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Ivan Havel; Mirko Křivánek

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ON MAXIMAL MATCHINGS IN Q_6 AND A CONJECTURE OF R. FORCADE

Ivan HAVEL and Mirko KŘIVÁNEK

Abstract: It is proved that every maximal matching in the cube Q_6 contains at least 24 edges. This fact disproves a conjecture by R. Forcade. The same result has been published by J.M. Laborde ([3]), who disproved the conjecture using a computer. Our proof is independent and does not use a computer.

Key words: n-dimensional cube, maximal matching.

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1. Introduction. In [1] a conjecture concerning the number of edges of the smallest maximal matching in the graph of the n-dimensional cube Q_n is formulated. According to the conjecture, there should exist a maximal matching in Q_6 containing 23 edges. In this paper, which is a modified version of [2], we prove that any maximal matching in Q_6 contains at least 24 edges; this fact disproves Forcade's conjecture. The same assertion was among other results published in [3]; the author announced in [3] that he had disproved Forcade's conjecture using a computer. The results contained in [2] were obtained independently of [3] and without help of a computer. We believe therefore that they could be of interest especially from the point of view of further progress in solving the difficult problem of obtain-

ing better estimates or determining the cardinality of the smallest maximal matching in Q_n.

2. Definitions. Statement of results. We deal with finite undirected graphs without loops and multiple edges. If $G = (V(G), E(G))$ is such a graph, then $M \subseteq E(G)$ is called a matching in G, if no two edges of M are adjacent. A matching M is a maximal matching in G, if $M \subsetneq M'$ holds for no matching M' in G.

For $U \subseteq V(G)$ we put $N_G(U) = \{v \in V(G); \exists u \in U$ such that $(u,v) \in E(G)\}$ and write frequently $N(U)$ instead of $N_G(U)$ and $N(u)$ instead of $N(\{u\})$.

An n-dimensional cube Q_n is a graph $Q_n = (V(Q_n), E(Q_n))$, where $V(Q_n) = \{(u_1,\dots,u_n);\ u_i \in \{0,1\},\ i = 1,\dots,n\}$, $E(Q_n) = \{(u,v);\ u,v \in V(Q_n)$, u and v differ in exactly one coordinate$\}$. Clearly, Q_n is a bipartite graph for any n.

Define further $V^{o}(Q_n) = \{u = (u_1,\dots,u_n) \in V(Q_n);\ \sum_{i=1}^{n} u_i \equiv 1 \pmod 2\}$, $V^{e}(Q_n) = V(Q_n) - V^{o}(Q_n)$. We say that $u, v \in V(Q_n)$ are of the same parity, if either $\{u,v\} \subseteq V^{o}(Q_n)$ or $\{u,v\} \subseteq V^{e}(Q_n)$. Put $\bar{0} = 1$, $\bar{1} = 0$ and for $u \in V(Q_n)$, $u = (u_1,\dots,u_n)$ put $\bar{u} = (\bar{u}_1,\dots,\bar{u}_n)$.

Let $m(Q_n) = \min\{|M|,\ M$ is a maximal matching in $Q_n\}$. The following assertions are proved in [1]:

Assertion 1. For $n \geq 1$, $m(Q_{n+1}) \leq 2m(Q_n)$.

Assertion 2. For $n \geq 1$, $m(Q_n) \geq 2^n \cdot n/(3n - 1)$.

Assertion 3. $\lim_{n \to \infty} m(Q_n)/2^n = 1/3$.

The following conjecture is also stated in [1]:

Conjecture. For $n \geq 1$, $m(Q_n) = \rbrack 2^n \cdot n/(3n - 1) \lbrack$.

It follows from the trivial identity $m(Q_3) = 3$ via Assertion 1 that $m(Q_6) \leq 24$, whereas Assertion 2 gives $m(Q_6) \geq 23$. According to the conjecture there should be $m(Q_6) = 23$; our intention is to prove $m(Q_6) = 24$.

For any matching M in Q_n we define "the set $X(M)$ of odd vertices not belonging to M" as follows:

$X(M) = \{u \in V^{\sigma}(Q_n);$ u is an end-vertex of no edge of $M\}$.

Theorem 1. If M is a maximal matching in Q_n, then

(1) $|X(M)| = 2^{n-1} - |M|$,

(2) $|N(X(M)| \leq |M|$,

(3) $u \in V^e(Q_n) \Rightarrow N(u) \cap (V^{\sigma}(Q_n) - X(M)) \neq \emptyset$,

(4) $u, v \in V^e(Q_n)$, $u \neq v$, $|N(u) \cap X(M)| = |N(v) \cap X(M)| =$

$= n - 1 \Rightarrow N(u) - X(M) \neq N(v) - X(M)$.

