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# SOME RESULTS ON INVERSE SPECTRA II

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Abstract: In this paper, we consider the following question: when a homeomorphism of limit spaces of two inverse spectra is induced by an isomorphism of cofinal subspectra? We prove two spectral theorems which generalize a number of A.V. Arhangel'skii's, B.A. Pasynkov's and E.V. Ščepin's results. Some related questions are considered, too.

Key words and phrases: Isomorphism of spectra, open mapping, continuous spectra, d-open mapping, almost continuous spectra, semiopen mapping, $\varkappa$-metrizable space.

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In the second part of the paper we introduce the new notion of a d-open mapping (Definition 5) and prove the spectral theorem for spectra with d-open projections (Theorem 3) which generalizes a similar Ščepin's result for spectra with open projections. We consider also the question: when a space of a regular weight $\tau > \aleph_0$ is representable as a limit of a spectrum $\{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ with d-open projections such that $w(X_\alpha) < \tau$ for every $\alpha < \tau$ ? Theorem 4 is a partial answer to this question. The spectra with semiopen projections are considered, too. We prove that a limit of an almost continuous spectrum $\{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ with semiopen projections has Souslin property iff a space $X_\alpha$ has Souslin property for each $\alpha < \tau$ (Theorem 6). Our last result (Theorem 7) is a generalization of Theorem 1 from [5]. With the aid of Theorem 7 we prove that a first-countable regular image of a dense subset of $\varkappa$-metrizable compact has a countable network (Corollary 4).

§ 2. <u>d-open mappings and the new spectral theorem</u>. There exists the following spectral theorem belonging to E.V. Ščepin. Let $\tau$ be a regular cardinal $> \aleph_0$ and S, T be regular spectra of the same length $\tau$ with open projections. If their limits are homeomorphic to a space X then there exists a closed cofinal subset A of $\tau$ such that the spectra $S_A$ and $T_A$ are isomorphic.

To prove it, Ščepin shows first that $\nabla c(X) \leq \tau$, i.e. the cardinality of each disjoint system consisting of open subsets of X is less than $\tau$.

Here we show that it is possible to replace the requirement on projections to be open by the weaker condition of d-openness (see definition 4 below), however, we need to retain the property $\nabla c(X) \leq \tau$ which does not follow from the d-openness of projections (see example 2).

The following definition is new.

<u>Definition 4</u>. We say that a continuous mapping $f:X \to Y$ is d-open if a set $f(\mathcal{O})$ is dense in some open subset of Y for each open subset $\mathcal{O}$ of X.

It is obvious that every continuous open mapping is d-open. In lemmas 5-9 below we establish some properties of d-open mappings.

Lemma 5. Let $f:X \rightarrow Y$ be a continuous mapping. Then the following conditions are equivalent:

(a) $f$ is a d-open mapping;

(b) $f^{-1}[\mathcal{O}] = [f^{-1}\mathcal{O}]$ for each open subset $\mathcal{O} \subseteq Y$.

Proof. Primarily we show that (a) implies (b). Let $\mathcal{O}$ be an open subset of $Y$. Since $[f^{-1}\mathcal{O}] \subseteq f^{-1}[\mathcal{O}]$ for each $\mathcal{O} \subseteq$ $\subseteq Y$, it is sufficient to show the inverse inclusion. Let $x \in X$ and $f(x) \in [\mathcal{O}]$. Let us assume that $x \notin [f^{-1}\mathcal{O}]$. Then $V =$ $= X \setminus [f^{-1}\mathcal{O}]$ is an open neighbourhood of $x$ in $X$. Consequently $f(V)$ is a dense subset of some open subset $W \subseteq Y$ so $[W] =$ $= [f(V)]$. However, $f(V) \cap \mathcal{O} = \Lambda$, hence $[f(V)] \cap \mathcal{O} = \Lambda$. Thus $[W] \cap \mathcal{O} = \Lambda$. It contradicts the fact that $f(x) \in W \cap [\mathcal{O}]$. So the inclusion $f^{-1}[\mathcal{O}] \subseteq [f^{-1}\mathcal{O}]$ is proved.

Now we show that (b) implies (a). Let $V$ be an open subset of $X$. Put $F = [f(V)]$. Then $f(V)$ is contained in the interior of $F$. Indeed, $\mathcal{O} = Y \setminus F$ is an open subset of $Y$ hence $f^{-1}[\mathcal{O}] =$ $= [f^{-1}\mathcal{O}]$. However, $V \cap f^{-1}\mathcal{O} = \Lambda$ so $V \cap [f^{-1}\mathcal{O}] = \Lambda$, i.e. $V \cap f^{-1}[\mathcal{O}] = \Lambda$. Consequently $f(V) \cap [\mathcal{O}] = \Lambda$ therefore $f(V)$ is contained in $\mathrm{Int}[f(V)]$. Thus lemma is proved.

When a d-open mapping is open? The following lemma is a partial answer to this question.

Lemma 6. Let $f$ be a d-open closed mapping of a regular space $X$ to a space $Y$. Then $f$ is open.

Proof. We prove that $[f^{-1}A] = f^{-1}[A]$ for each subset $A \subseteq Y$ which implies that $f$ is open. Indeed, let $A \subseteq Y$, $x \in X$ and $f(x) \in [A]$. Let us assume that $x \notin [f^{-1}A]$. Choose an open subset $\mathcal{O} \subseteq X$ such that $x \in \mathcal{O}$ and $[\mathcal{O}] \cap f^{-1}A = \Lambda$. Since $f$ is d-open, a set $f(\mathcal{O})$ is a dense subset of some open set $W \subseteq Y$, hence $[f(\mathcal{O})] = [W]$ . But $f$ is closed, hence $f([\mathcal{O}]) = [f(\mathcal{O})]$ and $f([\mathcal{O}]) = [W]$. Since $[\mathcal{O}] \cap f^{-1}A = \Lambda$ , we conclude that $[W] \cap A = \Lambda$ . It contradicts the fact that $f(x) \in W \cap [A]$. Thus $x \in [f^{-1}A]$ so $[f^{-1}A] = f^{-1}[A]$. This completes the proof.

The following lemma shows a way the d-open mappings arise on.

<u>Lemma 7</u>. Let $f: X \rightarrow Y$ be a continuous open mapping and $S$ be a dense subset of $X$. Then a mapping $g = f|S$ is d-open.

Proof. Let $V$ be an open subset of $S$. Then there exists an open subset $U$ of $X$ such that $U \cap S = V$. A set $V$ is dense in $U$ hence the set $g(V) = f(V)$ is dense in the open subset $W = f(U)$ of $Y$.

Corollary 2. Let $S$ be a dense subset of a product $X = \prod_{\alpha \in A} X_\alpha$ . Then a mapping $\pi_B | S$ is d-open for each subset $B \subseteq A$ ( $\pi_B$ is a natural projection of $X$ onto $X_B = \prod_{\alpha \in B} X_\alpha$ ).

Corollary 3. Let $S$ be a dense subspace of $X$. Then a natural embedding $i: S \hookrightarrow X$ is d-open.

