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METAMATHEMATICS OF THE ALTERNATIVE SET THEORY II

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Abstract: In the paper we continue in the investigation of metamathematics of the Alternative Set Theory which we began in [S 1]. We compare AST with formalizations of Cantor's set theory as far as it concerns equivalence of axiomatic systems and interpretability of one theory in the second one.

Key words: Alternative set theory, interpretation, KM^-, second an third order arithmetics, equivalence of axiomatic systems.

Classification: Primary 03E70, 03H99
Secondary 03H20

The alternative set theory (AST) as a formal system of axioms was introduced in [S 1] where even an introduction to the whole series can be found. We use the notions defined in [V] and [S 1].

In the fourth section we demonstrate that ZF_{Fin} is equivalent to the system of those axioms of AST which deal with sets only, and that the theories KM_{Fin} and $AST_{-5} + \neg A52$ are equivalent. Furthermore, we show some statements equivalent to the axiom of GB-class (A4) and to the schema of regularity (A8). At the end of this section we point out one difference between AST and the theory of semisets.

In the fifth section we investigate interpretations $\mathcal{a}$

corresponding to models $\mathfrak{A}$. We show that $\mathfrak{A}$ is an interpretation of AST_{-5} in TC + A6 + A7 + $\mathfrak{A} \models \not\approx \mathcal{T}_{Fin}$. Furthermore $\mathfrak{A}$ is an interpretation of the prolongation axiom iff $\mathfrak{A}$ is saturated (assuming $\mathfrak{A} \models \not\approx \mathcal{T}_{Fin}$) and every fully revealed model is saturated.

In § 6 we construct two particular interpretations using the **ultraproduct** construction and the method of trees and we show that the theories AST, KM^-, A_3 and TC + A51 + A61 are alike strong in the sense of interpretability. In the last section we demonstrate that the theories KM_{Fin} and AST_{-5} + A52 are strictly weaker than AST in the sense of interpretability using the statement that if T is a theory stronger than TC and if S is a recursive theory such that there is an interpretation of ZF_{Fin} + $Con(\overline{S})$ in T then the formula $Con_F(\mathcal{S})$ is provable in T.

§ 4. Equivalence of axiomatic systems. At the beginning of this section we show some statements which are equivalent to the axiom of GB-class. The axiom A4 guarantees the existence of GB-class such that the universal class is its sole η-element which is nearly universal. The following statements show that assuming A41, the axiom A4 is equivalent to the existence of the GB-class without η-elements which are proper semisets. Further, the following theorem shows that A4 is even equivalent to induction for finite set-formulas i.e. to the statement $(\forall \varphi \in FL_V)\ (V \models ((\varphi(0)\ \&\ (\forall x)(\forall y)(\varphi(x) \rightarrow \rightarrow \varphi(x \cup \{y\}))) \rightarrow (\forall x)\ \varphi(x)))$.

Lemma (BTC + A41). The following statements are equivalent:

(a) there is no nearly universal class different from V, i.e.

$(\forall x)(\mathrm{Nun}(X) \longrightarrow X = V)$

(b) axiom of replacement, i.e.

$(\forall F)(\forall x)(\mathrm{Set}(F''x))$

(c) there is no proper semiset, i.e.

$(\forall X)(\mathrm{Sms}(X) \longrightarrow \mathrm{Set}(X))$.

Proof. The implication (a) $\longrightarrow$(b) follows from the fact that $\mathrm{Set}(F''x) \longrightarrow \mathrm{Set}(F''(x \cup \{y\}))$. If $X \subseteq x$ then $X = (\mathrm{Id} \restriction X)''x$ and hence we get the second implication. To prove the remaining implication (c) $\longrightarrow$ (a) let us assume that $\mathrm{Nun}(X)$ & $x \in V-X$ holds. Put $Y = \{y \in X;\ y \subseteq x\}$. Obviously $Y \subseteq P(x)$ and the assumption $\mathrm{Set}(Y)$ contradicts the axiom of finity since for every $y \in Y$ we can choose $q \in (x-y)$ (because $x \notin Y$) and from this the formula $y \subset (y \cup \{q\}) \in Y$ follows.

Let us note that the first two implications are provable even in BTC, on the other hand for the proof of the last implication we used two particular cases of the axiom A41, namely the power-set axiom and the axiom of finity. These axioms are even necessary since e.g. Gödel-Bernays set theory is an extension of BTC + power-set axiom in which the negation of the axiom of finity (and hence even $\neg(\forall X)(\mathrm{Nun}(X) \longrightarrow X = V)$) holds.

Consequence (TC + A41). If S is a GB-class then the following conditions are equivalent:

(a) $(\forall X \,\eta\, S)(\mathrm{Nun}(X) \longrightarrow X = V)$

(b) $(\forall X \,\eta\, S)(\mathrm{Sms}(X) \longrightarrow \mathrm{Set}(X))$.

Let the symbol Sat denote the satisfaction class w.r.t.

the model $\{V,E,Id\}^{\eta}$ i.e.

$$\mathrm{Sat} = \{\langle\langle a_1,\dots,a_k\rangle, \varphi\rangle \,;\, \varphi\in FL_V \;\&\; V\models\varphi(a_1,\dots,a_k)\}.$$

<u>Theorem</u> (TC). The class $FL_V\times\{0\}\cup \mathrm{Sat}\times\{1\}$ is a GB-class and each its η-element is an η-element of every GB-class.

Proof. We have $E = \mathrm{Sat}''\{x\in y\}$ and $a = \mathrm{Sat}''\{x\in a\}$. Thus the well-known equalities e.g. $\mathrm{dom}(\mathrm{Sat}''\{\varphi(x,x_1,\dots,x_k)\}) = \mathrm{Sat}''\{(\exists x)\varphi\}$, $(\mathrm{Sat}''\{\varphi(x_1,x_2)\})^{-1} = \mathrm{Sat}''\{\varphi(x_2,x_1)\}$ and $\mathrm{Sat}''\{\varphi\} - \mathrm{Sat}''\{\psi\} = \mathrm{Sat}''\{\varphi\,\&\,\neg\psi\}$ show the first statement. The second one follows from the Bernays' metatheorem.

Let us now summarize some statements which are (in TC + A4) equivalent to the schema of regularity (A8):

(a) the schema of $\in$-induction i.e. the schema with the axiom $(\forall x)((\forall y\in x)(\Phi(y))\rightarrow\Phi(x))\rightarrow(\forall x)\Phi(x)$ for every set-formula Φ,

(b) the conjunction of the axiom of regularity (A81) and of the axiom of transitive closure (A82),

(c) the schema of regularity for (formal) finite set-formulas of the language FL_V i.e.

$$(\forall\varphi\in FL_V)(V\models((\exists x)(\varphi(x))\rightarrow(\exists x)(\varphi(x)\;\&\;(\forall y\in x)(\neg\varphi(y)))))$$

(d) the formula $(\exists S)(GB(S)\;\&\;(\forall X\,\eta\,S)(X\neq 0\rightarrow(\exists y\in X)(y\cap X=0)))$.

The equivalence of A8 and (a) was demonstrated in § 1 ch. I [V], the equivalence of (c) and (d) follows from the last theorem. The proof (b) $\rightarrow$ A8 can be found e.g. in § 1

[S 1], where we claimed that A82 is a particular case of A8. To be quite precise let us show it now: For every x with $(\forall y \in x)(\exists z)(\mathrm{Tran}(z)\ \&\ y \in z)$ there is a set b with $(\forall y \in x)(\exists z \in b)(\mathrm{Tran}(z)\ \&\ y \in z)$ - it is sufficient to collect sets with the desirable property and with the minimal type. Putting $c = P(\bigcup\{z \in b, \mathrm{Tran}(z)\})$ we get $x \in c\ \&$ $\&\ \mathrm{Tran}(c)$. Further since $V \models \mathcal{ZF}_{Fin}$, the statement (c) is equivalent to $V \models$ A 81 & A82 (the symbol Ai is used both for metaformula and its formalization; this would not lead to a misunderstanding) and moreover that last formula is equivalent to A81 & A82.

