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THE LCC-TOPOLOGY ON THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract: The topology on the function space is introduced and their properties are studied. This topology proved to be useful if relays and their generalization are studied.

Key words: Function space, projectively generated, Baire property, separation axioms, connectedness.

Classification: 54C35

Introduction. The purpose of this paper is to describe a topology on a function space which is useful when a relay and its generalization are studied [1, 2]. A relay can be considered as a non-linear operator from the set of real-valued continuous functions defined on a compact interval I to the space of two-valued right-side continuous functions defined on I equipped with the metric $\varrho(f,g) = \int_I |f(t) - g(t)| dt$. It is obvious that a relay is not a continuous operator if R^I is endowed by the compact-open topology. The topology considered in this note is one of the simplest topologies in which relay and its generalizations are continuous operators. In this note basic properties of the set Y^X endowed with this topology are examined.

§ 1. Notations and the definition of the level topology

1.1. Let X be a topological space and $\mathfrak{Z}$ be a covering of it. The star of a set Y in $\mathfrak{Z}$ is defined to be the set of all $Z \in \mathfrak{Z}$ intersecting Y and will be denoted by st $(Y, \mathfrak{Z})$.

Let us remind the definition of the hyperspace of lower semicontinuity. See also [3]. Let X be a topological space. The hyperspace $H_-(X)$ of lower semicontinuity is defined as follows: the underlying set of $H_-(X)$ is $\exp' X$ where $\exp' X = \exp X - \{0\}$ and $\{\text{st } (U, \exp' X) \mid Y \cap U \neq 0,\ U \text{ is an open set}\}$ is a local subbase at Y in $H_-(X)$.

For purposes of this paper let us denote $H_-^o(X)$ the space formed from $H_-(X)$ by adding of an empty set. Local bases of the points from $\exp' X$ are the same as in $H_-(X)$ and the local base of the empty set is one element set $\{\exp X\}$.

The set of all continuous maps on a topological space Y we shall denote Y^X. For a point $y \in Y$ we shall denote φ_y the mapping on Y^X into $H_-^o(X)$ defined by $\varphi_y(f) = f^{-1}(y)$.

1.2. Definition. The level topology (L-topology) on the set Y^X is the topology projectively generated by the family $\{\varphi_y\}_{y \in Y}$.

It is useful to know how the local subbase of the level topology on Y^X looks. Let $f \in Y^X$, $y \in Y$ and Q be a neighborhood from the local subbase of $\varphi_y(f)$. We shall denote the neighborhood of f determined by these parameters by W. We may suppose $\varphi_y(f)$ to be a non-void set. Its neighborhood Q is st $(G, \exp X)$, G being an open set with a non-void intersection with $\varphi_y(f)$. A mapping $g \in Y^X$ is an element of W iff the set $\varphi_y(g)$ is an element of Q; this is equivalent to $g^{-1}(y) \cap G \neq 0$. Let x be a point in G such that $f(x) = y$. The set G may be considered as a neighborhood of x. Then a map g is in the neighborhood W iff in the neighborhood G of x there is a point x' such that $f(x) = g(x')$. Thus V(x) being a neighborhood of a point x in X and $U(V(x))(f)$ being the set $\{g \in Y^X \mid$ there is x' in V(x) such that $g(x') = f(x)\}$, we can describe the local subbase in $f \in Y^X$ as follows:

<u>Proposition</u>. The local subbase in f is formed by the family $\{U(V(x))(f) \mid$ x is an arbitrary point in X and V(x) is an arbitrary neighborhood of $x\}$.

1.3. <u>Proposition</u>. Let Y be a T_1-space. Then the space Y^X endowed with the L-topology is T_1, too.

Proof. Let $f, g \in Y^X$, $f \neq g$ and $g \in U(V(x))(f)$ for an arbitrary point $x \in X$ and its arbitrary neighborhood V(x). Then there is a net $\{x_{V(x)} \in V(x) \mid V(x) \in \mathcal{V}(x)\}$ for each $x \in X$, $\mathcal{V}(x)$ denoting the local base in x, such that $g(x_{V(x)}) = f(x)$. The net converges to x and it follows from continuity of g that the net $\{g(x_{V(x)}) \mid V(x) \in \mathcal{V}(x)\}$ converges to g(x). Since Y is a T_1-space, every constant net has an only limit point and therefore $f(x) = g(x)$ for each $x \in X$. This contradicts the assumption that $f \neq g$.