<u>Lemma 8</u>. Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be d-open mappings. Then a mapping $h = g \circ f$ is d-open, too.

Proof. Let $\mathcal{O}$ be an open subset of $Z$. Then $g^{-1}[\mathcal{O}] = [g^{-1}\mathcal{O}]$ because $g$ is d-open. As $f$ is d-open and $g^{-1}\mathcal{O}$ is an open subset of $Y$, so $f^{-1}[g^{-1}\mathcal{O}] = [f^{-1}g^{-1}\mathcal{O}]$ . Thus $f^{-1}g^{-1}[\mathcal{O}] = [f^{-1}g^{-1}\mathcal{O}]$ . The lemma's conclusion follows from Lemma 5.

<u>Lemma 9</u>. Let $f: X \xrightarrow{onto} Y$, $g: Y \rightarrow Z$ be continuous mappings and $h = g \circ f$. If $f$ and $h$ are d-open then $g$ is d-open, too.

Proof. Let $V$ be an open subset of $Y$. Then $U = f^{-1}V$ is an open subset of X. Since h is d-open, h(U) is a dense subset in some open set W of Z. However g(V) = h(U) which completes the proof.

Now let us begin to consider the spectra with d-open projections. We recall once more that all projections of spectra under consideration are assumed to be onto (it should be noted that if a space X is a limit of some spectrum then X can be represented as a limit of a spectrum with projections onto).

Lemma 10. Let a space X be a limit of a spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta\in A}$. Then the following conditions are equivalent:

(a) $p_\alpha^\beta$ is a d-open mapping for each $\alpha, \beta \in A$ with $\alpha < \beta$;

(b) a limit projection $p_\alpha$ is a d-open mapping for every $\alpha \in A$.

Proof. (a) $\longrightarrow$ (b). Let $\alpha \in A$ and U be an open subset of $X_\alpha$. Let us assume that there exists a point $x \in X$ such that $p_\alpha(x) \in [U]$ but $x \notin [p_\alpha^{-1}U]$. Then there exist an element $\beta \in A$ and an open subset $V \subseteq X_\beta$ such that $x \in p_\beta^{-1}V$ and $p_\beta^{-1}V \cap p_\alpha^{-1}U = \Lambda$. Let $\gamma$ be an element of A such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Put $y = p_\gamma(x)$. Then $p_\alpha^\gamma(y) = p_\alpha(x) \in [U]$. However $y \notin [(p_\alpha^\gamma)^{-1}U]$ which contradicts the fact that $p_\alpha^\gamma$ is a d-open mapping. Thus $[p_\alpha^{-1}U] = p_\alpha^{-1}[U]$ hence $p_\alpha$ is d-open.

The fact that (b) implies (a) follows immediately from lemma 9. Thus our lemma is proved.

Combining lemmas 8 and 10 we get

Lemma 11. Let a space X be a limit of a spectrum

$S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\xi}$ where $p_\alpha^{\alpha+1}$ is a d-open mapping for every $\alpha < \xi$. Then all projections of a spectrum S (including limit ones) are d-open.

Recall that a continuous mapping $f: X \longrightarrow Y$ is said to be skeletal iff $f^{-1}(K)$ is nowhere dense subset of X for each nowhere dense subset $K \subseteq Y$. It is easily seen that every d-open mapping is skeletal. It is known (see [12]) that the equality $\nabla c(\varprojlim S) = \sup\{\nabla c(X_\alpha): \alpha < \tau\}$ holds for every continuous inverse spectrum $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ with skeletal projections onto. This result will be used in Lemma 12 below.

The following example shows that there exists a continuous well-ordered spectrum of the length $\omega_1$ consisting of separable metrizable spaces with skeletal projections which has no factorization property.

<u>Example 2</u>. Let I be the unit interval with the usual topology and Y be any nowhere dense subset of I such that $|Y| = \aleph_1$. Then there exist countable discrete disjoint subsets $A, B \subseteq I \setminus [Y]$ such that $[A] \cap [B] = [Y]$. Put $X = Y \cup A \cup B$ and let $\mathcal{T}_0$ be a subspace topology on X. Then A and B are open discrete subsets of the space $X_0 = (X, \mathcal{T}_0)$. We can enumerate the set Y so that $Y = \{x_\alpha : \alpha < \omega_1\}$. Now a chain $\{\mathcal{T}_\alpha : \alpha < \omega_1\}$ of strongly increasing topologies on X will be defined such that

1) each space $X_\alpha = (X, \mathcal{T}_\alpha)$ is regular second-countable;

2) the set A is dense in $A \cup Y$ in the space $X_\alpha$ for each $\alpha < \omega_1$;

3) for every $\alpha < \omega_1$ the set $A_\alpha = A \cup \{x_\beta : \beta < \alpha\}$ is

open and locally compact in the space $X_\alpha$;

4) $\mathcal{T}_\alpha | A_\beta = \mathcal{T}_\beta | A_\beta$ whenever $\beta < \alpha < \omega_1$.

The topology $\mathcal{T}_0$ on X has been defined. Let $\alpha < \omega_1$ and a topology $\mathcal{T}_\beta$ on X be defined for each $\beta < \alpha$. We begin with the case when $\alpha = \beta + 1$ for some ordinal $\beta$. Since the space $X_\beta$ is second-countable, the condition (2) implies that there exists a converging sequence $\{a_n : n \in N^+\} \subseteq A$ with a limit point $x_\alpha$. Obviously $A_\alpha$ is a countable open locally compact subspace of a regular space $X_\alpha$. Hence there exists a sequence $\xi = \{V_m : n \in N^+\}$ of pairwise disjoint open compact subsets of the space $A_\alpha$ such that $a_n \in V_n$ for each $n \in N^+$ and $\xi$ converges to $x_\alpha$.

For every $n \in N^+$ put $O_n = \{x_\alpha\} \cup \bigcup \{V_m : m \in N^+ \text{ and } n \le m\}$. Put also $\gamma = \{O_n : n \in N^+\}$. Now we can take the family $\mathcal{T}_\beta \cup \gamma$ as a base for a topology $\mathcal{T}_\alpha$. It is easily seen that the conditions (1)-(4) are satisfied.

In the case of a limit ordinal $\alpha$ we define a topology $\mathcal{T}_\alpha$ on X by taking the family $\bigcup_{\beta < \alpha} \mathcal{T}_\beta$ as a base for $\mathcal{T}_\alpha$. Then the conditions (1)-(4) are satisfied, too.