Let us note that we have proved that the axiom A8 is equivalent to the schema of regularity for finite set-formulas and in this point properties of the axioms A4 and A8 differ considerably since we shall see later that the axiom A4 is strictly stronger than the axiom A41. Further let us note that in ZF the axiom of transitive closure is provable (because the axiom of infinity is available) and hence as the axiom of regularity is accepted the axiom A81 alone. On the other hand, we shall show later that the axiom A82 is not provable in AST_{-8} + A81.

Now let us investigate the connection between AST and theories of finite sets which are obtained by formalizations of Cantor's ideas as far as it concerns the equivalence of axiomatic systems. Our first metatheorem asserts that ZF_{Fin} is equivalent to the system of those axioms of the alternative set theory which deal with sets only.

<u>Metatheorem</u>. ZF_{Fin} is equivalent to the theory with the

axioms A11, A3, A41 and A8.

Demonstration. The implication from right to left was shown in § 1 ch. I [V]. Using the equality $x \cup \{y\} = \bigcup\{x,\{y\}\}$ we see that the axiom A3 is a trivial consequence of the pairing and sum axioms. To prove A41 in ZF_{Fin} let us assume that $\Phi(x)$ is a set-formula and that there is a set x such that the conjunction $\Phi(0)$ & $(\forall y)(\forall q)(\Phi(y) \longrightarrow \Phi(y \cup \cup\{q\}))$ & $\neg\Phi(x)$ holds. Put $z = \{y;\ y \subseteq x\ \&\ \Phi(y)\}$, such a set exists by the power-set axiom and by the replacement schema. Evidently $0 \in z$ and moreover for $y \in z$ we have $x-y \neq 0$. Therefore for every $y \in x$ we can choose q with $q \in (x-y)$ and by the definition of z we get $y \subset (y \cup \{q\}) \in z$ which contradicts the axiom of finity.

Hence using the last statement and the first lemma of this section we see that the theory TC + A41 + A8 + + ¬ A52 (BTC + A41 + A8 + ¬ A52 respectively) is equivalent to the theory KM_{Fin} (GB_{Fin} respectively). But we are able to prove more, namely

<u>Metatheorem</u>. KM_{Fin} is equivalent to AST_{-5} + ¬ A52.

Demonstration. We have to prove the axioms A4, A6 and A7 in KM_{Fin}. The first required statement follows from the first lemma and the first theorem of this section. Let us proceed in KM_{Fin}. There is an one-one mapping of V onto N (cf. § 1 ch. II [V]) and thus we get A6 because N is well-ordered according to ¬ A52. Moreover we have Count(V) (cf. the definition of this notion in [S 1]) and hence there is no uncountable class and thence the axiom A7 is satisfied trivially.

AST admits proper semisets and in this aspect this theory is similar to the theory of semisets (TSS, see [V-H]). In TSS (with an axiom of regularity) the statement

$(*)\quad (\forall x)(\mathrm{Set}(X \cap x)) \longrightarrow (\forall x)(\mathrm{Set}(\mathrm{dom}(X) \cap x))$

is provable. Since the formula $(*)$ is simpler than our axiom A4 it seems to be convenient to strengthen AST assuming the axiom $(*)$ instead of A4. Unfortunately, this is impossible since such a theory would be inconsistent as the following result shows (cf. [So 3]).

<u>Theorem (AST)</u>. For every $\alpha \in (N - FN)$ there is an increasing function F with $\mathrm{dom}(F) \subseteq \alpha$ & $\neg\, \mathrm{Sms}(\mathrm{rng}(F))$ such that we have $(\forall x)(\mathrm{Set}(x \cap F))$.

Proof. The class Ω of all ordinal numbers was defined in § 3 ch. II [V] in such a way that $\neg\, \mathrm{Sms}(\Omega)$. Let $\alpha \in (N - FN)$ be given. Then the class $R = \{\beta; (\forall n \in FN)(\beta + n < \alpha)\}$ is a π-class. This class has no maximal element and hence it is no set. Therefore according to the last theorem of § 5 ch. II [V] we can by induction (cf. § 3 ch. II [V]) construct an increasing function G such that $\mathrm{dom}(G) = \Omega$ & & $\bigcup G''\Omega = R$. Moreover, by induction, we can construct an increasing function f_β for every $\beta \in \Omega$ so that

a) $f_\beta \subseteq \beta \times G(\beta)$

b) $f_{\beta+1} = f_\beta \cup \{\langle \beta, G(\beta)\rangle\}$

c) $(\forall \gamma \in (\beta \cap \Omega))(f_\gamma = f_\beta \restriction (\bigcup \mathrm{dom}(f_\gamma) + 1))$.

In fact, if β is a limit ordinal (i.e. $(\forall \gamma \in \Omega)(\gamma + 1 \neq \beta)$), then we put $Z = \{x; (\exists f)($"f is an increasing function" & $f \subseteq \beta \times G(\beta)$ & $(\forall y \in x)(y = f \restriction (\bigcup \mathrm{dom}(y) + 1))\}$ and $X = \{x; 0 \neq x \subseteq \{f_\gamma : \gamma \in (\beta \cap \Omega)\}$ & $\mathrm{Fin}(x)\}$. Thus X is a countable subsemiset of Z which is directed and Z is a set-theoretically definable class. Hence by the theorem on countable directed semisets (see § 4 ch. I [V]) we are able to choose $x \in Z$ so that $\{f_\gamma ; \gamma \in (\beta \cap \Omega)\} \subseteq x$ and put $f_\beta = \bigcup x$.

Finally we put $F = \bigcup \{f_\beta ; \beta \in \Omega\}$. Then $\mathrm{dom}(F) \subseteq \alpha$ and $\bigcup \mathrm{rng}(F) = N$ and, moreover, F is an increasing function. Therefore we have only to prove that the formula $(\forall x)(\mathrm{Set}(x \cap F)$ holds. Let $x \subseteq N^2$ be given. There is $\beta \in \Omega$ with $\mathrm{rng}(x) \subseteq \beta$ and using the properties of functions f_γ, we get $(F - f_\beta) \cap x = 0$ and thence the class $F \cap x = f_\beta \cap x$ is a set.

§ 5. Interpretations. Let A01 denote the formula $(\forall x,y,z)((x = y \;\&\; x \in z) \rightarrow y \in z)$. If $\mathcal{O}\!\mathcal{L} = \{A, \widetilde{E}, \check{I}\}^{\eta}$ is a model then for every X we define $X_{\mathcal{O}\!\mathcal{L}} = \{y; (\exists x \in X)(\mathcal{O}\!\mathcal{L} \models x = y)\}$ (the saturation of X). Let us recall that the interpretation $\mathcal{a}$ was defined in § 3 [S 1]. For every X we have evidently $\mathrm{Cls}^{\mathcal{a}}(X_{\mathcal{O}\!\mathcal{L}})$ and, moreover, if $\mathcal{O}\!\mathcal{L} \models$ A01 then $\mathrm{Cls}^{\mathcal{a}}(\widetilde{E}``\{z\})$ for every $z \in A$.