Proof. (1) Obviously $|V^{\sigma}(Q_n)| = |V^e(Q_n)| = 2^{n-1}$ holds and further, the end-vertices of any edge in Q_n are not of the same parity. Since no two edges of M are adjacent, (1) follows.

(2) Let $u \in X(M)$, $(u,v) \in E(Q_n)$. Suppose v to be an end-vertex of no edge of M; then $M \cup \{(u,v)\}$ is again a matching which contradicts the maximality of M. Hence $u \in X(M)$, $v \in N(u) \Rightarrow$ v is an end-vertex of an edge of M, and (2) follows immediately.

(3) can be proved similarly - it follows from $N(u) \subseteq X(M)$ for some $u \in V^e(Q_n)$ that M cannot be maximal - if we choose an arbitrary $v \in N(u)$, then $M \cup \{(u,v)\}$ is again a matching.

(4) Let $N(u) - X(M) = \{u'\}$, $N(v) - X(M) = \{v'\}$; the edges (u,u'), (v,v') belong to M and therefore $u' \neq v'$, q.e.d.

The following theorem disproves the conjecture from [1].

Theorem 2. For any maximal matching M in Q_6, $|M| \geq 24$.

Proof. Let M be a maximal matching in Q_6; according to Assertion 2, $|M| \geq 23$. Assume $|M| = 23$. Then we obtain for X(M) according to Theorem 1 that $|X(M)| = 9$ and (2) - (4) of Theorem 1 hold as well. However, we shall show in Theorem 3 that this is impossible.

Theorem 3. Let $X \subseteq V^{\sigma}(Q_6)$, $|X| = 9$.

Then either

(1) $|N(X)| > 23$, or

(2) there is $u \in V^e(Q_6)$ such that $N(u) \subseteq X$, or

(3) there are $u, v \in V^e(Q_6)$ such that $u \neq v$, $|N(u) \cap X| = |N(v) \cap X| = 5$ and $N(u) - X = N(v) - X$.

Proof of Theorem 3 is given in Part 3 of this paper.

3. The proof of Theorem 3. The proof essentially utilizes a well-known fact that Q_6 is a Cartesian product of Q_4 and Q_2.

Let us denote

$A = \{(u_1,u_2,u_3,u_4,0,0\}$; $(u_1,u_2,u_3,u_4) \in V(Q_4)\}$,

$B = \{(u_1,u_2,u_3,u_4,1,0\}$; $(u_1,u_2,u_3,u_4) \in V(Q_4)\}$,

$C = \{(u_1,u_2,u_3,u_4,0,1\}$; $(u_1,u_2,u_3,u_4) \in V(Q_4)\}$,

$D = \{(u_1,u_2,u_3,u_4,1,1\}$; $(u_1,u_2,u_3,u_4) \in V(Q_4)\}$.

Then obviously $V(Q_6) = A \cup B \cup C \cup D$ and $|A| = |B| = |C| = |D| = 16$; the subgraphs of Q_6 induced by any one of the sets A,B,C,D are isomorphic to Q_4 and there are exactly 16 vertex-disjoint circuits of the length 4 in Q_6, such that each of them contains exactly one vertex of each of the sets A,B,C and D. Let us denote this set of 16 circuits by $\mathcal{C}$. The sets of vertices A,B,C,D are joined in Q_6 only by edges belonging to circuits of $\mathcal{C}$ (e.g. there are 16 edges joining A with B, no edge between A and D, etc.).

For $u \in V(Q_6)$, $u = (u_1,u_2,u_3,u_4,u_5,u_6)$ put $\pi(u) = u_1 . 2^3 + u_2 . 2^2 + u_3 . 2 + u_4$; obviously π maps $V(Q_6)$ onto $[0,15]$. For $U \subseteq V(Q_6)$ put $\pi(U) = \{\pi(u); u \in U\}$. If $i \in [0,15]$, denote by $a_i (b_i, c_i, d_i)$ the vertex of A (B,C,D, respectively) with $\pi(a_i) = i$ and put $\tilde{a}_i = a_{15-i}$. (The first four coordinates of

$\tilde{a}_i$ are complements of those of a_i; the fifth and sixth coordinates of $\tilde{a}_i$ and a_i coincide). Define similarly $\tilde{b}_i$, $\tilde{c}_i$, $\tilde{d}_i$ for $i \in [0,15]$.

