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ON NUMBER OF COVERING ARCS IN ORDERINGS

V. KOUBEK, V. RÖDL

Abstract: Two of results - a distributive lattice on n-point set contains at most n $\log_2 n$ covering arcs. If the digraph of covering arcs of an ordering of n point set does not contain $\mathcal{K}_{a,2}$ ($\mathcal{K}_{a,2}$ is the digraph consisting of all arcs leading from a point set to two-element set) then it has at most $(1 + o(1))\frac{2}{3}\sqrt{a-1}\, n^{3/2}$ arcs.

Key words: Covering arc, transitive reduct, transitive closure, ordering, lattice, distributive lattice, algorithm.

Classification: 05C30, 05C20, 05A15

One of the possibilities of an economical description of an ordering is by means of its covering arcs - a directed graph (X,R) is a transitive reduct (or a Hasse diagram, or a graph of covering arcs) of an ordering $(X, \leq)$ if it is the smallest directed graph such that $(X, \leq)$ is a transitive and reflexive closure of (X,R). It is clear that if X is finite then for every ordering $(X, \leq)$ there exists its transitive reduct. The aim of this note is to give estimates of the maximal number of arcs in the transitive reduct for special classes of orderings. We give also some applications of these estimates.

In this note all sets (except the set $\mathbb{N}$ of all natural numbers) will be finite. If X is a finite set then $|X|$ denotes the size of X. For a directed graph (X,R) denote

$xR = \{y; (x,y) \in R\}$, $Rx = \{y; (y,x) \in R\}$ for each $x \in X$. If $(X, \leq)$ is an ordering then its transitive reduct is denoted by $\mathrm{Red}(X, \leq)$.

Recall that an ordering $(X, \leq)$ is a lattice if every couple of points $x, y \in X$ has the smallest upper bound (or supremum) - denote it by $x \vee y$, and the biggest lower bound (or infimum) - denote it by $x \wedge y$. A lattice $(X, \leq)$ is called distributive if for every triple x, y, z of points of X

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

holds.

If $\mathcal{G}$ is a finite set of directed graphs, then an ordering $(X, \leq)$ has property $\mathcal{P}(\mathcal{G})$ if for no graph $(Y,R) \in \mathcal{G}$ there is a one-to-one compatible mapping from (Y,R) to $\mathrm{Red}(X, \leq)$.

Define functions $d, \ell, p_{\mathcal{G}}$ from $\mathbb{N}$ to itself as follows: $d(n) = \max \{|R|; (X,R) = \mathrm{Red}(X, \leq), |X| = n, (X, \leq)$ is a distributive lattice$\}$,

$\ell(n) = \max \{|R|; (X,R) = \mathrm{Red}(X, \leq), |X| = n, (X, \leq)$ is a lattice$\}$, $p_{\mathcal{G}}(n) = \max \{|R|; (X,R) = \mathrm{Red}(X, \leq), |X| = n, (X, \leq)$ is an ordering with the property $\mathcal{P}(\mathcal{G})\}$.

For positive integers a, b define a directed graph $\mathcal{K}_{a,b} = (X,R)$ where $X = \{0, 1, \ldots, a+b-1\}$ and $R = \{(i,j); 0 \leq i < a, a \leq j < a+b\}$. Denote for $a \leq b$, $p_{a,b} = p_{\{\mathcal{K}_{a,b}, \mathcal{K}_{b,a}\}}$. We give the asymptotical estimates of functions $d, \ell, p_{a,b}$ for $b \geq a > 1$. First we give two easy observations.

<u>Lemma 1</u>: Every lattice has property $P(\{\mathcal{K}_{2,2}\})$.

<u>Lemma 2</u>: For every natural number n

$d(n) \leq \ell(n) \leq p_{2,2}(n)$

and for $a \leq c$, $b \leq d$ $p_{a,b}(n) \leq p_{c,d}(n)$.