Lemma (TC). If $\mathcal{O}\!\mathcal{L} \models$ A01 & A11 & A3 then $\mathrm{Fin}^{\mathcal{a}}(X^{\mathcal{a}}) \equiv (\exists X)(\mathrm{Fin}(X) \;\&\; X^{\mathcal{a}} = X_{\mathcal{O}\!\mathcal{L}})$; and further $\mathrm{Count}(X) \rightarrow (\mathrm{Count}^{\mathcal{a}}(X_{\mathcal{O}\!\mathcal{L}}) \vee \mathrm{Fin}^{\mathcal{a}}(X_{\mathcal{O}\!\mathcal{L}}))$. Supposing moreover the axiom A61 we get even $\mathrm{Count}^{\mathcal{a}}(X^{\mathcal{a}}) \rightarrow (\exists X)(\mathrm{Count}(X) \;\&\; X^{\mathcal{a}} = X_{\mathcal{O}\!\mathcal{L}})$.

Proof. If $\mathrm{Fin}(X) \;\&\; \neg\, \mathrm{Fin}^{\mathcal{a}}(X_{\mathcal{O}\!\mathcal{L}})$ then the class $\{q; q \subseteq X \;\&\; \neg\, \mathrm{Fin}^{\mathcal{a}}(q_{\mathcal{O}\!\mathcal{L}})\}$ is not empty and hence there is a minimal (w.r.t. $\subseteq$) element of this class which contradicts the assumption $\mathcal{O}\!\mathcal{L} \models$ A3.

Assuming $Fin^{\mathcal{A}}(X^{\mathcal{A}})$ we put $Z = \{y \in A; \breve{E}''\{y\} \subseteq X^{\mathcal{A}}$ & $\neg(\exists Y)(Fin(Y)$ & $\tilde{E}''\{y\} = Y_{\mathcal{A}})\}$. If there would exist no X with $Fin(X)$ & $X^{\mathcal{A}} = X_{\mathcal{A}}$ then $Z \neq 0$ & $Cls^{\mathcal{A}}(Z)$ & $(Z \subseteq P(X^{\mathcal{A}}))^{\mathcal{A}}$. Thus there would be $y \in Z$ with $(\forall z)(\tilde{E}''\{z\} \subset \tilde{E}''\{y\} \rightarrow z \notin Z)$ which contradicts the definition of Z.

If $R \subseteq A^2$ is a well-ordering of an infinite class X which has finite segments only, then
$\{x; (\exists y,z)(\langle y,z\rangle \in R$ & $\mathcal{A} \models x = \langle y,z\rangle)$ & $(\forall u,v)((\mathcal{A} \models y = u$ & $z = v) \rightarrow (\langle y,u\rangle \in R$ & $\langle z,v\rangle \in R))\}$
is an $\mathcal{A}$-well-ordering of the class $X_{\mathcal{A}}$ and, moreover, this ordering has only $\mathcal{A}$-finite segments by the first statement of this lemma and thence $Fin^{\mathcal{A}}(X_{\mathcal{A}}) \vee Count^{\mathcal{A}}(X_{\mathcal{A}})$. Let us note that if $\tilde{I}$ is a partialization of identity, then we have, moreover, $Count(X) \rightarrow Count^{\mathcal{A}}(X_{\mathcal{A}})$.

Let $R^{\mathcal{A}}$ be an $\mathcal{A}$-well-ordering of an $\mathcal{A}$-infinite class $X^{\mathcal{A}}$ having $\mathcal{A}$-finite segments only. Put $Z = \{\langle x,n\rangle; (\exists y)(y \approx n+1$ & $(\forall z \subset y)(z_{\mathcal{A}} \neq y_{\mathcal{A}} = \{z; (\langle z,x\rangle \in R^{\mathcal{A}})^{\mathcal{A}}\})$ & $n \in FN\}$. Evidently $X^{\mathcal{A}} = rng(Z)$ & $dom(Z) = FN$ & $(\forall n \in FN)(\forall x,y)((\langle x,n\rangle \in Z$ & $\langle y,n\rangle \in Z) \rightarrow \mathcal{A} \models x = y)$. Therefore, according to the axiom A61 we can choose $F \subseteq Z$ with $dom(F) = FN$ and for this function we get $(rng(F))_{\mathcal{A}} = X^{\mathcal{A}}$.

Lemma (TC). If $\mathcal{A} \vdash$ A01 & A11 & A3 then for every set-formula φ of the language FL and every $a_1, \ldots, a_n \in A$ we have $(V \vdash \varphi(\tilde{E}''\{a_1\}, \ldots, \tilde{E}''\{a_n\}))^{\mathcal{A}} \equiv \mathcal{A} \vdash \varphi(a_1, \ldots, a_n)$.
In particular, if φ is a formalization of a set-sentence Φ then $\Phi^{\mathcal{A}} \equiv \mathcal{A} \models \varphi$.

Proof can be done by induction. In fact, $(V \vdash \breve{E}''\{a_1\} \in \breve{E}''\{a_2\})^{\mathcal{A}} \equiv \breve{E}''\{a_1\} \in^{\mathcal{A}} \breve{E}''\{a_2\} \equiv (\exists a_3 \in \breve{E}''\{a_2\})(\tilde{E}''\{a_1\} = \tilde{E}''\{a_3\}) \equiv (\exists a_3 \in \breve{E}''\{a_2\})(\mathcal{A} \models a_1 = a_3) \equiv (\exists a_3)(\mathcal{A} \vdash a_1 =$

$= a_3$ & $a_3 \in a_2) \equiv \mathcal{O}\!\mathcal{L} \models a_1 \in a_2$. Further we have
$(V \models (\exists x)\ \varphi\ (x, \widetilde{E}''\{a_1\}, \dots, \widetilde{E}''\{a_n\}))^{\mathcal{a}} \equiv ((\exists x)(V \models \varphi\ (x, \widetilde{E}''\{a_1\},$
$\dots, \widetilde{E}''\{a_n\})))^{\mathcal{a}} \equiv (\exists a \in A)(V \models \varphi\ (\widetilde{E}''\{a\}, \widetilde{E}''\{a_1\}, \dots$
$\dots, \widetilde{E}''\{a_n\})^{\mathcal{a}} \equiv (\exists a \in A)(\mathcal{O}\!\mathcal{L} \models \varphi\ (a, a_1, \dots, a_n)) \equiv$
$\equiv \mathcal{O}\!\mathcal{L} \models (\exists x)\ \varphi\ (x, a_1, \dots, a_n)$.

<u>Metatheorem</u>. $\mathcal{a}$ is an interpretation of TC in TC + + $\mathcal{O}\!\mathcal{L} \models$ A01 & A11 & A3. Moreover, the formulas $(\mathcal{O}\!\mathcal{L} \models A41) \rightarrow \rightarrow A4^{\mathcal{a}}$, $A6 \rightarrow A6^{\mathcal{a}}$ and $(A6\ \&\ A7) \rightarrow A7^{\mathcal{a}}$ are provable in the later mentioned theory.

Demonstration. Let us proceed in the theory in question. According to the definition of $\mathcal{a}$-classes we have $((\forall X, Y, Z)((X = Y\ \&\ X \in Z) \rightarrow Y \in Z))^{\mathcal{a}}$. If $x \in X^{\mathcal{a}} - Y^{\mathcal{a}}$ then $\widetilde{E}''\{x\} \in^{\mathcal{a}} X^{\mathcal{a}}$. Assuming $\widetilde{E}''\{x\} \in^{\mathcal{a}} Y^{\mathcal{a}}$ we would get a set y with $y \in Y^{\mathcal{a}}$ & $\widetilde{E}''\{y\} = \widetilde{E}''\{x\}$ and hence it would the formula $\mathcal{O}\!\mathcal{L} \models x = y$ hold, which contradicts the definition of $\mathcal{a}$-classes. Therefore we have proved $A1^{\mathcal{a}}$.