Thus the chain $\{\mathcal{T}_\alpha : \alpha < \omega_1\}$ of regular second-countable topologies on X has been defined. Let $X_\alpha = (X, \mathcal{T}_\alpha)$ and $\pi_\beta^\alpha$ be an identity mapping of $X_\alpha$ onto $X_\beta$, $\beta < \alpha < \omega_1$. Then $\pi_\beta^\alpha$ is a continuous one-to-one mapping for each $\alpha$, $\beta < \omega_1$ with $\beta < \alpha$. Put $S = \{X_\alpha, \pi_\alpha^\beta\}_{\alpha, \beta < \omega_1}$. Obviously, the spectrum S is continuous and the space $\varprojlim S$ is naturally homeomorphic to the space $(X, \mathcal{T})$, where $\mathcal{T} = \bigcup_{\alpha < \omega_1} \mathcal{T}_\alpha$. Hence we identify the space $\varprojlim S$ and $(X, \mathcal{T})$. Let f be a function on X such that $f(A \cup Y) = 0$ and $f(B) = 1$. It is clear that $A \cup Y$

and B are open subsets of the space $(X,\mathcal{T})$ hence $f$ is continuous. It is also clear that $x_\beta$ belongs to the closure of the set B in the space $X_\alpha$, whenever $\alpha < \beta < \omega_1$. So the closures of the sets $A \cup Y$ and B in the space $X_\alpha$ are not disjoint for each $\alpha < \omega_1$. Thus the function $f$ does not admit a continuous factorization in the spectrum S. It remains to show that all projections of the spectrum S are skeletal.

But this follows easily from the fact that a limit projection $\pi_\alpha : \underleftarrow{\lim}\, S \to X_\alpha$ is skeletal for each $\alpha < \omega_1$. Indeed, the condition (2) implies that $A \cup B$ is an open dense discrete subset of the space $(X, \mathcal{T}_\alpha)$, i.e. $Y = X \setminus (A \cup B)$ is a maximal nowhere dense subset of $X_\alpha$ for each $\alpha < \omega_1$. Thus $\pi^\alpha_\beta$ is a skeletal mapping for each $\alpha, \beta < \omega_1$ with $\beta < \alpha$.

Let us continue our considerations of spectra with d-open projections.

Lemma 12. Let $\tau$ be an uncountable regular cardinal and a space X be a limit of a spectrum $S = \{X_\alpha, p^\beta_\alpha\}_{\alpha,\beta<\tau}$ with d-open projections. Let $f$ be a continuous function on X. Then there exist an ordinal $\alpha^* < \tau$ and a continuous function $g$ defined on $X_{\alpha^*}$ such that $f = g \circ p_{\alpha^*}$ ($p_{\alpha^*}$ is a limit projection of X onto $X_{\alpha^*}$).

Proof. Let $\mathcal{B}$ be a countable base of the usual topology on $\mathbb{R}$. Put $\mathcal{F} = \{\mathbb{R} \setminus U : U \in \mathcal{B}\}$. Fix an element $F \in \mathcal{F}$. Let $\gamma$ be a countable family consisting of open subsets of $\mathbb{R}$ such that $F = \bigcap\{[O] : O \in \gamma\}$. Since $f^{-1}O$ is open in X and $\mathrm{ce}(X) \leq \tau$, there exist an ordinal $\alpha_0 < \tau$ and an open subset $V_0 \subseteq X_{\alpha_0}$ such that $p_{\alpha_0}{}^{-1}V_0$ is dense in $f^{-1}O$. As $\aleph_0 < \mathrm{cf}(\tau) = \tau$ so there exists an ordinal $\alpha_F < \tau$ such that

$\alpha_\sigma < \alpha_F$ for each $\sigma \in \gamma$. For every $\sigma \in \gamma$ put $K_\sigma = [(p^{\alpha_F}_{\alpha_\sigma})^{-1}V_\sigma]$. Put also $K_F = \bigcap\{K_\sigma : \sigma \in \gamma\}$. Then $f^{-1}F = p^{-1}_{\alpha_F} K_F$. Indeed, the d-openness of projections of a spectrum S implies that the limit projections $p_\alpha : X \to X_\alpha$ are d-open, too. Hence $[f^{-1}\sigma] = [p^{-1}_{\alpha_\sigma} V_\sigma] = p^{-1}_{\alpha_\sigma}[V_\sigma]$ for every $\sigma \in \gamma$. Moreover the equality $F = \bigcap\{[\sigma] : \sigma \in \gamma\}$ implies that $f^{-1}F = \bigcap\{f^{-1}[\sigma] : \sigma \in \gamma\} = \bigcap\{[f^{-1}\sigma] : \sigma \in \gamma\} = \bigcap\{p^{-1}_{\alpha_\sigma}[V_\sigma] : \sigma \in \gamma\}$.

However, $p^{-1}_{\alpha_\sigma}[V_\sigma] = p^{-1}_{\alpha_F}(p^{\alpha_F}_{\alpha_\sigma})^{-1}[V_\sigma] = p^{-1}_{\alpha_F}[(p^{\alpha_F}_{\alpha_\sigma})^{-1}V_\sigma]$ which implies that $f^{-1}F = \bigcap\{p^{-1}_{\alpha_\sigma}[V_\sigma] : \sigma \in \gamma\} = \bigcap\{p^{-1}_{\alpha_F}[(p^{\alpha_F}_{\alpha_\sigma})^{-1}V_\sigma] : \sigma \in \gamma\} = p^{-1}_{\alpha_F}(\bigcap\{[(p^{\alpha_F}_{\sigma})^{-1}V_\sigma] : \sigma \in \gamma\}) = p^{-1}_{\alpha_F} K_F$.

Since $|\mathcal{F}| = |\mathcal{B}| = \aleph_0$, there exists an ordinal $\alpha^* < \tau$ such that $\alpha_F < \alpha^*$ for each $F \in \mathcal{F}$. Then $f^{-1}F = p^{-1}_{\alpha^*} \widetilde{K}_F$, where $\widetilde{K}_F = (p^{\alpha^*}_{\alpha_F})^{-1}K_F$ for every $F \in \mathcal{F}$.

We claim now that

(*) for each closed subset $\Phi$ of $\mathbb{R}$ there exists a closed subset $\widetilde{K}_\Phi \subseteq X_{\alpha^*}$ such that $f^{-1}\Phi = p^{-1}_{\alpha^*}\widetilde{K}_\Phi$.

Indeed, $\mathcal{B}$ is a base for $\mathbb{R}$ hence a family $\mathcal{F}$ is such that for each closed subset $\Phi \subseteq \mathbb{R}$ there exists a family $\gamma_\Phi \subseteq \gamma$ with $\Phi = \bigcap \gamma_\Phi$. It is obvious that then $f^{-1}\Phi = p^{-1}_{\alpha^*}\widetilde{K}_\Phi$ where $\widetilde{K}_\Phi = \bigcap\{\widetilde{K}_F : F \in \gamma_\Phi\}$.

For every point $r \in \mathbb{R}$ let $\widetilde{K}_r$ be a closed subset of $X$ such that $f^{-1}(r) = p^{-1}_{\alpha^*}\widetilde{K}_r$. A mapping $g : X_{\alpha^*} \to \mathbb{R}$ we define by the condition $g(x) = r$ for each $x \in \widetilde{K}_r$, $r \in \mathbb{R}$. This definition implies $f = g \circ p_{\alpha^*}$. We claim that $g$ is continuous.