Let us notice that the following holds: for $i,j \in [0,15]$, $(a_i,a_j) \in E(Q_6) \Longleftrightarrow (b_i,b_j) \in E(Q_6) \Longleftrightarrow (c_i,c_j) \in E(Q_6) \Longleftrightarrow (d_i,d_j) \in E(Q_6)$. From this we have e.g. $\pi(N_{Q_6}(a_i) \cap A) = \pi(N_{Q_6}(b_i) \cap B) = \dots = \pi(N_{Q_6}(d_i) \cap D)$. Similar relations, which easily follow from the structure of Q_6 and its decomposition into four Q_4 joined together by 16 circuits, will be used in the sequel without special references.

The next lemma (with an obvious proof) describes some structural properties of Q_4.

<u>Lemma 1.</u> (1) For any $u \in V^{\sigma}(Q_4)$ ($V^e(Q_4)$) there is just one $v \in V^{\sigma}(Q_4)$ ($V^e(Q_4)$, respectively) such that $N_{Q_4}(u) \cap N_{Q_4}(v) = \emptyset$.

(2) For $0 \le t \le 8$ define $\varphi(t)$ by the following table:

t	0	1	2	3	4	5	6	7	8
$\varphi(t)$	0	4	6	7	7	8	8	8	8

Then for any $U \subseteq V^{\sigma}(Q_4)$ ($V^e(Q_4)$, respectively), $|N_{Q_4}(U)| \geq \varphi(|U|)$. If $0 \le t \le 8$, then there is $U_t \subseteq V^{\sigma}(Q_4)$ ($V^e(Q_4)$, respectively) such that $|U_t| = t$ and $|N_{Q_4}(U_t)| = \varphi(|U_t|)$.

<u>Notation.</u> In the sequel we shall denote by X always a subset of $V^{\sigma}(Q_6)$ consisting of 9 elements, i.e. $X \subseteq V^{\sigma}(Q_6)$, $|X| = 9$. For $U \subseteq V(Q_6)$, $N(U)$ denotes $N_{Q_6}(U)$.

Let $X \subseteq V^{\sigma}(Q_6)$, $|X| = 9$. A characteristic vector $\chi(X)$ of X is a vector of 9 components, $\chi(X) = (r_1,\dots,r_9)$, where $r_1 = |X \cap A|$, $r_2 = |X \cap B|$, $r_3 = |X \cap C|$, $r_4 = |X \cap D|$, $r_5 = |N(X) \cap A|$,

$r_6 = |N(X) \cap B|$, $r_7 = |N(X) \cap C|$, $r_8 = |N(X) \cap D|$ and $r_9 = |N(X)|$.
The set of all characteristic vectors is denoted by R, hence

$$R = \{\chi(X);\ X \doteq V^{\sigma}(Q_6),\ |X| = 9\}.$$

For $r \in R$, $r = (r_1,\dots,r_9)$ the following relations obviously hold:

(a) $r_i \leq 8$, $i = 1,\dots,8$,

(b) $r_1 + r_2 + r_3 + r_4 = 9$,

(c) $r_5 + r_6 + r_7 + r_8 = r_9$.

Taking into account the obvious automorphisms of Q_n and (1) of Theorem 3, we conclude that in order to prove Theorem 3 it suffices to show that (2) or (3) of Theorem 3 holds for any X such that $r = \chi(X) \in R$, where r meets all the following conditions (d) - (j):

(d) $r_1 \geqslant \max(r_2,r_3,r_4)$,

(e) if $r_1 = r_4$, then $r_5 \geqslant r_8$,

(f) if $r_1 = r_2$, then $r_3 \geqslant r_4$,

(g) $r_2 \geqslant r_3$,

(h) if $r_2 = r_3$, then $r_6 \geqslant r_7$,

(i) $r_9 \leq 23$.

Let R_0 be the set of vectors from R fulfilling conditions (d) - (i); it is easy to see that for $r \in R_0$ the following condition holds as well:

(j) $r_5 \geqslant \varphi(r_1)$, $r_6 \geqslant \max(\varphi(r_2), r_1)$, $r_7 \geqslant \max(\varphi(r_3), r_1)$,

$r_8 \geqslant \max(\varphi(r_4), \max(r_2,r_3))$,

φ being defined in Lemma 1. The validity of (j) follows from an obvious identity $N(X) \cap A = N(X \cap A) \cap A \cup N(X \cap B) \cap A \cup N(X \cap C) \cap A$

and the similar ones for $N(X) \cap B$, $N(X) \cap C$ and $N(X) \cap D$.