The formula $A2^{\mathcal{a}}$ is a trivial consequence of the axiom A2; the statement $A3^{\mathcal{a}}$ follows from the assumption $\mathcal{O}\!\mathcal{L} \models$ A11 & & A3. The last lemma implies the implication $(\mathcal{O}\!\mathcal{L} \models A41) \rightarrow \rightarrow A4^{\mathcal{a}}$ according to the fourth section. If $\preceq$ is a well-ordering of V, we put $R = \{z; (\exists x, y)(x \preceq y\ \&\ \mathcal{O}\!\mathcal{L} \models z = = \langle x, y \rangle\ \&\ (\forall u, v)((\mathcal{O}\!\mathcal{L} \models x = u\ \&\ y = v) \rightarrow (x \preceq u\ \&\ y \preceq \preceq v)))\}$. Evidently $Cls^{\mathcal{a}}(R)$ and R is an $\mathcal{a}$-well-ordering of $V^{\mathcal{a}}$. This shows the implication $A6 \rightarrow A6^{\mathcal{a}}$.

If $X^{\mathcal{a}}$ and $Y^{\mathcal{a}}$ are $\mathcal{a}$-uncountable classes then by A6 we are able to choose minimal X, Y with $X_{\mathcal{O}\!\mathcal{L}} = X^{\mathcal{a}}$, $Y_{\mathcal{O}\!\mathcal{L}} = Y^{\mathcal{a}}$ and, moreover, X and Y are uncountable by the first lemma of this section. According to the axiom A7 there is a one-

one mapping F of X onto Y and thus the class $F^{\mathcal{A}} = \{z; (\exists x,y)(\langle x,y\rangle \in F \ \& \ \mathcal{A} \models z = \langle x,y\rangle\}$ is an $\mathcal{A}$-one-one mapping of $X^{\mathcal{A}}$ onto $Y^{\mathcal{A}}$.

Let us note that in particular the previous results show that $\mathcal{A}$ is an interpretation of TC + A4 in TC + $\mathcal{A} \models \models \mathcal{ZF}_{Fin}$ and therefore even in ZF + $\mathcal{A} \models \mathcal{ZF}_{Fin}$.

Following the definition of "$\aleph_1$-saturatued models" which is usual in the model theory in ZF, we define:

Definition (TC). A model $\mathcal{A} = \{A, \tilde{E}, \tilde{I}\}^{\eta}$ is saturated iff for every sequence $\{\varphi_n; n \in FN\}$ of formulas of the language FL_A we have $(\forall n)(\exists x \in A)(\mathcal{A} \models \varphi_0(x) \ \& \ \dots \ \& \ \& \ \varphi_n(x)) \rightarrow (\exists x \in A)(\forall n)(\mathcal{A} \models \varphi_n(x))$.

<u>Theorem</u> (TC + A61). If $\mathcal{A} \models \mathcal{ZF}_{Fin}$ then $\mathcal{A}$ is saturated iff the formula $A5^{\mathcal{A}}$ holds.

Proof. Let us assume at first that the formula $A5^{\mathcal{A}}$ holds and that $(\forall n)(\exists x \in A)(\mathcal{A} \models \varphi_0(x) \ \& \ \dots \ \& \ \varphi_n(x))$. For every $n \in FN$ we can choose an $\mathcal{A}$-natural number α_n so that the formula $(\exists x \in A)(\mathcal{A} \models x \in \overline{P}(\alpha_n) \ \& \ \varphi_0(x) \ \& \ \dots \ \dots \ \& \ \varphi_n(x))$ holds. According to the last results $\mathcal{A}$ is an interpretation of TC + A4 + A5 + A8 in our theory and hence there is an $\mathcal{A}$-natural number α with $(\forall n)(\mathcal{A} \models \alpha_n < \alpha)$ by the second theorem of the last part of § 4 ch. I [V] and by the first lemma of this section. Put $x_n^{\mathcal{A}} = \{y; \mathcal{A} \models y \in \overline{P}(\alpha) \ \& \ \varphi_0(y) \ \& \ \dots \ \& \ \varphi_n(y)\}$. Since $\mathcal{A} \models \mathcal{ZF}_{Fin}$ we get $(Set(x_n^{\mathcal{A}}) \ \& \ x_n^{\mathcal{A}} \neq 0 \ \& \ x_{n+1}^{\mathcal{A}} \subseteq x_n^{\mathcal{A}})^{\mathcal{A}}$ for every $n \in FN$. Thus we obtain $\bigcap \{x_n^{\mathcal{A}}; n \in FN\} \cap A \neq 0$ by the fifth theorem of the last part of § 4 ch. I [V].

On the other hand, let us suppose that $\mathcal{A}$ is a saturated model. According to the first lemma of this section for every $F^{\mathcal{A}}$ with $\mathrm{Count}^{\mathcal{A}}(F^{\mathcal{A}})$ there is a countable class X with $X_{\mathcal{O}\!\mathcal{L}} = F^{\mathcal{A}}$. Furthermore, for every $x \subseteq X$ there is $a \in A$ with $(\forall y \in x)(\mathcal{O}\!\mathcal{L} \models y \in a \ \&\ \mathrm{Fnc}(a))$ and thus there is $a \in A$ with $(\forall x \in X)(\mathcal{O}\!\mathcal{L} \models (\mathrm{Fnc}(a) \ \&\ x \in a))$ from which $F^{\mathcal{A}} \subseteq^{\mathcal{A}} \tilde{E}"\{a\}$ follows.

The following notions were defined in § 5 ch. II [V], § 2 [S-V 1] and § 2 [S-V 2]: A class is revealed iff for each countable $Y \subseteq X$ there is a set u so that $Y \subseteq u \subseteq X$ and a class X is called fully revealed iff for every normal formula $\varphi(z,Z)$ of the language FL the class $\{x; \varphi(x,X)\}$ is revealed. A class X is called a revealment of a class Y iff X is a fully revealed class such that for every normal formula $\varphi(Z)$ of the language FL we have $\varphi(X) \equiv \varphi(Y)$. Let us note that if $\mathcal{B}$ is a revealment of a model $\mathcal{O}\!\mathcal{L}$ then $\mathcal{O}\!\mathcal{L}$ and $\mathcal{B}$ are elementarily equivalent. In [S-V 2] it was shown that in AST every class has a revealment; in particular, there is a lot of models which are fully revealed.

<u>Theorem</u> (AST). If $\mathcal{O}\!\mathcal{L} \models$ A01 & A11 & A3 is fully revealed, then the formula $A5^{\mathcal{A}}$ holds.

Proof. If $\mathrm{Count}^{\mathcal{A}}(F^{\mathcal{A}})$ & $\mathrm{Fnc}^{\mathcal{A}}(F^{\mathcal{A}})$ then by the first lemma of this section there is a function G of FN into A with $F^{\mathcal{A}} = (\mathrm{rng}(G))_{\mathcal{O}\!\mathcal{L}}$. According to the prolongation axiom there is $g \supseteq G$. The class $\{\alpha \in \mathrm{dom}(g); (\exists x \in A)((g"\alpha)_{\mathcal{O}\!\mathcal{L}} \subseteq^{\mathcal{A}} \tilde{E}"\{x\} \ \&\ \mathcal{O}\!\mathcal{L} \models \mathrm{Fnc}(x))\}$ is revealed and it contains FN and hence we can choose $\alpha \notin$ FN and $a \in A$ with $(g"\alpha)_{\mathcal{O}\!\mathcal{L}} \subseteq^{\mathcal{A}} \tilde{E}"\{a\}$ & $\mathcal{O}\!\mathcal{L} \models \mathrm{Fnc}(a)$, from which $(\mathrm{Set}(\tilde{E}"\{a\})$ & $F^{\mathcal{A}} \subseteq \tilde{E}"\{a\}$ & $\mathrm{Fnc}(\tilde{E}"\{a\}))^{\mathcal{A}}$ follows.