Indeed, let $\Phi$ be a closed subset of $\mathbb{R}$ . Then $f^{-1}\Phi =$ $= p_{\alpha^*}^{-1}(g^{-1}\Phi)$. However, the property $(*)$ implies that $f^{-1}\Phi = p_{\alpha^*}^{-1}\widetilde{K}_\Phi$ where $\widetilde{K}_\Phi$ is closed in $X_{\alpha^*}$ . Hence $g^{-1}\Phi = \widetilde{K}_\Phi$ is closed in $X_{\alpha^*}$ . Thus g is continuous and the lemma is proved.

Lemmas 12 and 4 imply the main result of this paragraph.

Theorem 3. Let a space X of regular weight $\tau > \aleph_0$ be a limit of each of two almost regular spectra $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ and $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\tau}$ with d-open projections. Then there exists a closed cofinal subset $A$ of $\tau$ such that the spectra $S_A$ and $T_A$ are isomorphic.

Lemmas 7 and 12 imply the following

Corollary 3. Let S be a dense subset of some open subset of a product $\prod_{\alpha\in A} X_\alpha$ of separable spaces and f be a continuous function on S. Then there exist a countable subset $B\subseteq A$ and a continuous mapping $g\colon \pi_B(S) \longrightarrow \mathbb{R}$ such that $f =$ $= g\circ(\pi_B|S)$.

Corollary 3 is an improvement of a similar Gleason's result (see [91).

In connection with the fact that we have introduced the new class of d-open mappings, it naturally arises the following question. What are the spaces which can be represented as limits of spectra with d-open projections consisting of spaces of smaller weights? We will give sufficient conditions for such representability (Theorem 4). To do this we need a few notions and lemmas.

Definition 5. Let X be a space and $\lambda$ be an infinite cardinal. We will say that a closed subset $F\subseteq X$ is $\lambda$-pointed in X iff there exist a continuous mapping $f$ of X onto a space Y of weight $\leq \lambda$ and a closed subset $\Phi \subseteq Y$ such that $F = f^{-1}\Phi$ .

It is obvious that $\psi(F,X) \leq \lambda$ for each closed $\lambda$-pointed subset $F \subseteq X$. Inversely, if X is a normal space and F is a closed subset of X with $\psi(F,X) \leq \lambda$ then F is $\lambda$-pointed in X.

Lemma 13. Let $\tau$ be an infinite cardinal and F be a closed subset of a space X where $\psi(F,X) \leq \tau$ and $\ell(X) \leq \tau$. Then F is $\tau$-pointed in X.

Proof. Since $\psi(F,X) \leq \tau$ there exists a system $\mu$ consisting of closed subsets of X such that $X \setminus F = \bigcup \mu$ and $|\mu| = \tau$ . Fix an element $\Phi \in \mu$ . As $F \cap \Phi = \Lambda$ for each point $x \in \Phi$ there exists a continuous function $f_x$ on X such that $f_x(x) = 0$ and $f_x(F) = 1$. Put $O_x \{y \in X : f_x(y) < \frac{1}{2}\}$. Then $\{O_x : x \in \Phi\}$ is a cover of $\Phi$ by open subsets of X hence the inequality $\ell(X) \leq \tau$ implies that there exists a subset $P \subseteq \Phi$ such that $\Phi \subseteq \bigcup\{O_x : x \in P\}$ and $|P| \leq \tau$ . Put $f_\Phi = \Delta\{f_x : x \in P\}$ and $Y_\Phi = f_\Phi(X)$. Obviously, the image of F under a mapping $f_\Phi$ consists of one point $y_\Phi$ and $y_\Phi \notin f_\Phi(\Phi)$.

Put $f = \Delta\{f_\Phi : \Phi \in \mu\}$ and $Y = f(X)$. Then $Y \underset{top}{\hookrightarrow} Z = \prod_{\Phi \in \mu} Y_\Phi$ . For each $\Phi \in \mu$ let $\pi_\Phi$ be a natural projection of Z onto $Y_\Phi$ . Let z be a point of Z such that $\pi_\Phi(z) = y_\Phi$ for each $\Phi \in \mu$ . Then $z \in Y$ and $F \subseteq f^{-1}(z)$. However $y_\Phi \notin f_\Phi(\Phi)$ for each $\Phi \in \mu$ hence $z \notin f(\bigcup \mu) = f(X \setminus F)$. Thus $F = f^{-1}(z)$ which completes the proof.

Definition 6 (E.V. Ščepin). A $\varkappa$-pseudocharacter of a space X or shortly $\psi_{\varkappa}(X)$ is a minimal cardinal $\tau$ such that a pseudocharacter of every canonically closed subset of X does not exceed $\tau$ .

It is known that a $\varkappa$-pseudocharacter of any product of metric spaces is countable (see [1], Theorem 15). The following self-interesting lemma shows when there are a "large" number of d-open mappings of a given space onto spaces of smaller weights.

Lemma 14. Let $\tau$ be an uncountable cardinal and X be a space such that $\ell(X)\cdot\psi_{\varkappa}(X) \le \tau$ and $\tau^{\lambda} = \tau$ for each cardinal $\lambda < \nabla c(X)$. Let h be a continuous mapping of X onto a space Z with $w(Z) \le \tau$ . Then there exist a space Y with $w(Y) \le \tau$ a continuous mapping $g: Y \to Z$ and a d-open mapping $f: X \to Y$ such that $h = g \circ f$.

Proof. Put $Y_0 = Z$ and $f_0 = h$. Let $\mathcal{B}_0$ be a base for $Y_0$ with $|\mathcal{B}_0| \le \tau$ . Put $\mu = \nabla c(X)$ and $\tilde{\mathcal{B}}_0 = \{\bigcup\gamma : \gamma \subseteq \mathcal{B}_0$ and $|\gamma| < \mu\}$ . Then $|\tilde{\mathcal{B}}_0| \le \tau$ . We should note that $\mu$ is a regular cardinal (cf. [6], Theorem 3.1) and the lemma's conditions imply that $\mu \le \tau$ .

Now let $\alpha$ be an ordinal with $0 < \alpha < \mu$ and for each $\beta < \alpha$ we have already defined a space $Y_\beta$ a system $\tilde{\mathcal{B}}_\beta$ of open subsets of $Y_\beta$ , a continuous mapping $f_\beta : X \xrightarrow{onto} Y_\beta$ and a family $\{\pi^\beta_\gamma : \gamma < \beta < \alpha\}$ , where $\pi^\beta_\gamma$ is a continuous mapping of $Y_\beta$ onto $Y_\gamma$ such that $|\tilde{\mathcal{B}}_\beta| \le \tau$ , $f_\gamma = \pi^\beta_\gamma \circ f_\beta$ for $\gamma < \beta < \alpha$ and $w(Y_\beta) \le \tau$ for every $\beta < \alpha$ . It is easily seen that $\pi^\gamma_\delta \circ \pi^\beta_\gamma = \pi^\beta_\delta$ for all $\delta < \gamma < \beta < \alpha$ .