Let R_x be the set of vectors $(r_1, \ldots, r_9)$ whose components are nonnegative integers such that (a) - (j) hold. R_x is easy to construct by an elementary combinatorial argument; $R_x = \{\rho_1, \ldots, \rho_{31}\}$, where $\rho_1, \ldots, \rho_{31}$ are listed below.

ρ_1	7	1	1	0	8	7	7	1	23
ρ_2	6	2	1	0	8	6	6	2	22
ρ_3	6	2	1	0	8	7	6	2	23
ρ_4	6	2	1	0	8	6	7	2	23
ρ_5	6	2	1	0	8	6	6	3	23
ρ_6	5	3	1	0	8	7	5	3	23
ρ_7	5	2	2	0	8	6	6	2	22
ρ_8	5	2	2	0	8	7	6	2	23
ρ_9	5	2	2	0	8	6	6	3	23
ρ_{10}	5	2	1	1	8	6	5	4	23
ρ_{11}	4	4	1	0	7	7	4	4	22
ρ_{12}	4	4	1	0	8	7	4	4	23
ρ_{13}	4	4	1	0	7	8	4	4	23
ρ_{14}	4	4	1	0	7	7	5	4	23
ρ_{15}	4	4	1	0	7	7	4	5	23
ρ_{16}	4	3	2	0	7	7	6	3	23
ρ_{17}	4	3	1	1	7	7	4	4	22
ρ_{18}	4	3	1	1	8	7	4	4	23
ρ_{19}	4	3	1	1	7	8	4	4	23
ρ_{20}	4	3	1	1	7	7	5	4	23
ρ_{21}	4	3	1	1	7	7	4	5	23
ρ_{22}	4	2	2	1	7	6	6	4	23
ρ_{23}	4	2	1	2	7	6	4	6	23
ρ_{24}	4	1	1	3	7	4	4	7	22

ρ_{25}	4	1	1	3	8	4	4	7	23
ρ_{26}	4	1	1	3	7	5	4	7	23
ρ_{27}	4	1	1	3	7	4	4	8	23
ρ_{28}	4	1	0	4	7	4	4	7	22
ρ_{29}	4	1	0	4	8	4	4	7	23
ρ_{30}	4	1	0	4	7	5	4	7	23
ρ_{31}	4	1	0	4	7	4	5	7	23

Obviously $R_o \subseteq R_x$; as the next step in the proof, the elements of R_o will be found. But first we prove some auxiliary statements.

Lemma 2. Let $\chi(X) = r \in R_x$, $r = (r_1,\dots,r_9)$.
(a) If $r_1 = 4$ and $r_5 = 7$, then $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$.
(b) If $r_2 \geqslant 1$ and $r_6 = r_1$ or $r_3 \geqslant 1$ and $r_7 = r_1$, then $N(a_i) \cap A \subseteq X \cap A$ for some $i \in [0,15]$.

Proof. (a) From $r_1 = 4$ and $r_5 = 7$ we have $|N(X \cap A) \cap A| = 7$. As $N(X \cap A) \cap A \subseteq V^e(Q_6) \cap A$ and $|V^e(Q_6) \cap A| = 8$, $a_j \in (V^e(Q_6) \cap A) - (N(X \cap A) \cap A)$ for some $j \in [0,15]$. Further, $N(a_j) \cap (X \cap A) = \emptyset$ and if we put $a_i = \tilde{a}_j$, then $N(a_i) \cap A = X \cap A$.

(b) Assume $b_j \in X \cap B$ for some $j \in [0,15]$ and at the same time let $N(b_j) \cap B \subseteq N(X \cap A) \cap B$ not hold. Then we should have $|N(X) \cap B| > |X \cap A|$, hence $r_6 > r_1$. Therefore, $r_2 \geqslant 1$ and $r_6 = r_1$ imply that $b_i \in X \cap B$ for some $i \in [0,15]$ and $N(b_i) \cap B \subseteq N(X \cap A) \cap B$. Hence easily $N(a_i) \cap A \subseteq X \cap A$. Similarly, such an i is to be found also in the case $r_3 \geqslant 1$, $r_7 = r_1$.

Remark. (a) of Lemma 2 will be used below also for sets B,C,D; e.g. if $r_2 = 4$, $r_6 = 7$, then $N(b_i) \cap B = X \cap B$ for some $i \in [0,15]$ etc.

In the sequel we shall always denote by k (X being fixed) the number of circuits from $\mathcal{C}$, which have a vertex in common with both $X \cap B$ and $X \cap C$.