1.4. Remark. No stronger separation axiom is possible to prove for the space Y^X endowed with the L-topology. This statement becomes self-evident as soon as one realizes that any open subset of Y^X is a dense subset of Y^X.

§ 2. The definition of the level topology of compact convergence and its basic properties

2.1. Definition. The level topology of compact convergence (LCC-topology) on Y^X is the topology projectively generated from the L-topology and the topology of compact convergence on Y^X.

Speaking in the rest of this paper about a space Y^X of continuous mapping we shall always mean that the topology on this space is the LCC topology.

2.2. Proposition. If the space Y is a T_1-space then the space Y^X is also a T_1-space.

Proof. Any space projectively generated from T_1-spaces is a T_1-space.

2.3. The aim of this paragraph is to prove providing that X, Y are Tychonoff spaces and X is moreover a locally compact space that Y^X is a Tychonoff space. Two preparatory lemmas will be proved first.

If X and Y are completely regular spaces, their topologies may be projectively generated by families of pseudometrics. Let us suppose the topology of X is generated by the family $S = \{\sigma_\alpha\}_{\alpha \in A}$ and of Y by the family $R = \{\rho_\beta\}_{\beta \in B}$. The local base of a point $x \in X$ are sets $V(\sigma, \{\sigma_k\}_{k=1}^m)(x) = \{y \in X \mid \forall k = 1,\dots,m \text{ is } \sigma_k(x,y) < \sigma\}$ for all real positive σ and all finite subfamilies of S. The local base of a mapping $f \in Y^X$ are sets $U(\varepsilon, K, \{\rho_i\}_{i=1}^n, \{V(\sigma_j, \{\sigma_k^j\}_{k=1}^{m_j})(x_j)\}_{j=1}^m)(f) = \{g \in Y^X \mid \forall i = 1,\dots,n \text{ is } \rho_i(f(x),g(x)) < \varepsilon$ for $x \in K$ and for each $j = 1,\dots,m$ there is x_j' such that

$\sigma_k^j(x_j, x_j') < \delta_j$ for each $k = 1,\dots,m_j$ and $g(x_j') = f(x_j)\}$, for each real positive ε, each compact subset K of X, each finite subfamily of R and each finite family of points from X together with their arbitrary neighborhoods.

Lemma. Let the topology on the set X be projectively generated by the family of pseudometrics $S = \{\sigma_\alpha\}_{\alpha\in A}$. Let $x \in X$ and $V = V(\delta, \{\sigma_k\}_{k=1}^m)(x)$ be a neighborhood of x. Then $\overline{V} \subset \{y \in X \mid$ for each $k = 1,\dots,m$ is $\sigma_k(x,y) \leq \delta\}$.

Proof. Let $z \in \overline{V}$. For any finite subfamily $\{\sigma_j\}_{j=1}^n$ of S and any $\varepsilon > 0$ there is $y \in V$ such that $\sigma_j(y,z) < \varepsilon$ for $j = 1,\dots,n$. If the subfamily $\{\sigma_k\}_{k=1}^m$ is chosen then $\sigma_k(x,z) \leq \sigma_k(x,y) + \sigma_k(y,z) < \delta + \varepsilon$ for $k = 1,\dots,m$ and every positive ε. Thus $\sigma_k(x,z) \leq \delta$ and the lemma is proved.

Lemma. Let X, Y be Tychonoff spaces, X be locally compact, $f \in Y^X$, $U(\varepsilon, K, \wp, V(\delta, \{\sigma_k\}_{k=1}^m)(x))(f) = U$ be a neighborhood of f such that $V(\delta, \{\sigma_k\}_{k=1}^m)(x) \subset V(x)$ and $V(x)$ is a compact neighborhood of x. Then for $\varepsilon' < \varepsilon$ and $\delta' < \delta$ is $U_1 = \overline{U(\varepsilon', K, \wp, V(\delta', \{\sigma_k\}_{k=1}^m(x))(f)} \subset U$.