I. $\alpha = \beta + 1$

The condition $\ell(X)\cdot\psi_{\mathcal{X}}(X)\leq\tau$ with lemma 13 together imply that a set $[\mathcal{O}]$ is $\tau$-pointed in X for each open subset $\mathcal{O}\subseteq X$. Therefore for every $\mathcal{U}\in\widetilde{\mathcal{B}}_\beta$ there exist a continuous mapping $\varphi_{\mathcal{U}}$ of X onto a space $X_{\mathcal{U}}$ of weight $\leq\tau$ and a closed subset $F_{\mathcal{U}}\subseteq X_{\mathcal{U}}$ such that $[f_\beta^{-1}\mathcal{U}]=\varphi_{\mathcal{U}}^{-1}F_{\mathcal{U}}$. Put $\varphi_\beta=\Delta\{\varphi_{\mathcal{U}}:\mathcal{U}\in\widetilde{\mathcal{B}}_\beta\}$, $f_\alpha=f_\beta\,\Delta\,\varphi_\beta$ and $Y_\alpha=f_\alpha(X)$. Then $Y\underset{top}{\hookrightarrow}Y_\beta\times\prod_{\mathcal{U}\in\widetilde{\mathcal{B}}}Y_{\mathcal{U}}$ where $w(Y_\beta)\leq\tau$, $|\widetilde{\mathcal{B}}_\beta|\leq\tau$ and $w(Y_{\mathcal{U}})\leq\tau$ for every $\mathcal{U}\in\widetilde{\mathcal{B}}_\beta$. Hence $w(Y_\alpha)\leq\tau$. It is easy to see that there exists the unique continuous mapping $\pi_\beta^\alpha:Y_\alpha\to Y_\beta$ such that $f_\beta=\pi_\beta^\alpha\circ f_\alpha$. For each $\gamma<\beta$ put $\pi_\gamma^\alpha=\pi_\gamma^\beta\circ\pi_\beta^\alpha$. Now we claim that for every $\mathcal{U}\in\widetilde{\mathcal{B}}_\beta$ the equality $[f_\beta^{-1}\mathcal{U}]=f_\alpha^{-1}[(\pi_\beta^\alpha)^{-1}\mathcal{U}]$ holds. Let us prove it.

Let U be an arbitrary element of a system $\widetilde{\mathcal{B}}_\beta$ and $p_{\mathcal{U}}$ be a natural projection of a product $Y_\beta\times\prod_{V\in\widetilde{\mathcal{B}}_\beta}Y_V$ onto a factor $Y_{\mathcal{U}}$. Put $F=Y_\alpha\cap p_{\mathcal{U}}^{-1}F_{\mathcal{U}}$ (a set $F_{\mathcal{U}}$ was defined above). The equality $[f_\beta^{-1}\mathcal{U}]=\varphi_{\mathcal{U}}^{-1}F_{\mathcal{U}}$ implies that $[f_\beta^{-1}\mathcal{U}]=f_\alpha^{-1}F$. Put $W=(\pi_\beta^\alpha)^{-1}\mathcal{U}$. Then $f_\alpha^{-1}W=f_\beta^{-1}\mathcal{U}\subseteq f_\alpha^{-1}F$ hence $W\subseteq F$. So $[W]\subseteq F$ and $f_\beta^{-1}\mathcal{U}=f_\alpha^{-1}W\subseteq f_\alpha^{-1}[W]\subseteq f_\alpha^{-1}F=[f_\beta^{-1}\mathcal{U}]$. Thus $[f_\beta^{-1}\mathcal{U}]=f_\alpha^{-1}[W]=f_\alpha^{-1}[(\pi_\beta^\alpha)^{-1}\mathcal{U}]$. Let $\mathcal{B}_\alpha$ be a base for $Y_\alpha$ such that $|\mathcal{B}_\alpha|\leq\tau$. Put $\mathcal{B}'_\alpha=\mathcal{B}_\alpha\cup\{(\pi_\beta^\alpha)^{-1}\mathcal{U}:\mathcal{U}\in\widetilde{\mathcal{B}}_\beta\}$ and $\mathcal{B}_\alpha=\{\cup\gamma:\gamma\subseteq\mathcal{B}'_\alpha \text{ and } |\gamma|<\mu\}$.

II. $\alpha$ is a limit ordinal.

Put $f_\alpha=\Delta\{f_\beta:\beta<\alpha\}$ and $Y_\alpha=f_\alpha(X)$. Then $Y_\alpha\underset{top}{\hookrightarrow}\prod_{\beta<\alpha}Y_\beta$ hence $w(Y_\alpha)\leq\tau$. Obviously that for each $\beta<\alpha$ there exists the unique continuous mapping $\pi_\beta^\alpha:Y_\alpha\to Y_\beta$ such that $f_\beta=\pi_\beta^\alpha\circ f_\alpha$. Let $\mathcal{B}_\alpha$ be a base for $X_\alpha$ such that $|\mathcal{B}_\alpha|\leq\tau$. For every $\beta<\alpha$ put $\mathcal{B}_\beta^\alpha=\{(\pi_\beta^\alpha)^{-1}\mathcal{U}:\mathcal{U}\in\widetilde{\mathcal{B}}_\beta\}$ and $\mathcal{B}'_\alpha=\mathcal{B}_\alpha\cup\bigcup_{\beta<\alpha}\mathcal{B}_\beta^\alpha$. Finally put

$\widetilde{\mathcal{B}}_\alpha = \{\bigcup\gamma : \gamma \subseteq \mathcal{B}'_\alpha$ and $|\gamma| < \mu\}$. Then $|\widetilde{\mathcal{B}}_\alpha| \leq \tau$.

So we have completed our recursive construction. Put $f = \Delta\{f_\alpha : \alpha < \mu\}$ and $Y = f(X)$. Then $Y \underset{top}{\hookrightarrow} \prod_{\alpha<\mu} Y_\alpha$ so $w(Y) \leq \mu \cdot \tau = \tau$. For every $\alpha < \mu$ let $\pi_\beta$ be the unique continuous mapping of $Y$ onto $Y$ such that $f_\alpha = \pi_\alpha \circ f$. Put $g = \pi_0$. Then $h = f_0 = g \circ f$ hence it remains to show only that $f$ is d-open.