Theorem 3: $(1 + o(1)) \cdot \frac{n}{2} \cdot \log n \leq d(n) \leq n \cdot \log_2 n$.

Proof: First we prove the lower bound of $d(n)$. For this purpose we consider the lattice of all subsets of a set X with $|X| = k$. This lattice is distributive and for $Z, C \subset X$, (Z,V) is a covering arc iff $Z \subset V$ and $|Z| + 1 = |V|$. Thus there are $|V|$ covering arcs leading to V. Hence the number of covering arcs in this lattice is

$$\sum_{j=0}^{k} \binom{k}{j} j = \sum_{j=0}^{k} \frac{k}{2} \binom{k}{j} = \frac{k}{2} 2^k = k2^{k-1}$$

This lattice has 2^k points and so if $n = 2^k$ the $d(n) \geq \frac{n}{2} \log_2 n$. Let n be a positive integer. Then there exists exactly one increasing sequence $\{j_1, j_2, \ldots, j_k\}$ of non-negative integers with $n = \sum_{i=1}^{k} 2^{j_i}$. For each $i = 1,2,\ldots,k$, let $(X_i, \leq)$ be the distributive lattice of all subsets of the set $\{0,1,\ldots,j_i - 1\}$. We form a lattice $(X, \leq)$ such that X is a disjoint union of X_i and we define that for each $i = 1,2,\ldots,k-1$ the smallest element of $(X_{i+1}, \leq)$ is bigger than the biggest element of $(X_i, \leq)$. Then $(X, \leq)$ is a distributive lattice, $|X| = n$ and if $(X,R) = \mathrm{Red}(X, \leq)$ then $|R| = \sum_{i=1}^{k} j_i 2^{j_i - 1} + k - 1$. Further

$$\frac{n}{2} \log_2 n - \sum_{i=1}^{k} j_i 2^{j_i - 1} = \sum_{i=1}^{k} (2^{j_i - 1} \log_2 n - j_i 2^{j_i - 1}) =$$

$$= \sum_{i=1}^{k} (\log_2 n - j_i) 2^{j_i - 1} \underset{(1)}{\leq} \sum_{i=1}^{k} 2^{j_i - 1} + \sum_{i=1}^{k-1} (j_k - j_i) 2^{j_i - 1} \leq$$

$$\leq \sum_{i=1}^{k} 2^{j_i - 1} + \sum_{i=1}^{j_k - 1} j_k 2^{i-1} - \sum_{i=1}^{j_k - 1} i 2^{i-1} \underset{(2)}{<} \frac{n}{2} +$$

$$+ \sum_{i=1}^{j_k} 2^{i-1} \underset{(3)}{\leq} 2{,}5\, n$$

Here we used that

(1) $$j_k \leq \log_2 n < j_k + 1$$

(2) $$\sum_{i=1}^{t} i2^{i-1} = \sum_{i=2}^{t+1} (i-1)2^{i-1} - \sum_{i=1}^{t} i2^{i-1} = t2^t - 1 - \sum_{i=2}^{t} 2^{i-1}$$

and

(3) $$\sum_{i=1}^{j_k} 2^{i-1} \leq 2 \sum_{i=1}^{k} 2^{j_i - 1}$$

Thus we get the lower bound of $d(n)$.

We prove the upper bound of $d(n)$. Let $(X,\leq)$ be a distributive lattice with $(X,R) = \mathrm{Red}(X,\leq)$. We show that if for $x\in X$, $|Rx| = k$, then there exists a one-to-one mapping φ from the set of all subsets of Rx to X, hence $|X| \geq 2^k$ and so we have $|Rx| \leq \log_2 |X|$. Define $\varphi(\emptyset) = x$ and for $\emptyset \neq Z \subset Rx$, $\varphi(Z) = \bigvee Z$ (the smallest upper bound of Z; it exists because $(X,\leq)$ is a finite lattice). To end the proof we have to prove that φ is an injection. By distributivity we get $\bigwedge \{\varphi(Z), \varphi(V)\} = \varphi(Z \cap V)$ for each $Z,V \subset Rx$. Thus if φ is not injective then there exists a set $Z \subset Rx$ such that for some $y \in Rx - Z$ we have