Proof. Let $g \in U_1$. We shall show at first that there is a point $y \in X$ such that $\sigma_k(x,y) < \delta$ for $k = 1,\dots,m$ and $f(x) = g(y)$. We know that $V = \overline{V(\delta', \{\sigma_k\}_{k=1}^m)(x)} \subset \{z \in X \mid \sigma_k(z,x) \leq \delta'$ for $k = 1,\dots,m\} \subset V(\delta, \{\sigma_k\}_{k=1}^m)(x) \subset V(x)$ and that V is a compact set. Assuming $f(x)$ is not in $g(V)$ there exists a neighborhood of the set $g(V)$ which does not include $f(x)$. Choosing this neighborhood and the set V as parameters of a neighborhood of the mapping g, we have got the neighborhood of g which has an empty intersection with $U(\varepsilon', K, \wp, V(\delta', \{\sigma_k\}_{k=1}^m(x))(f)$ and this is in a contradiction with the assumption that g is in U_1.

It is a well-known fact that $\rho(f(x),g(x)) < \varepsilon$ for each $x \in K$ and thus $g \in U$.

Corollary. Let the spaces X, Y be the same as in the preceding lemma. Let $f \in Y^X$, $U(\varepsilon, K, \{\rho_i\}_{i=1}^{n})$, $\{V(\sigma_j, \{\sigma_k^j\}_{k=1}^{m_j})(x_j)\}_{j=1}^{m}(f) = U$ be a neighborhood of f such that the inclusion $V(\sigma_j, \{\sigma_k^j\}_{k=1}^{m_j})(x_j) \subset V(x_j)$ is valid for each $j = 1,\dots,m$ and $V(x_j)$ is a compact neighborhood of the point x_j. Let us denote

$U_r = U(r\varepsilon, K, \{\rho_i\}_{i=1}^{n}, \{V(r\sigma_j, \{\sigma_k^j\}_{k=1}^{m_j})(x_j)\}_{j=1}^{m})(f)$, for a real positive number r. Then for r from the interval $(0,1)$ is $\overline{U}_r \subset U$.

Proposition. Let X, Y be Tychonoff spaces and X be a locally compact space. Then Y^X is a Tychonoff space.

Proof. Let f be an element of Y^X, A be a closed subset of Y^X and do not let f be in A. There is a neighborhood of f which fulfils the assumptions of the above corollary and which has an empty intersection with the set A. It will be denoted again U. It is $\overline{U}_r \subset U$ for $r \in (0,1)$ and if $s \in (0,1)$ and $r < s$, then $\overline{U}_r \subset U_s \subset U$.

The function $F: Y^X \to [0,1]$ is defined so that $F(g) = 1$ for $g \in Y^X \setminus U$ and $F(g) = \inf\{r \in (0,1) \mid g \in U_r\}$ for $g \in U$. Then $F(f) = 0$ and $F(A) = 1$. The proof of continuity of that function is similar to that for the function from the Urysohn lemma. Hence the complete regularity of the space Y^X is proved. Using now the proposition 2.2 is the proof completed.

2.4. Proposition. Let (X,σ) and (Y,ρ) be metric spaces. Let (X,σ) be σ-compact and locally compact and let (Y,ρ) be complete. Then the space Y^X is a Baire space.

Proof. At first a few words about notations used in this paragraph. The local base of Y^X will be a modification of the local base from the previous paragraph. Using the fact that Y, X are metric spaces, the notation may be simplified: $U(\varepsilon,K,(\delta,\{x_j\}_{j=1}^m))(f) = \{g\in Y^X \mid \rho(f(x)),g(x))<\varepsilon$ for $x\in K$ and for each $j = 1,\dots,m$, there exists x_j' such that $g(x_j') = f(x_j)$ and $\sigma(x_j',x_j) < \delta\}$.

Substituting $\leqq$ for $<$ in the above notation the modified local base is obtained. An element of this new base is denoted: $\tilde{U}(\varepsilon,K,(\delta,\{x_j\}_{j=1}^m))(f)$.