Let $\mathcal{O}$ be an open non-empty subset of $Y$. A continuity of $f$ implies that $\nabla c(Y) \leq \nabla c(X) = \mu$. From our recursive construction we obtain that $\{(\pi_\beta^\alpha)^{-1}\mathcal{U} : \mathcal{U} \in \widetilde{\mathcal{B}}_\beta\} \subseteq \widetilde{\mathcal{B}}_\alpha$ for each pair $\alpha, \beta$ such that $\beta < \alpha < \mu$. Hence there exist an ordinal $\beta < \mu$ and an element $\mathcal{U}^* \in \widetilde{\mathcal{B}}_\beta$ such that $\pi_\beta^{-1}\mathcal{U}^* \subseteq \mathcal{O} \subseteq$ $\subseteq [\pi_\beta^{-1}\mathcal{U}^*]$. Indeed, for each $\gamma < \mu$ put $\mathcal{R}_\gamma = \{\pi_\gamma^{-1}\mathcal{U} : \mathcal{U} \in$ $\in \widetilde{\mathcal{B}}_\gamma\}$. Put also $\mathcal{R} = \bigcup_{\gamma<\mu}\mathcal{R}_\gamma$. Then $\mathcal{R}$ is a base for $Y$. For each point $y \in \mathcal{O}$ there exist an ordinal $\alpha(y) < \mu$ and an element $\mathcal{U}(y) \in \mathcal{B}$ such that $y \in \pi_{\alpha(y)}^{-1}\mathcal{U}(y) \subseteq \mathcal{O}$. Then $\mathcal{O} =$ $= \bigcup\{\pi_{\alpha(y)}^{-1}\mathcal{U}(y) : y \in \mathcal{O}\}$. As $\nabla c(Y) \leq \mu$ so there exists a set $P \subseteq \mathcal{O}$ such that $|P| < \mu$ and $\bigcup\{\pi_{\alpha(y)}^{-1}\mathcal{U}(y) : y \in P\}$ is dense in $\mathcal{O}$. The regularity of a cardinal $\mu$ implies that there exists an ordinal $\beta < \mu$ such that $\alpha(y) < \mu$ for every $y \in P$. Put $\mathcal{U}^* = \bigcup\{(\pi_{\alpha(y)}^\beta)^{-1}\ \mathcal{U}(y) : y \in P\}$.

Then $\mathcal{U}^* \in \widetilde{\mathcal{B}}_\beta$ and $\pi_\beta^{-1}\mathcal{U}^*$ is dense in $\mathcal{O}$.

Now we will show that $[f^{-1}\mathcal{O}] = f^{-1}[\mathcal{O}]$. Notice that $[f_\beta^{-1}\mathcal{U}] = f_\alpha^{-1}[(\pi_\beta^\alpha)^{-1}\mathcal{U}]$ for each $\beta < \mu$ and $\mathcal{U} \in \widetilde{\mathcal{B}}_\beta$ where $\alpha = \beta + 1$ (it had been proved after the construction of a space $Y$ was finished). Put $V = (\pi_\beta^\alpha)^{-1}\mathcal{U}^*$. Then $\pi_\alpha^{-1}V$ is a dense subset of $\mathcal{O}$ and $[f_\alpha^{-1}V] = f_\alpha^{-1}[V]$. Consequently $[\mathcal{O}] =$

$= [\pi_\alpha^{-1}V] \subseteq \pi_\alpha^{-1}[V]$ and $f^{-1}[\sigma] \subseteq f^{-1}\pi_\alpha^{-1}[V] = f_\alpha^{-1}[V] = [f_\alpha^{-1}V]$, i.e.

(1) $f^{-1}[\sigma] \subseteq [f_\alpha^{-1}\sigma]$.

Moreover, $\pi_\alpha^{-1}V \subseteq \sigma$ hence $f^{-1}\pi_\alpha^{-1}V \subseteq f^{-1}\sigma$. So $f_\alpha^{-1}V \subseteq f^{-1}\sigma$, i.e.

(2) $[f_\alpha^{-1}V] \subseteq [f^{-1}\sigma]$.

Inclusions (1) and (2) imply the equality $[f^{-1}\sigma] = f^{-1}[\sigma]$ which holds for every open subset $\sigma \subseteq Y$. Thus d-openness of f follows from lemma 5.

Definition 7 (A.V. Arhangel'skii). Let X be any space. Then $w_c(X)$ is a minimal cardinal $\tau$ such that there exists a perfect mapping of X onto a space of weight $\tau$.

The following lemma will be useful in the sequel (see [10], Proposition 3.7.10).

Lemma 15. Let $f:X \longrightarrow Y$ and $g:Y \longrightarrow Z$ be continuous mappings onto and Y, Z be Hausdorff spaces. If a mapping $h = g \circ f$ is perfect, then f and g are the same.

Lemma 16. Let $\tau$ be an uncountable cardinal and X be a space such that $w_c(X) \cdot \psi_{\varkappa}(X) \leq \tau$ and $\tau^\lambda = \tau$ for each $\lambda < \triangledown c(X)$. Let h be a continuous mapping of X onto a space Z of weight $\leq \tau$. Then there exist an open perfect mapping f of X onto a space Y of weight $\leq \tau$ and a continuous mapping $g:Y \longrightarrow Z$ such that $h = g \circ f$.

Proof. As $w_c(X) \leq \tau$ so there exists a perfect mapping $\varphi$ of X onto a space T of weight $\tau$ (in particular $\ell(X) \leq \tau$). Put $h' = \varphi \Delta h$ and $Z' = h'(X)$. Then $Z' \underset{top}{\hookrightarrow} T \times Z$ hence $w(Z) \leq \tau$. Moreover $h'$ is perfect as a diagonal product of perfect and continuous mappings. Applying lemma 14 to a space X and continuous mapping $h'$ we conclude that there exist a d-open mapping f of X onto a space Y of weight $\leq \tau$ and a continuous mapping $g':Y \to Z'$ such that $h' = g' \circ f$. Then f is a perfect mapping (lemma 15). Lemma 6 implies that f is open. Let r be the unique continuous mapping of $Z'$ onto Z such that $r \circ h' = h$. Put $g = r \circ g'$. Evidently, $h = g \circ f$. Thus the lemma is proved.

Theorem 4. Let $\mu$ be an uncountable regular cardinal and X be a space of weight $\mu$ such that $\ell(X) \cdot \psi_{\varkappa}(X) < \mu$ and $\tau^{c(X)} < \mu$ for each $\tau < \mu$. Then a space X is a limit of some well-ordered spectrum of length $\mu$ with d-open projections consisting of spaces of weights $< \mu$.

The above theorem follows from Theorem 1 and Lemma 14. In the same manner we formulate the following result which is an easy corollary of Theorem 1 and Lemma 16.

Theorem 5. Let $\mu$ be an uncountable regular cardinal and X be a space of weight $\mu$ such that $w_c(X) \cdot \psi_{\varkappa}(X) < \mu$ and $\tau^{c(X)} < \mu$ for each $\tau < \mu$. Then X is a limit of some well-ordered spectrum S of length $\mu$ with perfect open projections consisting of spaces of weights $< \mu$.

Question 2. Can one make a spectrum S in Theorem 5 continuous?

Now we proceed to the discussion of the following question. Let S be a well-ordered spectrum consisting of spaces with Souslin property. What kind of projections should a spectrum S have to insure us that a limit of S has Souslin property, too?

To give some sufficient conditions we need the following definition by I.A. Vainstein (see [11]).

Definition 9. A mapping $f:X \longrightarrow Y$ is called semiopen if an interior of $f(\mathcal{O})$ is non-empty for each non-empty open subset $\mathcal{O} \subseteq X$.

Definition 10 (E.V. Ščepin). Let $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ be a spectrum, $\alpha^* < \tau$ and $A \subseteq X = \varprojlim S$. We will say that $A$ does not depend on $\alpha^*$ if $p_{\alpha^*}(A) = (p_\alpha^{\alpha^*})^{-1} p_\alpha A$ for some $\alpha < \alpha^*$ where $p_\gamma$ is a limit projection of $X$ to $X_\gamma$ for every $\gamma < \tau$. Let $k(A)$ be a set of all ordinals $\alpha^* < \tau$ a set $A$ depends on. From the definition it follows that $0 \in k(A)$ for each non-empty $A \subseteq X$. We will say that $A$ is a set of a finite type iff $|k(A)| < \aleph_0$ and there exists $\alpha < \tau$ such that $A = p_\alpha^{-1} p_\alpha(A)$.