$$\varphi(Z) = \varphi(Z \cup \{y\}).$$

Assume that Z is a subset of Rx with the smallest size such that for some $y\in Rx - Z$, $\varphi(Z) = \varphi(Z\cup\{y\})$. If $|Z| = 1$ then $\varphi(Z) = z$ for $z\in Z$ and therefore $|Z| > 1$. Choose $z \in Z$ and put $V = Z - \{z\}$. Then $|V| < |Z|$, hence $\varphi(Z) \neq \varphi(V)$ and $\varphi(V \cup \{y\}) \neq \varphi(V)$. On the other hand, $\varphi(V) = \bigwedge\{\varphi(Z), \varphi(V\cup\{y\})\}$ and $\varphi(Z) = \varphi(Z\cup\{y\}) \geq \varphi(V\cup\{y\})$ because φ is compatible and so we have $\varphi(V) = \varphi(V\cup\{y\})$ – a contradiction. Thus φ is injective and hence $d(n) \leq n\cdot\log_2 n$.

<u>Conjecture</u>: $d(n) = (1 + o(1))\cdot \frac{n}{2}\cdot \log_2 n$.

Corollary 4: There exists an algorithm which for a directed graph (X,R) decides whether it is a distributive lattice and in the positive case constructs operations supremum and infimum in time proportional to $O(|X|^2 \log |X|)$.

Proof follows from Theorem 3 and Statements 1 and 2 in [4].

The best known algorithms deciding whether a bigroupoid $(X,\vee,\wedge)$ or a directed graph (X,R) is a lattice, require the same time - $O(|X|^{5/2})$ - see [4]. The analogous fact does not hold for a distributive lattice - there exists an algorithm deciding whether a bigroupoid is a distributive lattice in $O(|X|^2)$ time - see [3] whereas the best known algorithm deciding whether a directed graph is a distributive lattice is given in Corollary 4.

It was shown in [5], see also [4] that

$$(0) \qquad \ell(n) \geq (1 + o(1)) 8^{-1/2} n^{3/2}$$

and it follows by a result of W.G. Brown - see e.g. [1, § 12] that

$$p_{3,3}(n) \geq (1 + o(1)) \frac{1}{4} n^{5/3}$$

We conjecture that the equality holds in (0).
Now we prove

Theorem 5: Let a,b be given positive integers and $\varepsilon = \frac{a+b-2}{ab-1}$ then

$$p_{a,b}(n) \geq c \left\lfloor \frac{n}{2} \right\rfloor^{2-\varepsilon} \quad \text{for n sufficiently large and}$$

absolute constant c.

(Here we show $c = \frac{\left[0,1 \frac{a!\,b!}{a+b}\right]^{\frac{1}{ab-1}}}{e}$, where e = 2,71 .... is the base of natural logarithm but shall not make any attempt to find the best c with the above property.)

Note that the above theorem improves the inequality 12.1 in [1].

In the proof of the above theorem we shall use the following theorem of J. Spencer, which is a consequence of a theorem of L. Lovász - see [6].

First we introduce the following notions: Let Ω be a probability space and $A_1, A_2, \ldots, A_n$ events. The graph Γ with vertex set $\{1,2,\ldots,n\}$ is called <u>dependence graph</u> of $\{A_1, A_2, \ldots, A_n\}$ if $\{i,j\} \notin \Gamma$ iff A_i and A_j are mutually independent. $P(A_i)$ is a probability of A_i. The following is an easy consequence of Theorem 1.3 from [6].