Theorem (AST). Every two elementarily equivalent saturated models are isomorphic.

Proof can be done using the usual model-theoretic argument (cf. also the first part of § 1 ch. V [V]). If $\mathcal{A} = \{A, \widetilde{E}, \widetilde{I}\}^{\eta}$ and $\mathcal{B} = \{B, \widetilde{E}_1, \widetilde{I}_1\}^{\eta}$ then A and B can be well-ordered by type Ω since there is no countable saturated model. By induction we are able to construct a sequence $\{F_\alpha ; \alpha \in \Omega\}$ of countable functions so that for every α, $\beta \in \Omega$ it is

(a) $\mathrm{dom}(F_\alpha)$ ($\mathrm{rng}(F_\alpha)$ respectively) contains first α elements of A (B respectively)

(b) $\alpha < \beta \longrightarrow F_\alpha \subseteq F_\beta$

(c) if $\varphi \in FL$ and if $x_1, \dots, x_n \in \mathrm{dom}(F_\alpha)$ then
$\mathcal{A} \models \varphi(x_1, \dots, x_n) \equiv \mathcal{B} \models \varphi(F_\alpha(x_1), \dots, F_\alpha(x_n))$.

Let us note that the countability of functions F_α enables us to code these functions by sets and hence the usual induction works. In fact, if $a \in A - \mathrm{dom}(F_\alpha)$ is given, then we define $\Gamma = \{\varphi(z, a_1, \dots, a_n); \varphi \in FL \ \& \ a_1, \dots, a_n \in \mathrm{dom}(F_\alpha) \ \& \ \mathcal{A} \models \varphi(a, a_1, \dots, a_n)\}$. Then Γ is countable and if formulas $\varphi_1(z, a_1, \dots, a_{n_1}), \dots, \varphi_k(z, a_1, \dots, a_{n_k})$ are elements of Γ, then

$\mathcal{B} \models (\exists x)(\varphi_1(x, F_\alpha(a_1), \dots, F_\alpha(a_{n_1})) \ \& \ \dots \ \& \ \varphi_k(x, F_\alpha(a_1), \dots, F_\alpha(a_{n_k})))$ thus using the fact that $\mathcal{B}$ is saturated, there is $b \in B$ so that $\varphi(z, a_1, \dots, a_n) \in \Gamma \longrightarrow \mathcal{B} \models \varphi(b, F_\alpha(a_1), \dots, F_\alpha(a_n))$. This enables us to extend the mapping F_α.

Consequence. In AST + $A5^{\mathcal{A}}$ + $A5^{\mathcal{B}}$ + $\mathcal{A} \models \neg\mathcal{J}_{Fin}$ + + $\mathcal{A} \equiv \mathcal{B}$ we can prove $\Phi^{\mathcal{A}} \equiv \Phi^{\mathcal{B}}$ for every (even nonnormal) formula Φ .

Demontration. In the theory in question we are able to fix an isomorphism H of $\mathcal{A}$ onto $\mathcal{B}$. Further, by induction, we can show $\Psi^{\mathcal{A}}(X_1^{\mathcal{A}},\dots,X_n^{\mathcal{A}}) \equiv \Psi^{\mathcal{B}}(H''(X_1^{\mathcal{A}}),\dots$ $\dots,H''(X_n^{\mathcal{A}}))$.

§ 6. Interpretability. At first we are going to imitate the usual ultrapower construction (see e.g. [B-S]) in TC with the following two assumptions:

(a) A6 & $\mathcal{A} = \{A,\tilde{E},\tilde{I}\}^{\eta}$ & $\mathcal{A} \vdash$ A01 & A11 & A3

(b) All functions of FN into A are η-elements of a class $B = K\times\{0\}\cup S\times\{1\}$.

Let us realize that a class B satisfying our second condition can be fixed in TC + A51 for every countable A. In fact, $A\times FN$ is countable in this case and hence there are X and G with $(\forall Y\subseteq X)(\exists y)(Y = y\cap X)$ & $Fnc(G^{-1})$ & $G''X = A\times FN$. Therefore, all subclasses of $A\times FN$ are η-elements of the class $V\times\{0\}\cup\{\langle z,y\rangle;\ z\in G''(X\cap y)\}\times\{1\}$. Further let us note that all functions $F\subseteq A\times FN$ are η-elements of the class $V\times\{0\}\cup\{\langle x,f\rangle;\ x\in f\restriction FN\}\times\{1\}$ in TC + A5 and that both AST and KM^- are extensions of the theory in question ($KM^-\vdash$ A5 according to Set(ω) and according to the replacement schema).

Using the axiom of choice and our assumption (b) we can construct as usual (cf. e.g. § 4 ch. II [V]) a non-trivial ultrafilter Z with FN η Z, i.e. a class Z such that

(i) $(\forall X,Y\subseteq FN)((X\ \eta\ Z\ \&\ Y\ \eta\ Z) \rightarrow (X\cap Y)\ \eta\ Z)$,

(ii) $(\forall X\subseteq FN)(X\ \eta\ Z \equiv \neg(FN-X)\ \eta\ Z)$ and

(iii) $(\forall m \in FN)(\{n; n > m\} \eta Z)$.

We define

$\bar{A} = \{x \in K; Fnc(S''\{x\}) \& \{n; S''\{x\}(n) \in A\} \eta Z\}$

$\bar{E} = \{\langle x,y\rangle; x,y \in \bar{A} \& \{n; \mathfrak{A} \models S''\{x\}(n) \in S''\{y\}(n)\} \eta Z\}$

$\bar{I} = \{\langle x,y\rangle; x,y \in \bar{A} \& \{n; \mathfrak{A} \models S''\{x\}(n) = S''\{y\}(n)\} \eta Z\}$

and we put $\bar{\mathfrak{A}} = \{\bar{A},\bar{E},\bar{I}\}^{\eta} = Ul(\mathfrak{A},B,Z)$.

By induction we can prove Łoš's theorem (cf. e.g. ch. 5 [B-S]) and hence for every $\varphi \in FL$ and every $x_1,\dots,x_n \in \bar{A}$ we have $\bar{\mathfrak{A}} \models \varphi(x_1,\dots,x_k)$ iff $\{n; \mathfrak{A} \models \varphi(S''\{x_1\}(n),\dots$ $\dots,S''\{x_k\}(n))\} \eta Z$.

We are going to repeat from the proof of Łoš's theorem only the nontrivial step concerning existential quantifier. Let $\{n; \mathfrak{A} \models (\exists x)\ \varphi(x,S''\{x_1\}(n),\dots,S''\{x_k\}(n))\} \eta Z$. By the axiom of choice there is F with dom(F) = FN and such that $\{n; \mathfrak{A} \models \varphi(F(n),S''\{x_1\}(n),\dots,S''\{x_k\}(n))\} \eta Z$. Since all functions of FN into A are η-elements of B, there is $x \in \bar{A}$ with $S''\{x\} = F$ and thence by the induction hypothesis we get $\bar{\mathfrak{A}} \models \varphi(x,x_1,\dots,x_k)$.