Lemma 17. Let $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta<\tau}$ be an almost continuous spectrum with semiopen projections and $X = \varprojlim S$. Then the family of open subsets of finite type in $X$ forms a $\pi$-base for $X$.

Proof. Let $\mathcal{P}(\gamma)$ be the following statement: if $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\gamma}$ is an almost continuous spectrum of length $\gamma$ with semiopen projections then a limit of $T$ has a $\pi$-base consisting of sets of finite type (with respect to $T$). It is obvious that $\mathcal{P}(\gamma)$ holds for each $\gamma < \omega$. Let $\gamma \geq \omega$ and $\mathcal{P}(\gamma')$ holds for each $\gamma' < \gamma$.

I. $\gamma$ is a limit ordinal. Let $T = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta<\gamma}$ be an almost continuous spectrum with semiopen projections and $Y = \varprojlim T$. For each $\gamma' < \gamma$ put $T_{\gamma'+1} = \{Y_\alpha, q_\alpha^\beta\}_{\alpha,\beta\leq\gamma'}$; then $Y_{\gamma'} \cong \varprojlim T_{\gamma'+1}$. Further, $\mathcal{P}(\gamma'+1)$ implies that the open sets of finite type in $Y_{\gamma'}$ (with respect to $T_{\gamma'+1}$) form a $\pi$-base $\mathcal{B}_{\gamma'}$ for $Y_{\gamma'}$. Put $\mathcal{B}_{\gamma} = \{q_{\gamma'}^{-1} V: \gamma' < \gamma$ and $V \in \mathcal{B}_{\gamma'}\}$ where $q_{\gamma'}: Y \to Y_{\gamma'}$ is a limit projection for every $\gamma' < \gamma$. It is obvious that $\mathcal{B}_{\gamma}$ is a $\pi$-base for Y consisting of open sets of finite type with respect to a spectrum T.

II. $\gamma = \delta + 1$ where $\delta$ is a limit ordinal. Let $T = \{Y_{\alpha}\, q_{\alpha}^{\beta}\}_{\alpha,\beta<\gamma}$ be an almost continuous spectrum with semi-open projections.

Put $Y = \varprojlim \widetilde{T}$ where $\widetilde{T} = \{Y_{\alpha}\, q_{\alpha}^{\beta}\}_{\alpha,\beta<\delta}$. An almost continuity of a spectrum T implies that $Y_{\delta}$ is dense in Y. According to our inductive assumption $\mathcal{P}(\delta)$ holds hence the open sets of finite type in Y form a $\pi$-base $\widetilde{\mathcal{B}}_{\delta}$ for Y. Put $\mathcal{B}_{\delta} = \{U \cap Y_{\delta} : U \in \widetilde{\mathcal{B}}_{\delta}\}$. As $p_{\mu}^{\delta}$ is a mapping onto for every $\mu < \delta$ so $\mathcal{B}_{\delta}$ is a $\pi$-base for Y consisting of sets of finite type.

III. $\gamma = \delta + 1$ where $\delta$ is a non-limit ordinal. Let $T = \{Y_{\alpha}, q_{\alpha}^{\beta}\}$ be a spectrum as above. Let $\delta = \mu + 1$ and $T_{\delta} = \{Y_{\alpha}, q_{\alpha}^{\beta}\}_{\alpha,\beta<\delta}$. Then $Y_{\mu} = \varprojlim T_{\delta}$ and $\mathcal{P}(\delta)$ implies that the open sets of finite type in $Y_{\mu}$ (with respect to $T_{\delta}$) form a $\pi$-base $\mathcal{B}_{\mu}$ for $Y_{\mu}$. Let $O \neq \Lambda$ be some open subset of $Y_{\delta}$. Then there exists a non-empty open subset $U \subseteq Y_{\mu}$ such that $U \subseteq q_{\mu}^{\delta}(O)$. But $\mathcal{B}_{\mu}$ is a $\pi$-base for $Y_{\mu}$ hence we can choose an element $V \in \mathcal{B}_{\mu}$ such that $\Lambda \neq V \subseteq U$. Then a non-empty open subset $O' = O \cap (q_{\mu}^{\delta})^{-1} V$ of $Y_{\delta}$ is contained in $O$ and $q_{\mu}^{\delta}(O') = V$. Therefore $O'$ is an open set of finite type in $Y_{\delta}$ with respect to T. Thus $\mathcal{P}(\gamma)$ holds for every $\gamma$. This completes the proof.

Lemma 18. Let $S = \{X_{\alpha}, p_{\alpha}^{\beta}\}_{\alpha,\beta<\tau}$ be a spectrum and X =

$= \varprojlim S$. Let also A and B be disjoint subsets of finite type in X (with respect to S). Then there exists an ordinal $\alpha \in k(A) \cap k(B)$ such that $p_\alpha(A) \cap p_\alpha(B) = \Lambda$.

Proof. We will put $n = |k(A)| + |k(B)|$ and prove our lemma by induction. The case $A = \Lambda$ or $B = \Lambda$ is trivial hence we assume that A and B are non-empty sets. Then $0 \in k(A) \cap k(B)$ so $n \geq 2$. If $n = 2$ then $A = p_0^{-1} p_0(A)$ and $B = p_0^{-1} p_0(B)$ which implies that $p_0(A) \cap p_0(B) = \Lambda$ (we recall that $p_0$ is a mapping onto).

Now let us assume that the lemma's conclusion is proved for all $n \leq m$ where $m \geq 2$ and prove it for $n = m + 1$. Put $P = k(A) \cap k(B)$. Then $0 \in P$ hence $P \neq \Lambda$. Put $\alpha^* = \max P$. We claim that $p_{\alpha^*}(A) \cap p_{\alpha^*}(B) = \Lambda$. Indeed, assume the contrary. Put $Q = \{\alpha \in k(A) : \alpha^* \leq \alpha\}$, $R = \{\beta \in k(B) : \alpha^* \leq \beta\}$ and $\gamma = \max (R \cup Q)$. Then $\alpha^* < \gamma$ otherwise $A = p_{\alpha^*}^{-1} p_{\alpha^*}(A)$ and $B = p_{\alpha^*}^{-1} p_{\alpha^*}(B)$ which implies that $p_{\alpha^*}(A) \cap p_{\alpha^*}(B) = \Lambda$, i.e. a contradiction. Without loss of generality we may assume that $\gamma \in Q$. Then $k(\tilde{A}) = k(A) \setminus \{\gamma\}$ hence $|k(\tilde{A})| + |k(B)| = m$. It is obvious that $k(\tilde{A}) \cap k(B) = P$ and $p_\mu(\tilde{A}) \cap p_\mu(B) = p_\mu(A) \cap p_\mu(B) \neq \Lambda$ for each $\mu \in P$ (because $p_{\alpha^*}(A) \cap p_{\alpha^*}(B) \neq \Lambda$). So the inductive assumption implies that $\tilde{A} \cap B \neq \Lambda$. Moreover $\Lambda \neq p_\alpha(\tilde{A}) \cap p_\alpha(B) = p_\alpha(A) \cap p_\alpha(B)$. We choose a point $x \in p_\alpha(A) \cap p_\alpha(B)$. Let us consider two cases.