<u>Theorem 6</u>: Let A_L, B_K $(L \in \mathcal{L}, K \in \mathcal{K})$ be events in a probability space Ω with dependence graph Γ. Let $N(L,\mathcal{K})$ be the number of vertices of type $\mathcal{K}$ adjacent in Γ to a vertex corresponding to L. Set $N_{AB} = \max\{N(L,\mathcal{K});\ L \in \mathcal{L}\}$ and let N_{AA}, N_{BA}, N_{BB} be defined analogously. Suppose that to each event $A_L (B_K)$ there is associated some $y_L = y$ $(z_K = z)$ such that

$$yP(A_L) < 1, \quad zP(B_K) < 1$$

$$\ell n y > yP(A_L)N_{AA} + zP(B_K)N_{AB}$$

$$\ell n z > yP(A_L)N_{BA} + zP(B_K)N_{BB}$$

then

$$P\Big(\bigwedge_{L \in \mathcal{L}} \overline{A}_L \wedge \bigwedge_{K \in \mathcal{K}} \overline{B}_K\Big) > 0$$

Here by $\overline{A}$ we denote the event complementar to A.

<u>Proof of Theorem 5</u>: We employ the probabilistic method. Suppose, we are given positite integers a,b (without loss of generality assume $a \geq b \geq 2$). For a given sufficiently large n (this will be specified later) let $\mathbb{G}$ be a random subset of

$V_1 \times V_2$ (V_1, V_2 are disjoint sets $|V_1| = |V_2| = \lfloor \frac{n}{2} \rfloor = m$), where the elements of $\mathcal{G}$ (arcs) are chosen independently, each with probability $p = c_o m^{-\varepsilon}$ where $\varepsilon = \frac{a+b-2}{ab-1}$ and $c_o = $
$= [0,1 \frac{a!b!}{a+b}]^{\frac{1}{ab-1}}$. Set

$$\mathcal{L} = [V_1]^a \times [V_2]^b \cup [V_1]^b \times [V_2]^a$$

(where $[V_i]^a$ denotes the set of all a-element subsets of V_i; $[V_i]^b$ is defined similarly), and $\mathcal{K} = V_1$.

To every $L \in \mathcal{L}$ (i.e. to pair $(S,T) \in \mathcal{L}$) associate the event A_L, that $S \times T \subset \mathcal{G}$. Similarly to every $K \in \mathcal{K}$ (i.e. to $v \in V_1$) associate the event B_K that the number of arcs of $\mathcal{G}$ incident to v is at most $\frac{pe}{m}$ - here $e = 2{,}71 \ldots$ is the base of natural logarithm.

Then we have

(1) $\qquad P(A_L) = p^{ab}$ for every $L \in \mathcal{L}$

and

(2) $\qquad P(B_K) \leq \exp[(\frac{2}{e} - 1)pm]$

(2) follows by Chernoff inequality (see [1], 3.7) using elementary computation

$(P(B_K) = \sum_{j \leq k} \binom{m}{j} p^j (1-p)^{m-j} \leq \exp[(m-k)\log\frac{m(1-p)}{m-k} + k \log \frac{mp}{k}] \leq$
$\leq \exp[(\frac{2}{e} - 1)pm]$
for $k = \lfloor \frac{pm}{e} \rfloor$)

Let Ω be now a probability space with events A_L, $L \in \mathcal{L}$ and B_K, $K \in \mathcal{K}$ and let N_{AA}, N_{AB}, N_{BA}, N_{BB} be the numbers defined in Theorem 6. Then

$$N_{AA} \leq 2ab\binom{m}{a-1}\binom{m}{b-1} < \frac{2ab}{(a-1)!(b-1)!} m^{a+b-2}$$

$$N_{BA} \leq \binom{m-1}{a-1}\binom{m}{b} + \binom{m}{a}\binom{m-1}{b-1} < \frac{m^{a+b-1}}{a!b!}(a + b)$$