Lemma 3. Let $\chi(X) = r \in R_x$, $r_4 = 0$. Then $k = r_2 + r_3 - r_8$.

Proof. From $r_4 = 0$ we have $N(X) \cap D = (N(X \cap B) \cap D) \cup (N(X \cap C) \cap D)$, thus $|N(X \cap D)| = |N(X \cap B) \cap D| + |N(X \cap C) \cap D| - |(N(X \cap B) \cap D) \cap$ $\cap (N(X \cap C) \cap D)|$, therefore $r_8 = r_2 + r_3 - k$, q.e.d.

Lemma 4. $R_0 \subseteq \{ \varrho_1, \varrho_2, \varrho_3, \varrho_6, \varrho_7, \varrho_9, \varrho_{10}, \varrho_{11}, \varrho_{13}, \varrho_{16}, \varrho_{17}, \varrho_{21}, \varrho_{22}, \varrho_{23}, \varrho_{24}, \varrho_{28} \}$.

Proof. The proof will proceed in several steps.

(a) none of the vectors $\varrho_4, \varrho_5, \varrho_8, \varrho_{15}$ belongs to R_0. Suppose on the contrary $\chi(X) \in \{\varrho_4, \varrho_5, \varrho_8, \varrho_{15}\}$ for some X. According to Lemma 3, in these cases the number k of circuits from $\mathcal{C}$ having a vertex in common with both $X \cap B$ and $X \cap C$ is given by $k = r_2 + r_3 - r_8$. Further, the following holds:

(a.1) $k = r_2 \Rightarrow r_6 = r_7$ (since $k = r_2 \Rightarrow k = r_3$, hence $\pi(X \cap B) = = \pi(X \cap C)$, $\pi(N(X) \cap B) = \pi(N(X) \cap C)$ and $r_6 = r_7$).

(a.2) $k = r_3 \Rightarrow r_7 \leq r_6$ (since $k = r_3 \Rightarrow \pi(X \cap C) \subseteq \pi(X \cap B)$, hence $\pi(N(X) \cap C) \subseteq \pi(N(X) \cap B)$ and $r_7 \leq r_6$).

(a.3) $k < r_3$, $r_7 = r_1 \Rightarrow \varphi(r_2 + 1) \leq r_6$ (since for some $j \in [0,15]$ $c_j \in X \cap C$, $b_j \notin X \cap B$; from $r_7 = r_1$ we have $N(c_j) \cap C \subseteq N(X \cap A) \cap C$, hence $N(b_j) \cap B \subseteq N(X \cap A) \cap B$ and $N(\{b_j\} \cup X \cap B) \subseteq N(X) \cap B$, therefore $\varphi(r_2 + 1) \leq r_6$). To prove (a) notice that $r = \varrho_4, \varrho_5, \varrho_8$ and ϱ_{15} contradicts (a.2), (a.3), (a.1) and (a.3), respectively.

(b) none of the vectors $\varrho_{12}, \varrho_{18}, \varrho_{25}, \varrho_{29}$ belongs to R_0. Suppose $\chi(X) \in \{\varrho_{12}, \varrho_{18}, \varrho_{25}, \varrho_{29}\}$ for some X. In these cases $r_1 = 4$, $r_5 = 8$ and further either $r_2 = 1$, $r_6 = 4$ and

$r_7 < 8$ or $r_3 = 1$, $r_7 = 4$ and $r_6 < 8$. First we discuss the cases $r = \rho_{25}$ and $r = \rho_{29}$, when $r_2 = 1$ and $r_6 = 4$. Let X be such that $\chi(X) = r$. Then $X \cap B = \{b_i\}$ for some $i \in [0,15]$; $r_1 = r_6 = 4$ yields $N(X \cap A) \cap B = N(b_i) \cap B$ and $X \cap A = N(a_i) \cap A$. Thus neither $\tilde{a}_i \in N(X \cap A) \cap A$ nor $\tilde{a}_i \in N(X \cap B) \cap A$; since $r_5 = 8$, it has to be $\tilde{a}_i \in N(X) \cap A$, hence $\tilde{c}_i \in X \cap C$, $N(X \cap A) \cap C = N(c_i) \cap C \subseteq$ $\subseteq N(X) \cap C$, $N(\tilde{c}_i) \cap C \subseteq N(X) \cap C$, therefore $r_7 = 8$, which is a contradiction. In a similar manner we proceed if $r = \rho_{12}$ or $r = \rho_{18}$.