Let g be an element of $U = \tilde{U}(\varepsilon,K,(\delta,\{x_j\}_{j=1}^m))(f)$. Let us denote $\{x_j(U,g)\}_{j=1}^m$ an arbitrary set of the cardinality m and such that for each $j = 1,\dots,m$ it includes an element x_j' for which $g(x_j') = f(x_j)$ and $\sigma(x_j,x_j') \leq \delta$.

The local compactness and the σ-compactness of the space X implies existence of the increasing sequence of compact subsets of X the sum of which is X.

Let $\{H_i\}_{i=1}^{\infty}$ be a sequence of open and dense subsets of Y^X, f be a mapping from Y^X and $U = \tilde{U}(\varepsilon,K,(\delta,\{x_j\}_{j=1}^n))(f)$ be its neighborhood from the local base. To prove that Y^X is a Baire space, a mapping $h\in U$ must be found such that $h \in \bigcap_{i=1}^{\infty} H_i$.

The density of H_1 implies that there is $h_1\in H_1$ such that $h_1\in U$. There is a neighborhood of h_1 included in $H_1\cap U$, as this intersection is an open set. This neighborhood $U_1 =$ $= \tilde{U}(\varepsilon_1,K_{i_1},(\delta_1,\{x_j^1\}_{j=1}^{n_1}))(h_1)$ may be chosen such that

$\varepsilon_1 < \varepsilon/2$, $\{x_j(U,h_1)\}_{j=1}^{m} \subset \{x_j^1\}_{j=1}^{n_1} \subset K_{i_1}$, $K \subset K_{i_1}$, $\delta_1 < \delta/2$ and for each $x_j(U,h_1)$ holds: $\{x \mid \sigma(x,x_j(U,h_1)) < \delta_1\} \subset \subset \{x \mid \sigma(x,x_j) < \delta\}$.

Analogously there is a mapping $h_2 \in U_1 \cap H_2$ and there is its neighborhood included in $U_1 \cap H_2$. This neighborhood $U_2 = \tilde{U}(\varepsilon_2, K_{i_2}, (\delta_2, \{x_j^2\}_{j=1}^{n_2}))(h_2)$ may be chosen such that $\varepsilon_2 < \varepsilon_1/2$, $\{x_j^1(U_1,h_2)\}_{j=1}^{m_1} \subset \{x_j^2\}_{j=1}^{n_2} \subset K_{i_2}$, $K_{i_1} \subset K_{i_2}$, $\delta_2 < \delta_1/2$ and that for each $x_j^1(U_1,h_2)$ holds: $\{x \mid \sigma(x,x_j^1(U_1,h_2)) < \delta_2\} \subset \{x \mid \sigma(x,x_j^1) < \delta_1\}$. Analogously are defined mappings h_n and their neighborhoods U_n for all natural n.

Thus there is a sequence $\{U_n\}_{n=1}^{\infty}$ such that $U_{n+1} \subset U_n$ for every $n \in N$. This sequence forms a base of a filter $\mathcal{F}$ and this filter is a Cauchy filter in the uniformity of compact convergence of Y^X. The filter $\mathcal{F}$ converges to a mapping h. It will be shown that h is the mapping which is sought.

Let $U_n = \tilde{U}(\varepsilon_n, K_{i_n}, (\delta_n, \{x_j^n\}_{j=1}^{m_n}))(h_n)$ be a neighborhood from the sequence $\{U_n\}_{n=1}^{\infty}$. For each x_j^n the sequence $\{z_k\}_{k=0}^{\infty}$ will be constructed by induction such that $z_0 = x_j^n$ and supposing the element z_k has been defined, the element $z_{k+1} \in \in \{x_j^{n+k}(U_{n+k}, h_{n+k+1})\}_{j=1}^{m_{n+k}}$ is chosen such that $\sigma(z_k, z_{k+1}) \leq \leq \delta_{n+k}$ and $h_{n+k}(z_k) = h_{n+k+1}(z_{k+1})$. The sequence $\{z_k\}_{k=0}^{\infty}$ is a Cauchy sequence and is included in the closed δ_n-neighborhood of the point x_j^n. There is a limit point z of this sequence lying in this neighborhood.