$N_{AB} \leq \mathrm{Max}\{a,b\} = a$

$N_{BB} = 0$

The theorem states now: if there exist positive y,z such that

(3) $\quad yP(A_L) < 1, \quad zP(B_K) < 1$

(4) $\quad \ln y > \frac{2ab\, c_0^{ab}}{(a-1)!(b-1)!}\, y m^{a+b-2-ab\varepsilon} + az \exp[c_0(\frac{2}{e}-1)m^{1-\varepsilon}]$

(5) $\quad \ln z > \frac{(a+b)\, c_0^{ab}}{a!\,b!}\, y m^{a+b-1-ab\varepsilon}$

then there exists $G \in \mathcal{G}$ such that

i) the valency of arbitrary vertex $v \in V_1$ is more than $\frac{c_0}{e} m^{1-\varepsilon}$

ii) $S \times T$ is not a subset of G for any choice of $(S,T) \in \mathcal{L}$.

Set $y = 1,1$

$z = \exp[0,2\, c_0 m^{1-\varepsilon}]$

$c_1 = \frac{2ab\, c_0^{ab}}{(a-1)!(b-1)!}\, y$

$c_2 = \frac{(a+b)}{a!\,b!}\, y$

then (3),(4),(5) become

(3′) $\quad 1,1\, p^{ab} < 1, \ \exp[c_0(\frac{2}{e} - \frac{4}{5})m^{1-\varepsilon}] < 1$

(4′) $\quad \ln 1,1 > c_1 m^{-\varepsilon} + a \exp[c_0(\frac{2}{e} - \frac{4}{5})\, m^{1-\varepsilon}]$

(5′) $\quad 0,2 c_0 m^{1-\varepsilon} > c_2\, c_0^{ab}\, m^{1-\varepsilon}$

which is satisfied for $m \geq m_0(a,b)$ (where $m_0(a,b)$ is an absolute constant depending on a and b only)

It follows now from (i) and (ii) that there exists $G \in \mathcal{G}$ which fulfils $\mathcal{P}(\{\mathcal{K}_{a,b}, \mathcal{K}_{b,a}\})$ and, moreover, has more than $\frac{c_0}{e} m^{2-\varepsilon}$ arcs. This proves our Theorem.

It follows from (0) that $p_{a,2}(n) \geq (1 + o(1))8^{-1/2} n^{3/2}$. This can be slightly improved, as shown by the following

Proposition 7:

$$p_{a,2}(n) \geq (1 + o(1))2 \sqrt{\frac{n^3 \lfloor\sqrt{a}\rfloor^3}{(\lfloor\sqrt{a}\rfloor + 2)^3}}$$

We omit the tedious proof which is based on the convenient modification of the digraph, used in [4] for the proof of (0).

Note that the above proposition shows $p_{a,2} \geq c_a n^{3/2}$ where $c_a \to 2$ as $n \to \infty$. However, in Theorem 9 we show $p_{a,2} \leq$ $\leq c'_a n^{3/2}$ where c'_a tends to infinity as $a \to \infty$. We believe that the upper bound in Theorem 9 is closer to true and conjecture: $p_{a,2}(n) = (1 + o(1))d_a n^{3/2}$ where $\lim_{a \to \infty} d_a = \infty$

(6) Theorem 8: $p_{a,b}(n) \leq nb + (a-1)^{1/b} n^{2-1/b}$

Proof: Let $(X, \leq)$ be an ordering with the property $\mathcal{P}(\{\mathcal{K}_{a,b}, \mathcal{K}_{b,a}\})$ such that $|X| = n$. Let $x_1, x_2, \dots, x_n$ be in-degrees of all vertices of $(X,R) = \mathrm{Red}(X, \leq)$. Since every subset Z of X with $|Z| = b$ can be contained in at most a-1 neighborhoods of vertices of X we have

$$\sum_{i=1}^{n} \binom{x_i}{b} \leq (a-1) \binom{n}{b}$$

Using elementary computation this can be converted to (6) (as $p_{a,b}(n) \leq \sum_{i=1}^{n} x_i$)

