j}) = \sup\{|\langle x^*,u_{i,j}\rangle| : x^*\in\psi(u_{i,j})\}\leq$$

$$\leq\sup\{|\langle x^*,u_{i,j}\rangle| : x^*\in\psi(\Theta)+\frac{k\mu(r)}{16M}B_1^*(\Theta)\}\leq$$

$$\leq\sup\{|\langle x^*,u_{i,j}\rangle| : x^*\in\psi(\Theta)\}+\frac{k\mu(r)}{16M}\sup\{|\langle x^*,u_{i,j}\rangle| : x^*\in B_1^*\}\leq\frac{k\mu(r)}{8}+\frac{k\mu(r)}{16M}\|u_{i,j}\|\leq\frac{k\mu(r)}{4}$$

for all $(i,j)\in I\times I$, $(i,j)\geq(i_1 j_1)$.

Take $n,m\in N$ such that $\frac{1}{n}+\frac{1}{m}<\frac{k\mu(r)}{2}$. Choose $i_2\in I$, $i_2\geq i_1$, $i_2\geq j_1$ such that $\rho(i)\geq\max\{n,m\}$ for all $i\in I$, $i\geq i_2$. Take $(i_3,j_3)\in I\times I$, $(i_3,j_3)\geq(i_2,i_2)$ such that $\|u_{i_3,j_3}\|\geq r$. Then

$$d(x_{\rho(i_3)},F(x_{\rho(i_3)}))+d(x_{\rho(j_3)},F(x_{\rho(j_3)}))\geq$$

$$\geq\varphi(u_{i_3,j_3})\mu(\|u_{i_3,j_3}\|)-\psi_s(u_{i_3,j_3}).$$

Hence

$$\frac{1}{n}+\frac{1}{m}\geq\frac{1}{\rho(i_3)}+\frac{1}{\rho(j_3)}\geq d(x_{\rho(i_3)},F(x_{\rho(i_3)}))+$$

$$+d(x_{\rho(j_3)},F(x_{\rho(j_3)}))\geq\frac{3}{4}k\mu(r)-\frac{k\mu(r)}{4}=\frac{1}{2}k\mu(r).$$

This contradicts $\frac{1}{n}+\frac{1}{m}<\frac{1}{2}k\mu(r)$ and this proves our claim.

Since $\lim \|u_{i,j}\| = 0$, it follows that $\{x_{\varphi(i)}\}$ is a Cauchy net in the strong topology. Therefore $\{x_{\varphi(i)}\}$ converges strongly to an $x \in C$. Then for $i \in I$, we have

$$d(x,F(x)) \le \|x - x_{\varphi(i)}\| + d(x_{\varphi(i)},F(x_{\varphi(i)})) +$$
$$+ D(F(x_{\varphi(i)}),F(x)) \le 2\|x - x_{\varphi(i)}\| + \frac{1}{\varphi(i)}.$$

Hence $d(x,F(x)) = 0$. It follows that $x \in F(x)$ and this completes the proof.

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