I. $\beta \leq \alpha$. There exists a point $z \in A$ such that $p_\alpha(z) = x$. Then the equality $B = p_\beta^{-1} p_\beta(B)$ implies that $z \in A \cap B \neq \Lambda$ which contradicts the lemma's condition.

II. $\alpha < \beta$. There exists a point $y \in p_\beta(B)$ such that $p_\alpha^\beta(y) = x$. Then the equality $p_\beta(A) = (p_\alpha^\beta)^{-1} p_\alpha(A)$ implies that $y \in p_\beta(A) \cap p_\beta(B)$. Now we may choose a point $z \in A$ such that $p_\beta(z) = y$. Since $B = p_\beta^{-1} p_\beta(B)$ we conclude that $z \in A \cap \cap B \neq \Lambda$ which is a contradiction.

Thus $p_{\alpha^*}(A) \cap p_{\alpha^*}(B) = \Lambda$ which completes the proof.

Lemma 19. Let $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta \in A}$ be a spectrum with semiopen projections and $X = \varprojlim S$. Then the limit projections $p_\alpha : X \to X_\alpha$ are semiopen, too.

Proof. Let U be an open non-empty subset of X and $\alpha \in A$. Let $x \in \mathcal{U}$. Then there exist an element $\beta \in A$ and an open subset $V \subseteq X_\beta$ such that $x \in p_\beta^{-1} V \subseteq \mathcal{U}$. Further, there exists an element $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Put $W = (p_\beta^\gamma)^{-1} V$. Then $W \subseteq p_\gamma(\mathcal{U})$ and $p_\alpha^\gamma(W) \subseteq p_\alpha^\gamma p_\gamma(\mathcal{U}) = p_\alpha(\mathcal{U})$. Since W is an open non-empty subset of $X_\gamma$ and $p_\alpha^\gamma$ is semiopen, there exists an open non-empty subset $G \subseteq X_\alpha$ such that $G \subseteq p_\alpha(W)$. Thus $G \subseteq p_\alpha(\mathcal{U})$. Lemma is proved.

Theorem 6. Let $S = \{X_\alpha, p_\alpha^\beta\}_{\alpha,\beta < \tau}$ be an almost continuous spectrum with semiopen projections, $X = \varprojlim S$, and $c(X_\alpha) \leq \lambda$ for each $\alpha < \tau$ where $\lambda$ is an infinite cardinal. Then $c(X) \leq \lambda$. Analogously, if $(\mu, \lambda)$ is a precaliber (caliber) for every $X_\alpha$ x) then $(\mu, \lambda)$ is a precaliber (caliber) for X.

Proof. We will prove only the first part of the theorem using the standard method of quasi-disjoint families. Assume that $c(X) > \lambda$. Then there exists a disjoint family $\gamma$ consisting of non-empty open subsets of X with $|\gamma| = \lambda^+$. Since

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x) A pair $(\mu, \lambda)$ of cardinals is said to be a precaliber of a space X iff for every family $\gamma$ consisting of non-empty open subsets of X with $|\gamma| \geq \lambda$ there exists a subfamily $\gamma' \subseteq \gamma$ with finite intersection property such that $|\gamma'| \geq \mu$.

the family of all open subsets of finite type in $X$ (with respect to $S$) forms a $\pi$-base for $X$ (lemma 17) without loss of generality one can assume that all elements of $\gamma$ are of finite type. For every $\mathcal{U} \in \gamma$ put $p_{\mathcal{U}} = k(\mathcal{U})$. Since $\lambda^+$ is a regular cardinal and $|P_{\mathcal{U}}| < \aleph_0$ for each $\mathcal{U} \in \gamma$ there exist a finite set $P \subset \tau$ and a subfamily $\gamma' \subseteq \gamma$ with $|\gamma'| = \lambda^+$ such that $P_{\mathcal{U}} \cap P_V = P$ whenever $\mathcal{U}, V \in \gamma'$ and $P_{\mathcal{U}} \neq P_V$. Put $\alpha^+ = \max P$.

Then Lemma 18 implies that $p_{\alpha^*}(\mathcal{U}) \cap p_{\alpha^*}(V)$ for each different $\mathcal{U}, V \in \gamma'$. This contradicts the inequality $c(X_{\alpha^*}) \leq \lambda$ because $\mathrm{Int}_{p_{\alpha^*}}(\mathcal{U}) \neq \Lambda$ for each $\mathcal{U} \in \gamma'$ (Lemma 19). Therefore $c(X) \leq \lambda$. The theorem is proved.

Theorem 6 generalizes a similar Ščepin's result concerning the case when $S$ is a continuous spectrum with open projections.

<u>Definition 10</u>. Let $\{f: X \to X_f\}_{f \in \mathcal{E}}$ be a family of continuous mappings of $X$ and $\otimes \mathcal{E}$ be a homeomorphism. We will say that $\mathcal{E}$ is a $\sigma$-system iff $\otimes \gamma \in \mathcal{E}$ for each countable subfamily $\gamma \subseteq \mathcal{E}$ and every $f \in \mathcal{E}$ is a mapping onto (here $\otimes \gamma$ is a diagonal product of a family $\gamma$ considered as a mapping of a space $X$ onto its image).

Our last result generalizes Arhangel'skii theorem concerning the mappings of dense subspaces of products (see [5], Theorem 1).

<u>Theorem 7</u>. Let $\mathcal{E}$ be a $\sigma$-system consisting of open mappings of $X$ and $f(X)$ has a countable network for each $f \in \mathcal{E}$. Let $S$ be a dense subset of $X$, $\varphi$ be a continuous mapping of $S$ onto a regular space $Y$ and $M = \{y \in Y : \chi(y, Y) \leq \aleph_0\}$.

Then there exist a mapping $f \in \mathcal{E}$ and a continuous mapping $\psi : f(N) \to M$ such that $\varphi | N = \psi \circ (f|N)$ where $N = \varphi^{-1}M$. In particular, $n\,w(M) \le \aleph_0$.

Theorem 7 can be proved analogously to the same in [5]. However, to do this, one should reformulate lemmas 17 and 18 for $\mathfrak{S}$-systems. This reformulations do not present any difficulties.

Corollary 4. Let S be a dense subspace of a $\varkappa$-metrizable compact space X and a first-countable regular space X be a continuous image of S. Then $n\,w(Y) \le \aleph_0$.

Proof. As X is a $\varkappa$-metrizable compact space, there exists a $\mathfrak{S}$-system $\mathcal{E}$ of open mappings of X onto compact metric spaces (see [1], Theorem). Therefore Theorem 7 implies that $n\,w(Y) \le \aleph_0$.

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