For b = 2, we can strengthen the above theorem. Namely, we prove the following:

Theorem 9: $p_{a,2}(n) \leq (1 + o(1))\frac{2}{3}\sqrt{a-1}\, n^{3/2}$

Proof: Let ε_n denote the minimal number such that reduct of every ordering $(X,\leq)$, $|X| = n$ having property $\mathcal{P}(\{\mathcal{K}_{a,2}, \mathcal{K}_{2,a}\})$ $a > 1$ has at most

$$p_{a,2}(n) = (1 + \varepsilon_n)\frac{2}{3}\sqrt{a-1}\, n^{3/2}$$

arcs. We show that $\lim_{n\to\infty} \varepsilon_n = 0$. Let $1 > \varepsilon > 0$ be given and let $(X,\leq)$ be an ordering with property $\mathcal{P}(\{\mathcal{K}_{a,2}, \mathcal{K}_{2,a}\})$. Suppose that $X = \{0,1,\dots,n-1\}$ and that the natural order of integers extends that of $(X,\leq)$.

Set $(X,R) = \mathrm{Red}(X,\leq)$; we show that

$$|R| \leq (1 + \varepsilon_n)\frac{2}{3}\sqrt{a-1}\, n^{3/2} \leq (1+\varepsilon)\frac{2}{3}\sqrt{a-1}\, n^{3/2}$$

if n is large enough.

Let $x_0, x_1, \dots, x_{n-1}$ be indegrees of vertices $0,1,2,\dots,n-1$. Set $m = \alpha n$, where $\alpha = \frac{\varepsilon}{10}$, and for j, $m \leq j < n$ let the number of arcs of the form (i,j), $0 \leq i \leq m-1$ be denoted by y_j; the number of arcs of the form (i,j), $m \leq i$ be denoted by z_j.

Then, analogously, as in Theorem 8, we have

$$\sum_{i=0}^{m-1}\binom{x_i}{2} + \sum_{j=m}^{n-1}\binom{y_j}{2} \leq (a-1)\binom{m}{2}$$

and hence

$$\sum_{i=0}^{m-1} \bar{x}_i^2 + \sum_{j=m}^{n-1} \bar{y}_j^2 \leq (a-1)m^2$$

where $\bar{x}_i = x_i - 1$ and $\bar{y}_j = y_j - 1$

Set

$$\sum_{i=0}^{m-1} (\bar{x}_i)^2 = \sigma^2(a-1)m^2 = \sigma^2(a-1)\alpha^2 n^2$$

Then

$$\sum_{j=m}^{n-1} \bar{y}_j^2 \leq (1 - \sigma^2)m^2(a-1)$$

As the number of arcs of (X,R) is $\sum_{i=0}^{m-1} x_i + \sum_{j=m}^{n-1}(y_j + z_j)$

we have by Cauchy-Schwarz inequality that

$$|R| \leq n + \sigma \sqrt{a-1}\, m^{3/2} + (1-\sigma^2)^{1/2} \sqrt{a-1}\, (n-m)^{1/2} m +$$

$$= \sqrt{a-1}\, (1 + \varepsilon_{n-m}) \tfrac{2}{3} (n-m)^{3/2} \leq n +$$

$$+ \sqrt{a-1}\, n^{3/2} (\alpha + \tfrac{2}{3} (1+\varepsilon_{n-m})(1-\alpha)^{3/2})$$

because by an easy computation $0 \leq \sigma \leq \alpha^{1/2}$.