(c) $\rho_{14} \notin R_0$. If $\chi(x) = (4,4,1,0,7,7,5,4,23)$ for some X, then from $r_1 = 4$, $r_5 = 7$ according to Lemma 2(a) we obtain that there is $i \in [0,15]$ such that $N(a_i) \cap A = X \cap A$. From $X \cap C =$ $= \{c_i\}$ we should have $r_6 = 4$ (since $r_4 = 0$), but $r_6 = 5$; if $X \cap C = \{c_j\}$ for some $j \neq i$, we should have $N(X) \cap C =$ $= N(\{c_i, c_j\}) \cap C$, therefore $r_6 \geqslant 6$, which is a contradiction.

(d) $\rho_{19} \notin R_0$. Assume on the contrary that $\chi(X) =$ $= (4,3,1,1,7,8,4,4,23)$ for some X. From $r_1 = 4$, $r_5 = 7$ we obtain according to Lemma 2(a) that $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$. Further, from $r_3 = 1$, $r_7 = 4$ we have $X \cap C = \{c_i\}$ and $X \cap D = \{d_j\}$, where $j \in [0,15]$, $d_i \in N(d_j) \cap D$. $r_8 = 4$ yields $N(X \cap B) \cap D \subseteq N(d_j) \cap D$; let us show that $\tilde{b}_j \notin N(X) \cap B$. From $\tilde{b}_j \in N(X \cap B) \cap B$ it would necessarily follow that b_j, $\tilde{b}_j$ would have a common neighbour in $X \cap B$, which is impossible. Since obviously $\tilde{b}_j \notin N(X \cap D) \cap B = \{b_j\}$, it would have to be $\tilde{b}_j \in N(X \cap A) \cap B$, hence $\tilde{a}_j \in N(a_i) \cap A$, contradicting $d_i \in N(d_j)$.

(e) $\rho_{20} \notin R_0$. Let on the contrary $\chi(x) = (4,3,1,1,7,7,5,4,23)$ for some X. Lemma 2 (a) gives then $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$. From $r_7 = 5$ we have $X \cap C = \{c_i\}$ (otherwise $r_7 \geqslant 6$)

and it has to be $X \cap D = \{d_j\}$ for some $j \in [0,15]$ such that $(d_i,d_j) \notin E(Q_6)$ (if this were not true, we should have $|N(X) \cap C| = 4$). But then $\{d_i\} \cup (N(d_j) \cap D) \subseteq N(X) \cap D$, hence $r_8 \geqslant 5$, which is a contradiction.

(f) $\varrho_{26} \notin R_0$. If on the contrary $\chi(X) = (4,1,1,3,7,5,4,7,23)$ for some X, then from $r_1 = 4$ and $r_5 = 7$ according to Lemma 2 (a) $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$ and since $r_6 < 6$ and $r_7 = 4$, then necessarily $X \cap B = \{b_i\}$, $X \cap C = \{c_i\}$ and further $N(X) \cap B = N(b_i) \cap B \cup N(X \cap D) \cap B$, $N(X) \cap C = N(c_i) \cap C \cup N(X \cap D) \cap C$, hence $r_6 = r_7$, which is a contradiction.

(g) $\varrho_{27} \notin R_0$. If on the contrary $\chi(X) = (4,1,1,3,7,4,4,8,23)$ for some X, then $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$ according to Lemma 2 (a); the identities $r_2 = r_3 = 1$ and $r_6 = r_7 = 4$ imply $X \cap B = \{b_i\}$, $X \cap C = \{c_i\}$, $X \cap D \subseteq N(d_i) \cap D$, therefore $|N(X) \cap D| \leq |N(X) \cap A|$, which is a contradiction.

(h) neither ϱ_{30} nor ϱ_{31} belong to R_0. Assume on the contrary that for some X either $\chi(X) = \varrho_{30}$ or $\chi(X) = \varrho_{31}$. Then $r_1 = r_4 = 4$, $r_2 = 1$, $r_3 = 0$ and $r_5 = 7$. According to Lemma 2 (a) $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$. If $\pi(X \cap A) = \pi(X \cap D)$, then $r_7 = 4$ and $r_6 \neq 5$, since $N(X) \cap C = (N(X \cap A) \cup N(X \cap D)) \cap C$ and further either $r_6 = 4$ or $r_6 \geqslant 6$ (depending on whether $X \cap C = \{c_i\}$ or $X \cap C \neq \{c_i\}$), which is a contradiction. In the case $\pi(X \cap A) \neq \pi(X \cap D)$ we have $r_6 > 4$ and $r_7 > 4$, which again is a contradiction.