Now we show $A5^{\bar{\mathfrak{a}}}$. Let $(Count(F^{\bar{\mathfrak{a}}}) \& Fnc(F^{\bar{\mathfrak{a}}}))^{\bar{\mathfrak{a}}}$. There is $X = \{a_n; n \in FN\} \subseteq \bar{A}$ with $F^{\bar{\mathfrak{a}}} = X_{\bar{\mathfrak{A}}}$ according to the first lemma of the last section. Put $Z_n = \{m \geq n; (\exists a \in A)(\forall k < n)(\mathfrak{A} \models$ $\models Fnc(a) \& S''\{a_k\}(m) \in a)\}$ for every $n \in FN$. Evidently $Z_n \supseteq Z_{n+1}$ and $\bigcap\{Z_n; n \in FN\} = 0$ and moreover $Z_n \eta Z$ because $F^{\bar{\mathfrak{a}}}$ is a $\bar{\mathfrak{a}}$-function. Furthermore, there is a function G such that $dom(G) = FN \& (\forall x \in (Z_n - Z_{n+1}))(\forall k < n)(\mathfrak{A} \models S''\{a_k\}(x) \in$ $\in G(x) \& Fnc(G(x)))$. By our second assumption there is $a \in \bar{A}$ with $(\forall n \in FN)(S''\{a\}(n) = G(n))$. Evidently $\bar{\mathfrak{A}} \models Fnc(a)$ and for each $n \in FN$ we have $\{m; \mathfrak{A} \models S''\{a_n\}(m) \in S''\{a\}(m)\} \supseteq$

$\supseteq Z_n \eta Z$ from which $\overline{\mathfrak{U}} \models a_n \in a$ follows. As a consequence of the last statement we get $F^{\overline{\mathfrak{a}}} \subseteq^{\overline{\mathfrak{U}}} \overline{E}''\{a\}$ which finishes the proof of the statement $A5^{\overline{\mathfrak{U}}}$.

By § 3, $\check{\mathcal{F}}\mathcal{W}$ is a countable model of $\mathcal{Z}\mathcal{F}_{Fin}$. Thus the previous considerations and results of the fifth section show in particular the following statements.

Metatheorem. There is an interpretation of AST in TC + + A51 + A6 + A7 (and hence even in KM^- + A6 + A7).

Now we are going to construct an interpretation of KM^- in TC + A51 + A61 (and thus even in AST). In the constructi-on Gandy's and Zbierski's ideas are used (see [Z]).

Functions F and G are called isomorphic iff there is a one-one mapping H of $dom(F) \cup rng(F)$ onto $dom(G) \cup rng(G)$ such that $(\forall x,y)(y = F(x) \equiv H(y) = G(H(x)))$.

According to the axiom A51 there is a countable class A with $(\forall X \subseteq A)(\exists a)(X = a \cap A)$. Since the class FN^2 is countab-le, we can enumerate A so that $A = \{a_{n,m}; m \in FN\}$ and interprete $a_{n,m}$ as a code for $\langle n,m \rangle$. We put $\tilde{a} = \{\langle n,m\rangle; a_{n,m} \in a\}$ and let $A^{\#}$ denote the class of all a such that

(a) $\tilde{a}$ is a function which has no non-identical automorphism

(b) $(\exists ! n)(n \in (rng(\tilde{a}) - dom(\tilde{a})))$

(c) $\tilde{a}$ is well-founded, i.e. $(\forall X \subseteq dom(\tilde{a}))(X \neq 0 \rightarrow \neg X \subseteq \tilde{a}''X)$.

The element n with $n \in (rng(\tilde{a}) - dom(\tilde{a}))$ is called a-maxi-mal and elements x for which $\tilde{a}(x) = n$ are called a-almost ma-ximal. For $a \in A^{\#}$ we put $\tilde{a} \upharpoonright\upharpoonright x = \tilde{a} \upharpoonright \{y; (\exists n \in FN)\ (\underbrace{\tilde{a}(\ldots(\tilde{a}}_{n\text{-times}}(y)\ldots) = x\}$ and at the end we define $E^{\#}$ as the class of all ordered pairs $\langle a,b \rangle$ of elements of $A^{\#}$ such that there is b-almost maximal element n such that $\tilde{b} \upharpoonright\upharpoonright n$ is isomorphic to $\tilde{a}$

and $I^{\#}$ as the class of all ordered pairs $\langle a,b\rangle \in (A^{\#})^2$ such that $\widetilde{a}$ and $\widetilde{b}$ are isomorphic.

Lemma (TC + A51). If $a,b \in A^{\#}$ have the same $\#$-elements (i.e. $(E^{\#})''\{a\} = (E^{\#})''\{b\}$) then $\langle a,b\rangle \in I^{\#}$.

Proof. Under our assumption for every a-almost maximal element n there is a b-almost maximal element m so that $\widetilde{a} \upharpoonright\upharpoonright n$ and $\widetilde{b} \upharpoonright\upharpoonright m$ are isomorphic. This element is determined uniquely according to our requirement (a); from the same reason there is only one such isomorphism. Therefore we can extend these "partial" isomorphisms to an isomorphism of $\widetilde{a}$ onto $\widetilde{b}$.

Lemma (TC + A51). If X is a subclass of $A^{\#}$ which is either finite or countable, then there is $a \in A^{\#}$ which $\#$-contains exactly all elements of X, i.e.

$(\forall b \in A^{\#})(\langle b,a\rangle \in E^{\#} \equiv (\exists c \in X)(\langle c,b\rangle \in I^{\#}))$.

Proof. Let $\trianglelefteq$ be a well-ordering of X and if X is countable then we require, moreover, that $\trianglelefteq$ is a well-ordering of type ω. By induction w.r.t. $\trianglelefteq$ to every $a \in X$ we construct $a_1 \in A^{\#}$ so that for every $b < a$, $\widetilde{a}$ is isomorphic to $\widetilde{a}_1$ and we have (i) $0 \notin (\mathrm{dom}(\widetilde{a}_1) \cup \mathrm{rng}(\widetilde{a}_1))$, (ii) if $n \in (\mathrm{dom}(\widetilde{a}_1) \cup \mathrm{rng}(\widetilde{a}_1)) \cap (\mathrm{dom}(\widetilde{b}_1) \cup \mathrm{rng}(\widetilde{b}_1))$ then $\widetilde{a}_1 \upharpoonright\upharpoonright n = \widetilde{b}_1 \upharpoonright\upharpoonright n$ and (iii) if $\widetilde{a}_1 \upharpoonright\upharpoonright m$ is isomorphic to $\widetilde{b}_1 \upharpoonright\upharpoonright n$ then $\widetilde{a}_1 \upharpoonright\upharpoonright m = \widetilde{b}_1 \upharpoonright\upharpoonright$. We put $F = \bigcup\{\widetilde{a}_1; a \in X\} \cup \{\langle 0,x\rangle;$ "x is a_1-maximal for some $a \in X$"$\}$. Thus F satisfies the above formulated conditions (a) - (c) if we write F instead of $\widetilde{a}$. By our assumption which $\#$ has to satisfy there is $a \in A^{\#}$ with $\widetilde{a} = F$ and we are done.

We put $\mathcal{O}\!\iota^{\#} = \{A^{\#}, E^{\#}, I^{\#}\}\eta$. Let us note that $\mathcal{O}\!\iota^{\#}$

depends on the choice of A and an enumeration of A, but we shall neglect this dependence since all models of the form we deal with are isomorphic.