Thus we get

(7) $$p_{a,2}(n) \leq n + \sqrt{a-1}\, \alpha\, n^{3/2} + p_{a,2}((1-\alpha)n)$$

Moreover, by Theorem 8

(8) $$p_{a,2}(n) \leq 2\, n + \sqrt{(a-1)}\, n^{3/2} \leq 2 \sqrt{a-1}\, n^{3/2}$$

for $n \geq n_o$

Let n be so large that

(9) $$\tfrac{\varepsilon}{8} n^{3/2} \geq (2n_o)^{3/2},\ \tfrac{n}{\alpha} \leq \tfrac{\varepsilon}{4} n^{3/2} \sqrt{a-1}$$

and let t be the largest integer that

(10) $$(1-\alpha)^t n \geq n_o$$

(11) Then we have clearly $(1-\alpha)^t n \leq 2n_o$

Combining (7),(8),(9),(10) and (11) we get

$$p_{a,2}(n) \leq \sum_{i=0}^{t-1} (1-\alpha)^i n + \sqrt{a-1}\, \alpha n^{3/2} \sum_{i=0}^{t-1} (1-\alpha)^{i3/2} +$$

$$+ p_{a,2}((1-\alpha)^t n) \leq \frac{n}{\alpha} + \sqrt{a-1}\, n^{3/2} \frac{\alpha}{1-(1-\alpha)^{3/2}} +$$

$$+ 2\sqrt{a-1}(1-\alpha)^{t3/2} \cdot n^{3/2} \leq \frac{n}{\alpha} + 2\sqrt{a-1}(2n_o)^{3/2} +$$

$$+ \sqrt{a-1}\, n^{3/2} \frac{\alpha}{1-(1-{}^{3}/{}_{2}\,\alpha + \binom{3/2}{2}\alpha^2)} \leq$$

$$\leq \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{1}{3/2 - 3/8\,\alpha}\right) \sqrt{a-1}\, n^{3/2} < (1+\varepsilon)\tfrac{2}{3} \sqrt{a-1}\, n^{3/2}$$

The first inequality is obtained by t-1 iterations of (1).

Notice that in the proofs of Theorems 8 and 9 we used only facts that the ordering $(X, \leq)$ fulfils either $\mathcal{P}(\{\mathcal{K}_{a,b}\}$ or $\mathcal{P}(\{\mathcal{K}_{b,a}\})$. Thus the following holds:

Corollary 10: For every $a,b \geq 2$, $a \geq b$

$$P(\{\mathcal{K}_{a,b}\})(n),\ P(\{\mathcal{K}_{b,a}\})(n) \leq nb + (a-1)^{1/b} n^{2-1/b}$$

$$P(\{\mathcal{K}_{a,2}\})(n),\ P(\{\mathcal{K}_{2,a}\})(n) \leq (1 + o(1)) \tfrac{2}{3} \sqrt{a-1}\, n^{3/2}$$

Note: Theorems 8 and 9 hold also if, more generally, transitive reduct of ordered set is replaced by the transitive reduct of directed graph.

We mention in closing one application of Theorems 3 and 8. We say that an acyclic directed graph has the property D, or L, or $\mathcal{P}(G)$ if its transitive and reflexive closure is a distributive lattice, or a lattice, or it has the property $\mathcal{P}(\mathcal{G})$. A directed graph has the property D, or L, or $\mathcal{P}(\mathcal{G})$ if its quotient graph by the decomposition into strongly connected components has the property D, or L or $\mathcal{P}(\mathcal{G})$.

Corollary 11: If a directed graph (X,R) has the property D (or $\mathcal{P}(\{\mathcal{K}_{a,b}, \mathcal{K}_{b,a}\})$ for $a \leq b$ then there exists an algorithm constructing its transitive closure in $O(|X|^2 \cdot \log X)$ (or $O(X^{3-1/a})$ resp.) time.

Corollary 12: If a directed acyclic graph (X,R) has the property D (or $\mathcal{P}(\{\mathcal{K}_{a,b}, \mathcal{K}_{b,a}\})$ for $a \leq b$ then there exists an algorithm constructing its transitive reduct (i.e. the transitive reduct of its transitive and reflexive closure) in

$O(|X|^2 \log |X|)$, (or $O(|X|^{3-1/a}$ resp.) time.

The proof follows from Theorems 3 and 8 if we use the result in [3].

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