From (a) - (h) the desired inclusion follows. It is possible to show by constructing suitable sets X that the converse inclusion and therefore the equality $R_0 = \{\varrho_1, \varrho_2, \varrho_3, \varrho_6, \ldots, \varrho_{23}\}$ holds as well.

Now we proceed to the proof of the main assertion:

If $\chi(X) = r \in R_0$, then X fulfils (2) or (3) of Theorem 3.

We first discuss separately three cases:

(1) Let $\chi(X) = r \in \{\varrho_7, \varrho_9\}$, i.e. either $r = (5,2,2,0,8,6,6,2,22)$ or $r = (5,2,2,0,8,6,6,3,23)$. Two possibilities are to be considered: (1.a) Assume $N(a_i) \cap A \subseteq X \cap A$, $b_i \in X \cap B$, $c_i \in X \cap C$ for some $i \in [0,15]$. Then, of course, $N(a_i) \subseteq X$ and (2) of Theorem 3 is fulfilled. (1.b) Let (1.a) not hold; since according to Lemma 3 the number k of circuits from $\mathcal{C}$ satisfies $k \geqslant 1$, then $b_i \in X \cap B$, $c_i \in X \cap C$ for some $i \in [0,15]$. But $N(a_i) \cap A \subseteq X \cap A$ does not hold, hence $(N(b_i) \cap B) - (N(X \cap A) \cap B) \neq \emptyset$. From $r_1 = 5$, $r_6 = 6$ we obtain $|(N(b_i) \cap B) - (N(X \cap A) \cap B)| = 1$. Since $r_2 = 2$, let $j \in [0,15]$ be such that $j \neq i$ and $b_j \in X \cap B$. From $r_6 = 6$ we have $N(b_j) \cap B \subseteq N(\{b_i\} \cup X \cap A) \cap B$. Further, $\mathcal{T}(N(\{b_i\} \cup X \cap A) \cap B) = \mathcal{T}(N(\{c_i\} \cup X \cap A) \cap C) = \mathcal{T}((N(X \cap C) \cup N(X \cap A)) \cap C)$, and, since $r_3 = 2$, also $c_j \in X \cap C$. Hence $k \geqslant 2$, and consequently the case (1.b) cannot occur for $r = \varrho_9$. Since necessarily $|N(a_i) \cap N(a_j) \cap A| = 2$, we obtain $|N(a_i) \cap N(a_j) \cap (X \cap A)| = 1$ and $a_\ell \in N(a_i) \cap N(a_j)$, $a_\ell \notin X \cap A$ for some $\ell \in [0,15]$. Further $|(N(a_i) \cap A) \cap (X \cap A)| = |(N(a_j) \cap A) \cap (X \cap A)| = 3$, hence $|N(a_i) \cap X| = |N(a_j) \cap X| = 5$ and at the same time (a_i, a_ℓ), $(a_j, a_\ell) \in E(Q_6)$; X fulfils (3) of Theorem 3, q.e.d.

(2) Let $\chi(X) = r \in \{\varrho_{17}, \varrho_{21}\}$, i.e. either $r = (4,3,1,1,7,7,4,4,22)$, or $r = (4,3,1,1,7,7,4,5,23)$. According to Lemma 2(a), $N(a_i) \cap A = X \cap A$ for some $i \in [0,15]$; $r_3 = 1$ and $r_7 = 4$ necessarily imply $X \cap C = \{c_i\}$. $b_i \in X \cap B$ would mean $N(a_i) \subseteq X$ and (2) of Theorem 3 would be fulfilled. Assume therefore $b_i \notin X \cap B$. Let $X \cap D = \{d_j\}$; $j \neq i$ (because $(c_i, d_i) \in E(Q_6)$). It has to be $d_j \in N(d_i)$ - otherwise $|N(X) \cap C| \geqslant 5$ - and therefore also $a_j \in N(a_i)$, $a_j \in X$. Further, $|X \cap B \cup \{b_i\}| = 4$,

$|N(X \cap B \cup \{b_i\}) \cap B| = 7$. In a similar manner as in the proof of Lemma 2 (a) we can show that $N(b_\ell) \cap B = X \cap B \cup \{b_i\}$ for some $\ell \in [0,15]$. It must be $\ell = j$ ($\ell \neq j$ would imply $|N(X) \cap D| \geq 6$, since $N(\{d_j, d_\ell\}) \cap D \subseteq N(X) \cap D$, contradicting $r_8 \in \{4,5\}$). But then $|N(X) \cap D| = 4$ and therefore it is sufficient to consider the case $r = \varrho_{17}$. Then $X \cap B = N(b_j) \cap B - \{b_i\}$, therefore $|N(a_i) \cap X| = |N(b_j) \cap X| = 5$; $(a_i b_i)$, $(b_j, b_i) \in E(Q_6)$. X fulfils (3) of Theorem 3, q.e.d.