Showing that $h(z) = h_n(x_j^n)$, it is proved that for each x_j^n there is $x_j^{n'}$ such that $\sigma(x_j^n, x_j^{n'}) \leq \delta_n$ and $h(x_j^{n'}) = = h_n(x_j^n)$. To show that $h(z) = h_n(x_j^n)$, it is sufficient to prove that $\wp(h(z),h_n(x_j^n)) = 0$. It is true that $\wp(h(z), h_n(x_j^n)) \leq \wp(h(z),h(z_k)) + \wp(h(z_k),h_{n+k}(z_k) + \wp(h_{n+k}(z_k),h_n(x_j^n)) = \wp(h(z),h(z_k)) + \wp(h(z_k),h_{n+k}(z_k))$, because $h_{n+k}(z_k) = h_n(x_j^n)$. The terms $\wp(h(z),h(z_k))$, $\wp(h(z_k),h_{n+k}(z_k))$ converge to zero as $k \to +\infty$ because h is a continuous mapping and because the sequence $\{h_{n+k}\}_{k=0}^{\infty}$ converges to h in the topology of pointwise convergence. Thus $\wp(h(z),h_n(x_j^n)) = 0$.

It is left to prove that $\wp(h(x),h_n(x)) \leq \varepsilon_n$ for each $x \in K_{i_n}$. As a limit point of the filter $\mathcal{F}$ is also a cluster point of this filter, h is an element of $\overline{U}_n^{TCC} \subset \{g \in Y^X \mid \wp(h_n(x),g(x)) \leq \varepsilon_n$ for $x \in K_{i_n}\}$.

Let us sum up what has been proved. It has been proved that $h \in U_n \subset U$ for each natural number n and that $U_n \subset H_n$. Hence $h \in \bigcap_{i=1}^{\infty} H_i \cap H$.

2.5. In this paragraph it will be proved that the space R^R is separable. At first we shall remind the definition of a piecewise linear function and a well-known lemma.

Definition. We shall call a continuous function f from R into R piecewise linear if there is a finite set S_f such that for each $x \in D(f) \setminus S_f$ a neighborhood of x can be found on which the function f is linear.

Lemma. Let $f:[T_0,T_1] \to R$ be a continuous function. Then for each $\varepsilon > 0$ there is a piecewise linear function $g_\varepsilon:[T_0,T_1] \to R$ such that $g_\varepsilon(T_0) = f(T_0)$, $g_\varepsilon(T_1) = f(T_1)$ and $\sup_{x \in [T_0,T_1]} |f(x) - g(x)| < \varepsilon$.

Proposition. For each $f \in R^R$ and each neighborhood U of f in the LCC-topology there is a piecewise linear function $g \in U$.

Proof. It may be supposed that $U = U(\varepsilon, K, (\sigma, \{x_j\}_{j=1}^m))(f)$, $K = [T_0, T_1]$, $\{x_j\}_{j=1}^m \subset K$ and $x_1 < \ldots < x_m$. Then on the intervals $[x_j, x_{j+1}]$ for $j = 1, \ldots, m - 1$ and on the intervals $[T_0, x_1]$, $[x_m, T_1]$ there are piecewise linear functions possessing the properties of the function g . Concatenation of these functions produces a piecewise linear function defined on the interval $[T_0, T_1]$. A new piecewise linear function can be defined on R such that its restriction to the interval $[T_0, T_1]$ is identical with the former function and that this latter function is an element of U.

Proposition. Let Q denote the set of all piecewise linear functions $f \in R^R$ such that S_f's are subsets of rational numbers and coefficients of linear parts of these functions are also rational numbers.

Then

i) Q is a countable set

ii) Q is a dense subset of the subspace of all piecewise linear functions.

Proposition. The function space R^R is separable.

Proof. The countable set Q is dense in the subspace of all piecewise linear functions and this subspace is dense in the space R^R. Hence Q is a dense subset of R^R.