Metatheorem. $\mathfrak{A}^{\#}$ is an interpretation of KM^{-} in TC + + A51 + A61 and of KM^{-} + A6 + A7 in AST.

Demonstration. We have $\mathfrak{A}^{\#} \models$ A01 by the definition of $E^{\#}$ and $I^{\#}$ and the formula $\mathfrak{A}^{\#} \models$ All is a consequence of the last but one lemma. For every $a \in A^{\#}$ there is $X \subseteq A^{\#}$ which is at most countable and with $X_{\mathfrak{A}^{\#}} = (E^{\#})''\{a\}$ and hence the $\mathfrak{A}^{\#}$-pairing, $\mathfrak{A}^{\#}$-sum and $\mathfrak{A}^{\#}$-infinity axioms follows from the last lemma. Using the previous results, we have only to show the $\mathfrak{A}^{\#}$-axiom of replacement and $A81^{\mathfrak{A}^{\#}}$. If $F^{\mathfrak{A}^{\#}}$ with $Fnc^{\mathfrak{A}^{\#}}(F^{\mathfrak{A}^{\#}})$ and $a \in A^{\#}$ be given then by A61 we can choose $Y \subseteq A^{\#}$ which is at most countable so that $Y_{\mathfrak{A}^{\#}} = F^{\mathfrak{A}^{\#}}$ and thus even the $\mathfrak{A}^{\#}$-axiom of replacement is a consequence of the last lemma. If $\mathfrak{A}^{\#} \models (\forall x \in a)(\exists y \in a)(y \in x) \,\&$ $\& \,(\exists x)(x \in a)$ then we can construct $X \subseteq \mathrm{dom}(\tilde{a})$ with $\tilde{a}''X = X$ which contradicts the condition (c) of the definition of $A^{\#}$.

We have shown that there is an interpretation of AST in KM^{-} + A6 + A7 and vice versa. Let us remind that in [M-S] an interpretation $\ast$ of $KM^{-} + V = L + V = HC$ in KM^{-} is constructed in such a way that there is a well-ordering of $\ast$-classes (represented by a formula) such that every initial segment of this well-ordering is codable as a $\ast$-class. In particular, there is an interpretation of KM^{-} + A6 + A7 in KM^{-} and thus even the theories AST and KM^{-} are alike strong in the sense of interpretability. Furthermore, considering the well-ordering in question, we get an interpretation of

AST + A62 in AST where A62 is the axiom corresponding to the schema of choice use e.g. in the second order arithmetic:

A 62 Schema of choice. For every formula $\Phi(z,Z)$ we accept the axiom

$$(\forall n \in FN)(\exists X)\ \Phi(n,X) \rightarrow (\exists Y)(\forall n \in FN)\ \Phi(n,Y''\{n\}).$$

Moreover, the previous considerations show also which axioms of the alternative set theory are for this theory specific in the sense of interpretability. As convenient we accepted the axioms of TC which express the essence of the theory of sets and classes and which are not specific for the alternative set theory since they are accepted e.g. in Kelley-Morse's axiomatization of Cantor's set theory. We have shown that there are only two candidates for really proper axioms of the alternative set theory - some kind of the prolongation axiom and some kind of the axiom of choice. We shall see later that some kind of the prolongation axiom is even necessary, but the essentiality of the axiom A61 remains as an open problem. The axioms of GB-class and the axiom of cardinalities are only auxiliar in the sense that the theory with these axioms can be modelled in the theory without them. Therefore, in particular, these axioms can be consistently added to other axioms of the alternative set theory (cf. the claims in § 1 and § 6 ch. I [V]).

Remark 1. The countable model $\{Def, E \cap Def^2\}^{\eta}$ is elementarily equivalent to the model $\{V,E\}^{\eta}$; on the other hand there is a lot of countable models which are not elementarily equivalent to $\{V,E\}^{\eta}$. Every countable model can be reconstructed in $\mathcal{O}\!\mathcal{C}^{\#}$ as $\mathcal{a}^{\#}$-set. The composition of interpretations $\bar{\mathcal{a}}$ and $\mathcal{a}^{\#}$ ($\ast$, say) is an interpretation of AST in AST depending on the parameter $\mathcal{O}\!\mathcal{C}$ (we have defined $\overline{\mathcal{O}\!\mathcal{C}}$ as Ul($\mathcal{O}\!\mathcal{C}$,B,Z)). Thus we can fix the parameter $\mathcal{O}\!\mathcal{C}$ so that there is a set-formula Φ such that the formula $\bar{\Phi}$ & $\neg\, \Phi^{\ast}$ holds. On the other hand, we are able to choose $\mathcal{O}\!\mathcal{C}$ so that for every set-formula φ of the language FL we have $(V \models \varphi) \equiv (V \models \varphi)^{\ast}$ and in this case the interpretation $\ast$ is faithful (cf. [V-H]), i.e. for every formula Ψ we have AST $\vdash \Psi$ iff AST $\vdash \Psi^{\ast}$. To show this we are going to demonstrate AST $\vdash \Psi \equiv \Psi^{\ast}$. The model $\{V^{\ast}, E^{\ast}, I^{\ast}\}$ is elementarily equivalent to $\{V,E\}^{\eta}$, we have A5, A5* and $V \models \mathcal{Z}\mathcal{F}_{Fin}$, thence it suffices to use a result of the last section.

Remark 2. We have demonstrated that KM^- and AST are mutually interpretable. Using the same ideas we are able to show that GB^- and AST_{-2} + A21 + A22 are alike strong in the sense of interpretability, too. In fact, $\bar{\mathcal{a}}$ is an interpretation of AST_{-2} + A21 in GB^- + A6 + A7 + $\mathcal{O}\!\mathcal{C} \models \mathcal{Z}\mathcal{F}_{Fin}$ and furthermore $FN^{\bar{\mathcal{a}}}$ is the minimal (w.r.t. $\subseteq$) class X such that $(\forall x \in X)(\exists y \in X)(\overline{\mathcal{O}\!\mathcal{C}} \models y = x+1)$ & $(\exists x \in X)(\overline{\mathcal{O}\!\mathcal{C}} \models x = 0)$ & & $X = X_{\overline{\mathcal{O}\!\mathcal{C}}}$.

By [Sh 1] the theories GB^- and ZF^- are equiconsistent and hence even AST_{-2} + A21 + A22 is equiconsistent to ZF^-. On the other hand, there is no interpretation of AST_{-2} + + A21 + A22 in ZF^- since such an interpretation would give an interpretation of GB^- in ZF^- which is absurd. (In detail: GB^- is finitely axiomatizable and therefore there would be a finite theory $T \subseteq ZF^-$ such that GB^- and T could be interpreted one in the second one. Further, ZF^- is a reflective theory and thence there would be in ZF^- a model (which would be a set, of course) of T and thus there would be even a model of $\overline{ZF^-}$, which contradicts Gödel's theorem.)

Remark 3. We dealt with set theories, but, of course, we can reformulate previous results to statements concerning arithmetic . Peano arithmetic (A_1) is mutually interpretable with ZF_{Fin} (this statement goes back to [Sk] where a one-one correspondence between natural numbers and finite sets is constructed). KM_{Fin} and the second order arithmetic (A_2) can be interpreted one in the second one (this follows e.g. from [M-St] where mutual interpretability of KM_{Fin} with the schema of choice, ZF^-, A_2 and A_2 without schema of choice is proved using the results of [Z] and [B-H-F] (cf. also [E]) because the interpretations of the last theory in KM_{Fin} and of KM_{Fin} in KM_{Fin} with the schema of choice are trivial). Furthermore even the theories A_3 and KM^- are alike strong in the sense of interpretability (A_3 is mutually interpretable with KM^- with the axiom and schema of choice by [M-St] and the same is true for the lastly mentioned theory and KM^- according to [M-S]). Let us note that a result of this kind concerning interpretability of GB_{Fin} is described in [Ri].