(3) Let $\chi(X) = r = \varrho_{28}$, i.e. $r = (4,1,0,4,7,4,4,7,22)$. According to Lemma 2 (a), $N(a_i) \cap A = X \cap A$, $N(d_j) \cap D = X \cap D$ for some $i, j \in [0,15]$. As $r_2 = 1$ and $|N(X) \cap B| = r_6 = 4$, we have $i = j$ and $X \cap B = \{b_i\}$, therefore $|N(a_i) \cap X| = |N(d_i) \cap X| = 5$ and at the same time $c_i \notin X$, (a_i, c_i), $(c_i, d_i) \in E(Q_6)$; X fulfils (3) of Theorem 3, q.e.d.

The remaining cases are covered by the next two propositions:

<u>Lemma 5.</u> Let $\chi(X) = r \in R_0 - \{\varrho_7, \varrho_9\}$, $r = (r_1, \dots, r_9)$. If $\varphi(r_2 + 1) > r_6$ and $\varphi(r_3 + 1) > r_7$, then $N(a_i) \subseteq X$ for some $i \in [0,15]$ and X fulfils (2) of Theorem 3.

<u>Proof.</u> Obviously r meets the assumptions of (a) or (b) of Lemma 2; therefore $N(a_i) \cap A \subseteq X \cap A$ for some $i \in [0,15]$. Then, however, $N(b_i) \cap B \subseteq N(X \cap A) \cap B$, hence $N(\{b_i\} \cup (X \cap B)) \cap B \subseteq N(X) \cap B$. From $b_i \notin X$ it would follow $\varphi(r_2 + 1) \leq |N(\{b_i\} \cup (X \cap B)) \cap B| \leq |N(X) \cap B| = r_6$, which is a contradiction. Therefore $b_i \in X$, in a similar way $c_i \in X$, hence $N(a_i) \subseteq X$, q.e.d.

<u>Lemma 6.</u> Let $\chi(X) = r \in R_0 - \{\varrho_7, \varrho_9\}$, $r = (r_1, \dots, r_9)$. If $r_4 = 0$, then $N(a_i) \subseteq X$ for some $i \in [0,15]$ and X fulfils (2) of Theorem 3.

<u>Proof.</u> Let X be such that $\chi(x) = r \in R_0 - \{\varrho_7, \varrho_9\}$, $r_4 = 0$. This means $r \in \{\varrho_1, \varrho_2, \varrho_3, \varrho_6, \varrho_{11}, \varrho_{13}, \varrho_{16}\}$ and in these cases

$k = r_3$ according to Lemma 3 and further $1 \leq k < 3$, $r_7 = \max(r_1, \varphi(r_3))$. According to Lemma 2, $N(a_i) \cap A \subseteq X \cap A$. Let first $r \neq \varphi_{16}$, then $r_3 = 1$ and assume $j \in [0,15]$ be such that $X \cap C = \{c_j\}$. If $N(a_j) \cap A - X \cap A \neq \emptyset$, then $i \neq j$ and also $N(c_j) \cap C - N(X \cap A) \cap C \neq \emptyset$; this gives $r_7 \geqslant r_1 + 1$. From $N(c_i) \cap C \subseteq N(X \cap A) \cap C$ we obtain $r_7 \geqslant \varphi(r_3 + 1)$, contradicting $r_7 = \max(r_1, \varphi(r_3))$.

For $r = \varphi_{16} = (4,3,2,0,7,7,6,3,23)$ we proceed as follows: if $c_i \notin X \cap C$, then $|N(X \cap C \cup \{c_i\}) \cap C| = 6$ and at the same time $|X \cap C \cup \{c_i\}| = 3$, contradicting $\varphi(3) = 7$. Hence $c_i \in X \cap C$ and since $k = r_3$, we conclude that $b_i \in X \cap B$ holds as well, therefore $N(a_i) \subseteq X$, q.e.d.

This completes the proof of Theorem 3.

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Matematický ústav ČSAV
Žitná 25
115 67 Praha 1
Československo

VÚMS
Loretánské nám. 3
110 00 Praha 1
Československo

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