2.6. The aim of this paragraph is to prove that the local character of R^R is countable and that generally the space Y^X is not normal.

Proposition. The local character of the space R^R is countable.

Proof. Let E_f denote the set of strict local extremes of $f \in R^R$, $\mathcal{K}$ system of all closed intervals which ends are integers, R_f the union of E_f and of the set of rational numbers. Then the family $\{U(1/n, K, (1/n, \{x_j\}_{j=1}^m))(f) \mid n \in N, \{x_j\}_{j=1}^m \subset R_f$ and $K \in \mathcal{K}\}$ is countable and forms a local base at f.

Lemma. The set of all constant mappings in Y^X is closed. The subspace formed by this set is discrete.

Proof. The set of all constant mappings is closed in the topology of compact convergence and thus also in the LCC-topology. The second part of the lemma follows from the definition of the LCC-topology.

Proposition. The space R^R is not normal.

Proof. As R^R is separable, there is only a continuum of continuous real functions defined on R^R. If the space R^R were a normal one, it would have to be at least 2^c of continuous real functions defined on R^R in order that each continuous function defined on the set of all constant mappings may be extended on the whole space R^R.

2.7. In this paragraph the arcwise connectedness of the space X^R will be examined. The final result is that providing X is an arcwise connected space, the space X^R is also an arcwise connected. The final result will follow from several propositions which will be proved at first.

Let $f:R \longrightarrow X$ be a continuous mapping. The mapping F: $: [0,\pi/2] \longrightarrow X^R$ is defined such that providing $t \neq \pi/2$

$$F(t)(x) = f(0) \text{ for } |x| \leq \operatorname{tg} t$$
$$= f(x - \operatorname{tg} t) \text{ for } x > \operatorname{tg} t$$
$$= f(x + \operatorname{tg} t) \text{ for } x < -\operatorname{tg} t$$

and for $t = \pi/2$ is $F(\pi/2)(x) = f(0)$.

<u>Proposition</u>. The mapping $F: [0,\pi/2] \rightarrow X^R$ is continuous in the L-topology on the space X^R.

Proof. Let $t \in [0, \pi/2]$ and $U = U(\sigma, \{x_i\}_{i=1}^n)$ is a neighborhood of $F(t)$. A neighborhood V of the point t is sought for which $F(V) \subset U$.

At the first part of the proof it will be assumed that $t \neq \pi/2$ and will be proved that supposing $u \in [0, \pi/2)$ is such that $|\operatorname{tg} u - \operatorname{tg} t| < \sigma$, then there is x_i' for each $i = 1,\dots$ $\dots,n$ such that $|x_i' - x_i| < \sigma$ and $F(u)(x_i') = F(t)(x_i)$.

Let x_i from the set $\{x_i\}_{i=1}^n$ be chosen. Without loss of generality it may be supposed that $x_i \geq 0$. Then either
1) $x_i \leq \operatorname{tg} t$ or 2) $x_i > \operatorname{tg} t$.

Ad 1) $F(t)(x_i) = f(0)$ and either $x_i \leq \operatorname{tg} u$ and thus $F(u)(x_i) = f(0)$ or $x_i > \operatorname{tg} u$ and x_i' may be set equal to tg u.

Ad 2) $F(t)(x_i) = f(x_i - \operatorname{tg} t)$ and x_i' may be set equal to $(x_i + \operatorname{tg} u - \operatorname{tg} t)$. Then $x_i' > \operatorname{tg} u$ and $F(u)(x_i') =$ $= f(x_i' - \operatorname{tg} u) = f(x_i - \operatorname{tg} t)$.

Continuity of the function tangens on the interval $[0,\pi/2)$ implies there is a neighborhood V of t such that for each $u \in V$ is $|\operatorname{tg} u - \operatorname{tg} t| < \sigma$.

The second part of the proof is for $t = \pi/2$. Then the neighborhood V is the set $\{u \in [0,\pi/2] \mid \operatorname{tg} u > x_i$ for every

$i = 1,\ldots,n\}$.

<u>Proposition</u>. The mapping F is continuous in the topology of compact convergence on X^R.