Thus AST is mutually interpretable with A_3 and AST_{-2} + + A21 + A22 is equiconsistent to A_2, but there is no interpretation of AST_{-2} + A21 + A22 in A_2.

§ 7. <u>Non-interpretability</u>. We are going to show that there are no interpretations of AST in some theories using the following result.

<u>Metatheorem</u>. Let T be a theory stronger than TC, let S be a recursive theory and let $*$ be an interpretation of ZF_{Fin} + Con($\overline{S}$) in T. Then $T \vdash Con_F(\mathcal{S})$.

Demonstration. Assuming that $*$ is an interpretation of A11 & A3 in T we define in T

$$\Theta_1(K,S) \equiv (\forall y \in K)(Set^*(S''\{y\}) \;\&\; (\forall z)((z \in y \equiv (z \in K \;\& \; S''\{z\} \in^* S''\{y\}) \;\&\; (\forall z^*)(\exists z \in K)(z^* \in^* S''\{y\} \rightarrow \rightarrow z^* =^* S''\{z\}))))$$

$$\Theta(x,X) \equiv (\exists K,S)(\Theta_1(K,S) \;\&\; x \in K \;\&\; X = S''\{x\})$$

and proceeding in T we get the following statements.

(a) $(\forall x \in FV)(\exists K \supseteq x)(\exists S)\ \Theta_1(K,S)$.

If $\Theta_1(K,S) \;\&\; (\forall x \in \overline{P}(n))(x \stackrel{\wedge}{\leq} m \rightarrow x \in K)$ then according to $A3^*$ there are K_1, S_1 with $\Theta_1(K_1,S_1) \;\&\; (\forall x \in \overline{P}(n))(x \stackrel{\wedge}{\leq} m{+}1 \rightarrow \rightarrow x \in K_1)$. Since even formulas with the symbol $*$ form classes, our statement can be shown by double induction.

(b) $(x \in (K_1 \cap K_2 \cap FV) \;\&\; \Theta_1(K_1,S_1) \;\&\; \Theta_1(K_2,S_2)) \rightarrow \rightarrow S_1''\{x\} =^* S_2''\{x\}$.

Let us suppose that our statement holds for every $y \in x$ and let $y^* \in^* S_1''\{x\}$. Then there is $y \in x$ such that $y^* =^* S_1''\{y\}$ according to the definition of Θ_1 and therefore $y^* =^* =^* S_2''\{y\} \in^* S_2''\{x\}$ by the assumption and thence $y^* \in^* S_2''\{x\}$ according to the $*$-equality axiom. Thus $(\forall y^*)(y^* \in^* S_1''\{x\} \equiv y^* \in^* S_2''\{x\})$ and hence our statement is a consequence of $A11^*$.

(c) For every restricted formula $\Phi(Z_1,\ldots,Z_n)$ we have

$(\forall x_1,\dots,x_n \in FV)(\forall X_1,\dots,X_n)((\Theta(x_1,X_1)$ & ... &
& $\Theta(x_n,X_n)) \longrightarrow (\Phi(x_1,\dots,x_n) \equiv \Phi^*(X_1,\dots,X_n)))$.

If $x,y \in FV$ then by (a) there are K, S with $\Theta_1(K,S)$ & & $x,y \in K$ and by the definition of Θ_1 we have $S''\{x\} \in^* \in^* S''\{y\} \equiv x \in y$. For every X, Y with $\Theta(x,X)$ & $\Theta(y,Y)$ we have $X =^* S''\{x\}$ & $Y =^* S''\{y\}$ according to (b) and thence $X \in^* Y \equiv x \in y$ is a consequence of All* and $*$-equality axiom. Thus our statement can be shown by induction w.r.t. the complexity of the formula Φ.

If $*$ is an interpretation of ZF_{Fin} in T and if we are able to fix a constant d such that in T we have "d is a finite proof of inconsistency of finite $s \subseteq \mathcal{S}$" then $d \in FV$ and $*$ is an interpretation of $ZF_{Fin} + \neg Con(\overline{S})$ by (a) and (c) (let us realize that if Φ is a restricted formula and if $x \in FV$ & $\Theta(x,x^*)$ then $((\exists y)\ \Phi(x,y))^{\mathcal{F}\mathcal{V}} \equiv (\exists y \in FV)\Phi(x,y)$ and $(\exists y \in FV)\ \Phi(x,y) \longrightarrow (\exists y^*)\ \Phi^*(x^* y^*)$). Thus if $*$ is an interpretation fulfilling our assumptions then either T is inconsistent or we cannot fix a constant d as described above.

Consequence. Let T be a theory stronger than TC, let S be a consistent recursive theory stronger than TC and let $T \vdash Con_F(\mathcal{S})$. Then there is no interpretation of T in S.

Demonstration. The composition of the interpretation $\mathcal{F}\mathcal{V}$ and such an interpretation (restricted to sets) would give us an interpretation of $ZF_{Fin} + Con(\overline{S})$ in S. Hence according to the last metatheorem we would obtain $S \vdash Con_F(\mathcal{S})$, however, this contradicts Gödel's theorem.

<u>Theorem</u> (TC + A51). The theory $\mathcal{K}\mathcal{M}_{Fin}$ is finitely consistent.

Proof. The class FV is at most countable since $N^{\mathcal{FV}} = FN$ and according to the fact that $\mathcal{FV}$ is an interpretation of KM_{Fin} in TC and $KM \vdash V \hat{\approx} N$. Hence according to A51 all subclasses of FV can be coded by sets. Therefore the existence of a model of the theory $\mathcal{KM}_{Fin}$ follows from the metatheorem of the third section.

Thus combining the previous results we see that there is no interpretation of AST in KM_{Fin}.

In the last section we showed that AST has an interpretation in TC + A51 + A61 and further in § 4 we saw that the difference between AST and KM_{Fin} lies largely in the fact that in KM_{Fin} the negation of the axiom A52 is provable. Therefore it is natural to ask whether the axiom A52 is not strong enough for constitution of the alternative set theory, in particular, whether there is an interpretation of AST in TC + A52 + A61. The following result shows that this is not the case (it can be proved even in the theory AST_{-5} + A51 and therefore there is no interpretation of AST_{-5} + A51 in AST_{-5} + A52).

<u>Theorem</u> (AST). The theory $\mathcal{AST}_{-5}$ + A52 is finitely consistent.

Proof. Fix $\alpha \in N-Fn$ and let T be the theory with the language $\in$, = , α and as axioms of T we accept all statements of this language which are true in the model $\{V,E\}^{\eta}$. Thus T is finitely consistent and hence there is a model $\mathcal{U} = \{A,\hat{E},\hat{I},\hat{\alpha}\}^{\eta}$ of $\mathcal{T}$ so that A is countable. Every subclass of A can be coded by a set using the axiom A5 and hence it is sufficient to realize that $\mathcal{U}$ is an interpretation of AST_{-5} + A52 in AST. This follows from the first metatheorem of § 5 since $FN^{\mathcal{A}} \subseteq^{\mathcal{A}} \tilde{E}"\{\tilde{\alpha}\}$ and $\neg \mathrm{Set}^{\mathcal{A}}(FN^{\mathcal{A}})$.

R e f e r e n c e s

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