Proof. Let us define mappings $\Phi : [0, \pi/2] \times R \to X$, $\Phi(t,x) = F(t)(x)$, $\varphi : [0, \pi/2] \times R \to R$, for $t \neq \pi/2$ is $\varphi(t,x) = 0$ if $|x| \leq tg\ t$, $\varphi(t,x) = (x - tg\ t)$ if $x > tg\ t$, $\varphi(t,x) = (x + tg\ t)$ if $-x > tg\ t$ and for $t = \pi/2$ is $\varphi(\pi/2,x) = 0$. The mapping φ is continuous and thus $\Phi = f \circ \varphi$ is also continuous. Using the theorem of exponential correspondence gives the required result.

<u>Proposition</u>. The mapping F is continuous in the LCC-topology.

Proof. It follows from the properties of the projectively generated topology.

Let $g \in X^R$, $x_0 \in R$. Let us define a mapping $G: [0, \pi/2] \to X^R$ such that providing $t \neq \pi/2$

$$G(t)(x) = \begin{cases} g(x_0) & \text{for } |x - x_0| \leq tg\ t \\ g(x - tg\ t) & \text{for } |x - x_0| > tg\ t \\ g(x + tg\ t) & \text{for } |x_0 - x| > tg\ t \end{cases}$$

and for $t = \pi/2$, $G(\pi/2)(x) = g(x_0)$.

<u>Proposition</u>. The mapping G is continuous in the LCC-topology.

<u>Corollary</u>. Let $f \in X^R$, $g \in X^R$ and $f = \text{const}\ (g(x_0))$. Then there are continuous mappings $G: [0, \pi/2] \to X^R$ and $H: [0, \pi/2] \to X^R$ such that $G(0) = g$, $G(\pi/2) = f$, $H(0) = f$ and $H(\pi/2) = g$.

Proof. The map G is the mapping from the preceding proposition and $H = G \circ \psi$ where $\psi(t) = \pi/2 - t$ for $t \in$ $\in [0, \pi/2]$.

The corollary may be also stated as follows: Let f, g be mappings from X^R. If f is a constant mapping and if there is a point $x_0 \in R$ such that $f(x_0) = g(x_0)$ then the mappings f and g lie in the same arc-component of the space X^R.

It follows from this formulation that mapping f and g from X^R are in the same arc-component if there is a point $x_0 \in R$ such that $f(x_0) = g(x_0)$.

Proposition. If the space X is arcwise connected, then the space X^R is arcwise connected.

Proof. Let f, g be mappings from X^R. Let us choose points x_0 and y_0, $x_0 \neq y_0$. Arcwise connectedness of X implies there is a mapping $h \in X^R$ such that $h(x_0) = f(x_0)$ and $h(y_0) =$ $= g(y_0)$. The mappings f and h are in the same arc-component and also the mappings g and h are in the same arc-component. Thus f and g are in the same arc-component and the space X^R is arcwise connected.

Corollary. If the space X is arcwise connected, then the space X^R is connected.

2.8. Various algebraic operations are possible to define pointwise on the space R^X. It is easy to verify that the operation $f \longrightarrow -f$ is in the LCC-topology continuous. Let us now consider the operation of addition. If this operation were continuous, the space R^X would be a topological group. That it is not the case, demonstrates the following example. In this example even the space X is the space of real numbers, i.e. the space possessing a lot of nice topological properties, implying for example that R^R is a Tychonoff space.

Let f and g be two different constant mappings from R^R. Let $U = U(\varepsilon, K, (\delta, \{x_i\}_{i=1}^n))(f)$, $V = V(\eta, L, (\xi, \{y_i\}_{i=1}^m))(g)$ be their arbitrary neighborhoods from the local bases. There are $f' \in U$ and $g' \in V$ such that $f' > f$ for $x \in R - \{x_i'\}_{i=1}^n$ and $g' > g$ for $x \in R - \{y_i'\}_{i=1}^m$ and $\{x_i'\}_{i=1}^n \cap \{y_i'\}_{i=1}^m = 0$. Thus $f' + g' > f + g$ for all real numbers and $f' + g'$ cannot lie in any small enough neighborhood of $f + g$.

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